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Polynomial Interpolation on the Unit Sphere and Some Properties of Its Integral Means

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Abstract. We study Hermite interpolation on the unit sphere. We give poised Hermite schemes on parallel circles with odd and even number of points on each circle. We also prove continuity and convergence properties of integral means of Hermite interpolation polynomials.

1. Introduction

Let $\mathcal{P}_n(\mathbb{R}^k)$ be the space of all polynomials of degree at most n in $\mathbb{R}^k$. It is known that $\dim\mathcal{P}_n(\mathbb{R}^k) = \binom{n+k}{k}$. We denote by S the unit sphere in $\mathbb{R}^3$, i.e., $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. The polynomials of degree at most n in $\mathbb{R}^3$, when restricted to S, form a vector space, say $\mathcal{P}_n(S)$, with

$$\dim \mathcal{P}_n(S) = (n+1)^2, \quad n \geq 0.$$

In one variable, the Lagrange and Hermite interpolation polynomials of functions at given points always exist. When the interpolated function is fixed and is sufficiently smooth, the interpolation polynomial is continuous with respect to the interpolation points (see Theorem 2.3 below). Moreover, if an array of interpolation points is suitably distributed, then the sequence of interpolation polynomials converges uniformly to the function.

Multivariate polynomial interpolation problems are more difficult. The problem of Hermite interpolation means to find a polynomial which matches, on a set of distinct points, values of a function and its partial derivatives. We deal with the case in which the number of interpolation conditions is equal to the dimension of the polynomial space. If the interpolation problem has a unique solution, then we say that the problem is poised. Unlike the univariate Hermite interpolation, the multivariate Hermite interpolation is not always poised. Moreover, it is difficult to check whether a particular Hermite scheme is poised. In addition, the above-mentioned continuity property of Hermite interpolation is not true in the multivariate case without additional assumptions (see for instance [1, 9]). In other words, multivariate interpolation is unstable when the interpolation points coalesce. Similarly, it is not easy to find an array of interpolation points such that the interpolation polynomials of smooth functions converge uniformly to the function (see

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for example [2, 3, 8, 12]). The trouble here is that the interpolation operator has bad behavior when interpolation points tend to an algebraic hypersurface. Furthermore, multivariate interpolation polynomials lack compact formulas and error formulas which are the main tools to prove the continuity and convergence properties. It is to be expected that weaker properties hold true. In [16], we study the continuity and convergence properties of integral means of Bojanov-Xu interpolation [4]. It is a kind of bivariate Hermite interpolation in which the interpolation points are equidistributed on $d = [\frac{n}{2}] + 1$ concentric circles centered at the origin. Let $\{s_1, \ldots, s_d\}$ be a set of multi-radii and let $\mathbb{H}[\{s_1, \ldots, s_d\}; f]$ be the Bojanov-Xu interpolation polynomial of a function f. We prove that the integral means of the interpolation polynomial over a fixed circle and a fixed annulus are continuous functions of the radii s_i's. We also give a distribution of the radii such that the integral means are convergent. More precisely, Theorem 3.2 in [16] asserts that the following two maps are continuous:

$$(s_1, \ldots, s_d) \in [\rho_1, \rho_2]^d \longmapsto \frac{1}{2\pi} \int_0^{2\pi} \mathbb{H}[\{s_1, \ldots, s_d\}; f](r\cos\theta, r\sin\theta)d\theta$$

and

$$(s_1, \ldots, s_d) \in [\rho_1, \rho_2]^d \longmapsto \int_{A(\rho_1,\rho_2)} \mathbb{H}[\{s_1, \ldots, s_d\}; f](x, y)dxdy.$$

