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Remarks on the Eigenpairs of Some Jacobi Matrices

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Abstract. In this note we show that the results recently established by D. Bozkurt and B.B. Altındağ can be derived from the well-known literature.

1. Preliminaries

The eigenvalues of the Jacobi matrix

$$A_n = \begin{pmatrix} a+b & b & & & & \\ b & a & b & & & \\ & b & \ddots & \ddots & & \\ & & \ddots & \ddots & b & \\ & & & b & a & b \\ & & & & b & a+b \end{pmatrix}_{n\times n}, \tag{1}$$

are known for more than 60 years. Indeed we can find the spectrum of A_n in [13] or [5, p.46]

$$\lambda_k = a + 2b\cos\frac{k\pi}{n}, \quad \text{for } k = 1, \ldots, n. \tag{2}$$

For the eigenvectors, we can find a complete description in [9, Theorem 3]. Another interesting matrix is

$$B_n = \begin{pmatrix} a & 2b & & & & \\ b & a & b & & & \\ & b & \ddots & \ddots & & \\ & & \ddots & \ddots & b & \\ & & & b & a & b \\ & & & & 2b & a \end{pmatrix}_{n\times n}. \tag{3}$$

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Since the eigenvalues of

$$J_n = \begin{pmatrix} 0 & 2 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & 0 & 1 \\ & & & & 2 & 0 \end{pmatrix}_{n\times n} . \tag{4}$$

are

$$\tilde{\mu}_k = -2\cos\frac{(k-1)\pi}{n-1}, \quad \text{for } k = 1, \ldots, n,$$

the eigenvalues of $B_n = aI_n + bJ_n$ are

$$\mu_k = a - 2b\cos\frac{(k-1)\pi}{n-1}, \quad \text{for } k = 1, \ldots, n.$$

We can find the eigenvalues of J_n for example in [10, 11]. They were also fully determined in [4, Table 1], as well as the corresponding eigenvectors. Notice that since $p(x) = a + bx$ is a polynomial, if $(\tilde{\mu}, u)$ is an eigenpair of J_n, then $(p(\tilde{\mu}), u)$ is an eigenpair of $B_n = p(J_n)$ (cf. [8, Theorem 1.1.6.]). For more general results and motivation the reader is referred to [6, 7].

The eigenpairs of A_n and B_n have been rediscovered by several authors. Knowing the eigenpairs of a tridiagonal matrix is essential for the computation of the powers of that matrix. This topic has attracted many researchers in the recent years. Our aim is to discuss the recent paper [3] by Bozkurt and Altındağ about the powers of slight sign changes to A_n and B_n.

2. Eigenpairs of two Jacobi matrices

In [3], Bozkurt and Altındağ considered the two matrices

$$\tilde{A}_n = \begin{pmatrix} a+b & b & & & & \\ b & a & -b & & & \\ & -b & \ddots & \ddots & & \\ & & \ddots & \ddots & -b & \\ & & & -b & a & b \\ & & & & b & a+b \end{pmatrix}_{n\times n} , \tag{5}$$

and

$$\tilde{B}_n = \begin{pmatrix} a & 2b & & & & \\ b & a & -b & & & \\ & -b & \ddots & \ddots & & \\ & & \ddots & \ddots & -b & \\ & & & -b & a & b \\ & & & & 2b & a \end{pmatrix}_{n\times n} . \tag{6}$$

While for the eigenvalues the minus signs are irrelevant, because each factor of the characteristic polynomial contains the product of the entries $(i, i+1)$ and $(i+1, i)$, for the eigenvectors the situation is different. Indeed, this symmetrization process is well-known for generic tridiagonal matrices (see, for example, [12]) or in more general instances as the acyclic matrices [1, 2].

Let us define the diagonal matrix D_n by

$$\mathrm{diag}\,(1, 1, -1, 1, \ldots, -1, 1, 1)$$

if n is odd, and

$$\mathrm{diag}\,(-1, -1, 1, \ldots, -1, 1, 1)$$

otherwise. It is clear that

$$D_n A_n D_n = \tilde{A}_n \quad \text{and} \quad D_n B_n D_n = \tilde{B}_n \, . \tag{7}$$

Therefore if (λ, u) is an eigenpair for A_n, $(\lambda, D_n u)$ is an eigenpair for $\tilde{A}_n$. Analogously, for B_n and $\tilde{B}_n$. This means that the eigenvectors of $\tilde{A}_n$ and $\tilde{B}_n$ follow immediately. Actually, using the similarity relations (7), all the main results of [3] follow immediately from [4].

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