Here f belongs to $C^{d-1}(A(\rho_1, \rho_2))$, where $A(\rho_1, \rho_2)$ is the annulus defined by two circles of radii $0 < \rho_1 < \rho_2$. Moreover, it is also showed in [16] that if $f \in C^M(A(\rho_1, \rho_2))$ and the Lebesgue constant $\Delta(\{s_{1d}^2, \ldots, s_{dd}^2\}, [\rho_1^2, \rho_2^2])$ grows at most like a polynomial of degree N in d with $A_d = \{s_{1d}, \ldots, s_{dd}\} \subset [\rho_1, \rho_2]$, then

$$\sup_{r\in[\rho_1,\rho_2]} \Big|\frac{1}{2\pi} \int_0^{2\pi} \mathbb{H}[A_d; f](r\cos\theta, r\sin\theta)d\theta - \frac{1}{2\pi} \int_0^{2\pi} f(r\cos\theta, r\sin\theta)d\theta\Big| = o\Big(\frac{1}{n^{M-N}}\Big)$$

and

$$\Big| \int_{A(\rho_3,\rho_4)} \mathbb{H}[A_d; f](x, y)dxdy - \int_{A(\rho_3,\rho_4)} f(x, y)dxdy\Big| = o\Big(\frac{1}{n^{M-N}}\Big), \quad \rho_1 \le \rho_3 < \rho_4 \le \rho_2.$$

The aim of this paper is to investigate Hermite interpolation on the unit sphere. Our study is inspired from [10, 19] in which the authors studied the following problem of Lagrange interpolation on S:
Problem. Let $A = \{\mathbf{a}_i : 1 \le i \le (n + 1)^2\}$ be a set of distinct points on S. Find conditions on A such that there is a unique polynomial $p \in \mathcal{P}_n(S)$ satisfying

$$p(\mathbf{a}_i) = f_i, \quad 1 \le i \le (n + 1)^2,$$

where $\{f_i\}$ is an arbitrary given data.

Xu in [19] gave a large amount of interpolation sets solving the problem. More precisely, the $(n + 1)^2$ interpolation points are distributed on $n + 1$ distinct latitudes (parallel circles on S), and each latitude contains an odd number of equidistant points. The number of points on two latitudes can be different. In particular, Corollary 3.2 in [19] points out that, when $n = 2m$, the set A consisting of $(2m + 1)^2$ points lying on $2m + 1$ latitudes, each of them contains $2m + 1$ equidistant points, solves the problem. In [10], the authors gave an analogous result, but each parallel circle contains an even number of points. To investigate the problem, the authors use the spherical coordinates

$$(\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta), \quad 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi,$$

to write a function f defined on S into the form

$$\widetilde{f}(\theta, \phi) := f(\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta). \tag{1}$$

Hence $\widetilde{f}(\theta, \phi)$ will be a trigonometric polynomial in θ and ϕ when f is a polynomial on S. Then, using the uniqueness of trigonometric interpolation for equidistant points and certain spaces of functions, they reduced the interpolation constraints to conditions relating to Chebyshev systems. Then they proved factorization theorems and used them to solve the problem.

In this paper, we generalize [19, Corollary 3.2] and a special case of [10, Theorem 3.1] in which all latitudes contain the same number of points. Here, Lagrange interpolation is extended to Hermite interpolation that can be viewed as a result of coalescing latitudes with respect to the variable θ. Hence the derivatives in θ appear in the constraints for Hermite interpolation. The first main results are Theorem 3.2 (with odd number of points on each circle) and Theorem 3.5 (with even number of points on each circle). To prove the theorems, we modify the methods given in [10, 19]. Some arguments in [10, 19] are repeated suitably in this paper. Next we wish to study the continuity and convergence properties of the Hermite interpolation polynomials. Note that an explicit formula or an error formula for Hermite interpolation is not available yet. Hence strong versions are very difficult to obtain. It is of interest to know whether there are similar results as above-mentioned properties of Bojanov-Xu interpolation. Fortunately, the nice distribution of interpolation points enables us to write the integral mean of the Hermite interpolation polynomials on S into a sum of univariate Hermite interpolation polynomials. It is the key to show desired results. Theorems 4.2 and 4.3 show continuity and convergence properties of the integral means of Hermite interpolation polynomials on S in the case of odd number of points. The analogous results for the case of even number of points are given in Theorems 4.6 and 4.8. It is worth pointing out that our theorem are analogous of that mentioned above of Bojanov-Xu interpolation. Finally, a problem of Hermite interpolation was studied in [13] in which we gave poised interpolation schemes and investigated some continuity properties of interpolation polynomials. For recent account of Hermite interpolation on algebraic hypersurfaces, we refer the reader to [6, 7] and the references therein.

The paper is organized as follows. In Section 2, we recall some facts about univariate Hermite and Lagrange interpolations. Some results relating the vanishing of derivatives are also given. Section 3 deals with problems of Hermite interpolation on the sphere. In the final section, we study integral means of the Hermite interpolation polynomials over circles and spherical zones.

2. Preliminaries

2.1. Univariate Hermite interpolation

Let $t_1, \dots, t_\lambda$ be λ distinct real numbers. Let $\nu_1, \dots, \nu_\lambda$ be λ positive integers and $d = \nu_1 + \cdots + \nu_\lambda$. The following theorem is well-known.

Theorem 2.1. *Given a function g for which $g^{(\nu_i-1)}(t_i)$ exists for $i = 1, \dots, \lambda$. Then there exists a unique $p \in \mathcal{P}_{d-1}(\mathbb{R})$ such that*

$$p^{(j)}(t_i) = g^{(j)}(t_i), \quad 1 \le i \le \lambda,\ 0 \le j \le \nu_i - 1.$$

The polynomial p in Theorem 2.1 is denoted by $\mathbf{H}[\{(t_1, \nu_1), \dots, (t_\lambda, \nu_\lambda)\}; g]$ and is called the Hermite interpolation polynomial. The coefficient of t^{d-1} in $\mathbf{H}[\{(t_1, \nu_1), \dots, (t_\lambda, \nu_\lambda)\}; g]$ denoted by $g[(t_1, \nu_1), \dots, (t_\lambda, \nu_\lambda)]$ is called the divided difference.

In studying Hermite interpolation, it is convenient to use interpolation sets in which elements may be repeated. For example, if $A = \{1, -2, 3, -2, 1, 1\}$, then we can write $A = \{(1, 3), (-2, 2), (3, 1)\}$. More generally, any multipoint set $A = \{s_1, \dots, s_d\}$ can be identified with the set of pairs of nodes and multiplicities $\{(t_1, \nu_1), \dots, (t_\lambda, \nu_\lambda)\}$. Here, (t_i, ν_i) means that t_i is repeated ν_i times. Hence we can write $\mathbf{H}[\{s_1, \dots, s_d\}; g]$ and $g[s_1, \dots, s_d]$ instead of $\mathbf{H}[\{(t_1, \nu_1), \dots, (t_\lambda, \nu_\lambda)\}; g]$ and $g[(t_1, \nu_1), \dots, (t_\lambda, \nu_\lambda)]$ respectively. In the case where the s_i's are pairwise distinct, the interpolation polynomial becomes the ordinary Lagrange interpolation polynomial and will be denoted by $\mathbf{L}[A; g]$.

The divided difference is continuous with respect to interpolation points (see for instance [5, Corollary 1.5]).

Lemma 2.2. *Let $I \subset \mathbb{R}$ be an interval and $g \in C^{d-1}(I)$. Then the function*

$$(s_1, \dots, s_d) \in I^d \longmapsto g[s_1, \dots, s_d]$$

is continuous.

Using the Newton formula

$$\mathbf{H}[\{s_1, \dots, s_d\}; g](t) = g[s_1] + g[s_1, s_2](t - s_1) + \cdots + g[s_1, \dots, s_d](t - s_1) \cdots (t - s_{d-1})$$

and Lemma 2.2, one can prove the continuity property of the univariate Hermite interpolation.

Theorem 2.3. *Let $I \subset \mathbb{R}$ be an interval and $g \in C^{d-1}(I)$. Then the map*

$$(s_1, \dots, s_d) \in I^d \mapsto \mathbf{H}[\{s_1, \dots, s_d\}; g] \in \mathcal{P}_{d-1}(\mathbb{R})$$

is continuous. Here the topology in $\mathcal{P}_{d-1}(\mathbb{R})$ is induced by any norm on $\mathcal{P}_{d-1}(\mathbb{R})$.

From the formula for Hermite interpolation polynomial given in [5, Theorem 1.1], we proved in [14] a factorization property of the generalized Vandermonde determinant.

Lemma 2.4. *Let $\nu_1, \dots, \nu_\lambda$ and d be positive integers such that $\nu_1 + \cdots + \nu_\lambda = d$. Let $T = \{(t_1, \nu_1), \dots, (t_\lambda, \nu_\lambda)\}$ and let $\mathcal{F} = \{g_1, \dots, g_d\}$ be given sufficiently smooth functions. We denote by* $\mathrm{VDM}(\mathcal{F}; T)$ *the generalized Vandermonde determinant*

$$\mathrm{VDM}(\mathcal{F}; T) = \begin{vmatrix} g_1(t_1) & g_2(t_1) & \cdots & g_{d-1}(t_1) & g_d(t_1) \\ g_1'(t_1) & g_2'(t_1) & \cdots & g_{d-1}'(t_1) & g_d'(t_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_1^{(\nu_1-1)}(t_1) & g_2^{(\nu_1-1)}(t_1) & \cdots & g_{d-1}^{(\nu_1-1)}(t_1) & g_d^{(\nu_1-1)}(t_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_1(t_\lambda) & g_2(t_\lambda) & \cdots & g_{d-1}(t_\lambda) & g_d(t_\lambda) \\ g_1'(t_\lambda) & g_2'(t_\lambda) & \cdots & g_{d-1}'(t_\lambda) & g_d'(t_\lambda) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_1^{(\nu_\lambda-1)}(t_\lambda) & g_2^{(\nu_\lambda-1)}(t_\lambda) & \cdots & g_{d-1}^{(\nu_\lambda-1)}(t_\lambda) & g_d^{(\nu_\lambda-1)}(t_\lambda) \end{vmatrix}.$$

Then

$$\mathrm{VDM}(\mathcal{F}; T) = \Big(\prod_{k=1}^{\lambda} \prod_{i=0}^{\nu_k - 1} i! \Big) \prod_{1 \le i < j \le \lambda} (t_j - t_i)^{\nu_i \nu_j} \mathrm{D}(\mathcal{F}; T),$$

where

$$\mathrm{D}(\mathcal{F}; T) = \begin{vmatrix} g_1[t_1] & g_2[t_1] & \cdots & g_d[t_1] \\ g_1[(t_1, 2)] & g_2[(t_1, 2)] & \cdots & g_d[(t_1, 2)] \\ \vdots & \vdots & \ddots & \vdots \\ g_1[(t_1, \nu_1)] & g_2[(t_1, \nu_1)] & \cdots & g_d[(t_1, \nu_1)] \\ g_1[(t_1, \nu_1), (t_2, 1)] & g_2[(t_1, \nu_1), (t_2, 1)] & \cdots & g_d[(t_1, \nu_1), (t_2, 1)] \\ \vdots & \vdots & \ddots & \vdots \\ g_1[(t_1, \nu_1), \dots, (t_\lambda, \nu_\lambda)] & g_2[(t_1, \nu_1), \dots, (t_\lambda, \nu_\lambda)] & \cdots & g_d[(t_1, \nu_1), \dots, (t_\lambda, \nu_\lambda)] \end{vmatrix}.$$

Here the factor $\prod_{1 \le i < j \le \lambda} (t_j - t_i)^{\nu_i \nu_j}$ does not appear when $\lambda = 1$.

Using Lemmas 2.2 and 2.4, we easily obtain the following result. Here and in the sequel, we sometimes use the tuple $(t_1, \dots, t_d)$ instead of the set $\{t_1, \dots, t_d\}$.

Lemma 2.5. *Let $d, \nu_1, \dots, \nu_\lambda$ be positive integers such that $\nu_1 + \cdots + \nu_\lambda = d$. Let $\mathcal{F} = \{g_1, \dots, g_d\}$ be a subset of $C^{d-1}(I)$ with $I = [a, b]$. Let $T^0 = \Big((t_1^0, \nu_1), \dots, (t_\lambda^0, \nu_\lambda)\Big)$ be tuples of points in I. Let $T^N = \Big(t_1^N, \dots, t_d^N\Big)$ be tuples of d distinct points in I such that the j-th element of T^N tends to the j-th element of T^0, i.e.,*

$$\lim_{N\to\infty} t_j^N = t_1^0 \text{ for } 1 \le j \le \nu_1, \quad \lim_{N\to\infty} t_j^N = t_l^0 \text{ for } \nu_1 + \cdots + \nu_{l-1} + 1 \le j \le \nu_1 + \cdots + \nu_l,\ 2 \le l \le \lambda.$$

Then

$$\lim_{N\to\infty} \frac{\mathrm{VDM}(\mathcal{F}; T^N)}{\prod_{1\le i<j\le d}(t_j^N - t_i^N)} = \frac{\mathrm{VDM}(\mathcal{F}; T^0)}{\Big(\prod_{k=1}^{\lambda}\prod_{i=0}^{\nu_k-1} i!\Big)\prod_{1\le i<j\le\lambda}(t_j^0 - t_i^0)^{\nu_i\nu_j}}.$$

2.2. The Lebesgue inequality

Let $A = \{s_1, \dots, s_d\}$ be a set of d distinct points in $I = [a, b]$. Let us define the Lebesgue constant

$$\Delta(A, I) = \sup_{t\in[a,b]} \sum_{i=1}^{d} \left| \prod_{j=1, j\ne i}^{d} \frac{t - s_j}{s_i - s_j} \right|.$$

It is known that $\Delta(A, I)$ is the norm of the Lagrange operator $\mathbf{L}[A; \cdot] : g \in C(I) \to \mathcal{P}_{d-1}(\mathbb{R})$, where the space $C(I)$ of all continuous functions on I is endowed with the usual sup-norm. The Lebesgue inequality shows that

$$\sup_{t\in I} |f(t) - \mathbf{L}[A; f](t)| \le \Big(1 + \Delta(A, I)\Big)\mathrm{dist}_I\Big(f, \mathcal{P}_{d-1}(\mathbb{R})\Big), \tag{2}$$

where $\mathrm{dist}_I\Big(f, \mathcal{P}_{d-1}(\mathbb{R})\Big) = \inf\{\sup_I |f - p| : p \in \mathcal{P}_{d-1}(\mathbb{R})\}$. By the Jackson theorem in [17, Theorem 1.5], if $f \in C^M(I)$, then there exists a constant C_0 depending only on a, b and M such that

$$\mathrm{dist}_I(f, \mathcal{P}_{d-1}) \le \frac{C_0}{(d-1)^M}\omega(D^M f; \frac{1}{d-1}), \tag{3}$$

where $\omega(g; \frac{1}{k}) = \sup\{|g(s) - g(t)| : s, t \in I, |s - t| \le \frac{1}{k}\}$ is the modulus of continuity. On the other hand, $\Delta(A, I)$ grows at least like $\log d$. The optimal growth can be obtained when A is the zero set of orthogonal polynomials on $I = [-1, 1]$. Recently, Calvi and Phung proved in [8] that the Lebesgue constant of the first d points of a $\mathfrak{R}$-Leja sequence grows like $O(d^3 \log d)$.

Remark that the Lebesgue constant is invariant under affine transformations of $\mathbb{R}$. Let $\ell(t) = \alpha t + \beta$ with $\alpha \ne 0$, $J = \ell(I)$ and $B = \ell(A)$. Then it is easy to verify that

$$\Delta(A, I) = \Delta(B, J).$$

Hence, from sets of points in $[-1, 1]$ with Lebesgue constants growing moderately, we can construct analogous sets in $[a, b]$. Finally, we say that $\Delta(A_d, I)$ with $A_d = \{s_1^d, \dots, s_d^d\} \subset I$ grows at most like a polynomial of degree N in d if there exists a constant $C > 0$ such that $\Delta(A_d, I) \le Cd^N$ for $d \ge 1$.

2.3. Vanishing of derivatives of functions

We need the following elementary results. For completeness, we give their proofs.

Lemma 2.6. *Let k be a natural number. Let g and h be k-times differentiable functions at $t_0 \in \mathbb{R}$. If $g(t_0) \ne 0$ and $(gh)^{(i)}(t_0) = 0$ for $i = 0, \dots, k$, then $h^{(i)}(t_0) = 0$ for $i = 0, \dots, k$.*

Proof. By hypothesis that $g(t_0)h(t_0) = 0$ and $g(t_0) \neq 0$, we have $h(t_0) = 0$. Assume that the assertion holds for $i = 0, \dots, j-1$ with $j \leq k$; we will prove it for j. By Leibniz's formula, we get

$$0 = (gh)^{(j)}(t_0) = g(t_0)h^{(j)}(t_0) + \sum_{i=1}^{j} \binom{j}{i} g^{(i)}(t_0)h^{(j-i)}(t_0) = g(t_0)h^{(j)}(t_0).$$

Hence $h^{(j)}(t_0) = 0$, which completes the proof. □

Lemma 2.7. *Let g be a k-times differentiable function in a neighborhood of $\theta_0 \in (0, \pi)$. If, for every $i = 0, \dots, k$, $\frac{d^i}{d\theta^i} g(\cos\theta)\Big|_{\theta=\theta_0} = 0$, then $g^{(i)}(\cos\theta_0) = 0$ for every $i = 0, \dots, k$.*

Proof. The assertion is trivial when $k = 0$. Assuming the assertion holds up to $k-1$, we will prove it for k. We first note that $(\cos\theta)^{(j)} = \cos(\theta + j\pi/2)$. Using Faa di Bruno's formula in [18], we obtain

$$0 = \frac{d^k}{d\theta^k} g(\cos\theta)\Big|_{\theta=\theta_0} = \sum \frac{k!}{n_1! \cdots n_k!} g^{(n)}(\cos\theta_0) \prod_{j=1}^{k} \Big(\frac{\cos(\theta_0 + \frac{j\pi}{2})}{j!}\Big)^{n_j},$$

where $n = n_1 + \cdots + n_k$ and the sum is over all values of $n_1, \dots, n_k \in \mathbb{N}$ such that $n_1 + 2n_2 + \cdots + kn_k = k$. Note that $n \leq k$ and $n = k$ only if $n_1 = k, n_2 = \cdots = n_k = 0$. Hence, by induction hypothesis, all terms in the last sum vanish except the term corresponding to $n_1 = k, n_2 = \cdots = n_k = 0$. It follows that $(-1)^k g^{(k)}(\cos\theta_0)\sin^k\theta_0 = 0$, and hence $g^{(k)}(\cos\theta_0) = 0$. The proof is complete. □

Corollary 2.8. *Let $g, g_1, \dots, g_d$ be k-times differentiable functions in a neighborhood of $\theta_0 \in (0, \pi)$ such that*

$$\frac{d^i}{d\theta^i} g(\cos\theta)\Big|_{\theta=\theta_0} = \Big(\sum_{j=1}^{d} \frac{d^i}{d\theta^i} g_j(\cos\theta)\Big)\Big|_{\theta=\theta_0}, \quad i = 0, \dots, k.$$

Then

$$g^{(i)}(\cos\theta_0) = \sum_{j=1}^{d} g_j^{(i)}(\cos\theta_0), \quad i = 0, \dots, k.$$

3. Hermite interpolation on the sphere

3.1. Hermite interpolation with odd number of points on circles

In this subsection, we always assume that n is an even number, i.e., $n = 2m$. Applying generalized Rolle's theorem, we can use similar arguments as in the proof of [19, Lemma 2.4] to obtain the following result.

Lemma 3.1. *Let k and m be positive integers such that $k \leq 2m$. Let p_{2m-k} be a polynomial of degree at most $2m-k$, and let q_{k-1} be a polynomial of degree at most $k-1$. If the function $h(t) = p_{2m-k}(t) + (1-t^2)^{m-k+1/2} q_{k-1}(t)$ has $2m+1$ roots, taking multiplicity into account, in $[-1, 1]$, then $p_{2m-k} = q_{k-1} = 0$.*

For $\alpha \in [0, 2)$, we denote by $\Theta_{\alpha,m}$ the set of angles

$$\Theta_{\alpha,m} = \Big\{\varphi_j^\alpha : \varphi_j^\alpha = \frac{(2j+\alpha)\pi}{2m+1}, \quad j = 0, 1, \dots, 2m\Big\}.$$

Theorem 3.2. *Let m be a positive integer. Let $\theta_1, \dots, \theta_l$ be l distinct numbers in $(0, \pi)$. Let $\mu_1, \dots, \mu_l$ be positive integers such that $\mu_1 + \cdots + \mu_l = 2m + 1$. Then, for suitably defined function f on the sphere, there exists a unique polynomial $p \in \mathcal{P}_{2m}(S)$ such that*

$$\frac{\partial^i}{\partial\theta^i}\widetilde{p}(\theta,\phi)\Big|_{\theta=\theta_j} = \frac{\partial^i}{\partial\theta^i}\widetilde{f}(\theta,\phi)\Big|_{\theta=\theta_j}, \quad 1 \le j \le l,\ 0 \le i \le \mu_j - 1,\ \phi \in \Theta_{\alpha,m},$$

where $\widetilde{f}$ and $\widetilde{p}$ are defined as in (1).

Proof. The number of interpolation conditions is equal to $(2m+1)^2$ that matches the dimension of $\mathcal{P}_{2m}(S)$. Hence it suffices to show that if $p \in \mathcal{P}_{2m}(S)$ satisfying the following condition

$$\frac{\partial^i}{\partial\theta^i}\widetilde{p}(\theta,\phi)\Big|_{\theta=\theta_j} = 0, \quad 1 \le j \le l,\ 0 \le i \le \mu_j - 1,\ \phi \in \Theta_{\alpha,m}, \tag{4}$$

then $p = 0$. From relation (2.1) in [19], we can write

$$\widetilde{p}(\theta,\phi) = a_0(\cos\theta) + \sum_{k=1}^{2m}\Big[(\sin\theta)^k a_k(\cos\theta)\cos k\phi + (\sin\theta)^k b_k(\cos\theta)\sin k\phi\Big],$$

where $a_k(t)$ and $b_k(t)$ are polynomials of degree $2m-k$. For every $\phi \in \Theta_{\alpha,m}$, using [19, Lemma 2.2], we have

$$\widetilde{p}(\theta,\phi) = a_0(\cos\theta) + \sum_{k=1}^{m}\Big[(\sin\theta)^k g_k(\cos\theta)\cos k\phi + (\sin\theta)^k h_k(\cos\theta)\sin k\phi\Big], \tag{5}$$

where $g_k(t) = a_k(t) + (1-t^2)^{m-k+1/2}u_{2m-k+1}(t)$, $h_k(t) = b_k(t) + (1-t^2)^{m-k+1/2}v_{2m-k+1}(t)$ for $k = 1, \dots, m$, and

$$u_{2m-k+1}(t) = a_{2m-k+1}(t)\cos\alpha\pi + b_{2m-k+1}(t)\sin\alpha\pi,$$

$$v_{2m-k+1}(t) = a_{2m-k+1}(t)\sin\alpha\pi - b_{2m-k+1}(t)\cos\alpha\pi.$$

Taking derivatives of the function $\theta \mapsto \tilde{p}(\theta,\phi)$ to order $0, 1, \dots, \mu_j - 1$ at θ_j and using relation (4), we obtain trigonometric polynomials of degree at most m in ϕ that vanish at $2m+1$ distinct points in $\Theta_{\alpha,m}$. The uniqueness of the trigonometric interpolation implies, for $1 \le j \le l,\ 0 \le i \le \mu_j - 1,\ 1 \le k \le m$,

$$\frac{d^i}{d\theta^i}a_0(\cos\theta)\Big|_{\theta=\theta_j} = 0$$

and

$$\frac{d^i}{d\theta^i}\Big((\sin\theta)^k g_k(\cos\theta)\Big)\Big|_{\theta=\theta_j} = \frac{d^i}{d\theta^i}\Big((\sin\theta)^k h_k(\cos\theta)\Big)\Big|_{\theta=\theta_j} = 0.$$

Lemma 2.6 now yields

$$\frac{d^i}{d\theta^i}g_k(\cos\theta)\Big|_{\theta=\theta_j} = \frac{d^i}{d\theta^i}h_k(\cos\theta)\Big|_{\theta=\theta_j} = 0.$$

By Lemma 2.7, above relations show that

$$a_0^{(i)}(\cos\theta_j) = g_k^{(i)}(\cos\theta_j) = h_k^{(i)}(\cos\theta_j) = 0, \quad 1 \le j \le l,\ 0 \le i \le \mu_j - 1,\ 1 \le k \le m.$$

The uniqueness of univariate Hermite interpolation forces $a_0 = 0$. Using Lemma 3.1 for g_k and h_k, we get $a_k = b_k = u_{2m-k+1} = v_{2m-k+1} = 0$ for $k = 1, \dots, m$. Hence $a_k = b_k = 0$ for $0 \le k \le 2m$, which completes the proof. □

The polynomial p defined in Theorem 3.2 is called a Hermite-type interpolation polynomial of f and is denoted by $\mathbb{H}^{\text{odd}}[\{(\theta_1,\mu_1),\dots,(\theta_l,\mu_l);\Theta_{\alpha,m}\};f]$.

If $\vartheta_1,\vartheta_2,\dots,\vartheta_{2m+1}\in(0,\pi)$ are not assumed to be distinct, then we can write

$$\{\vartheta_1,\vartheta_2,\dots,\vartheta_{2m+1}\}=\{(\theta_1,\mu_1),\dots,(\theta_l,\mu_l)\}.$$

We will write $\mathbb{H}^{\text{odd}}[\{\vartheta_1,\vartheta_2,\dots,\vartheta_{2m+1};\Theta_{\alpha,m}\};f]$ for $\mathbb{H}^{\text{odd}}[\{(\theta_1,\mu_1),\dots,(\theta_l,\mu_l);\Theta_{\alpha,m}\};f]$. Otherwise, the interpolation polynomial becomes the Lagrange interpolation polynomial $\mathbb{L}^{\text{odd}}[\{\vartheta_1,\vartheta_2,\dots,\vartheta_{2m+1};\Theta_{\alpha,m}\};f]$.

3.2. Hermite interpolation with even number of points on circles

In this subsection, we always assume that n is an odd number, i.e., $n=2m-1$. The following result generalizes [10, Proposition 2.4] which plays a crucial role in our proof.

Proposition 3.3. *Let r and s be two integers such that $r>s>0$. For $\epsilon_1\in\{-1,1\}$ and $\epsilon_2\in\{0,1\}$, let*

$$g(t)=p_r(t^2)+t^{\epsilon_1}(1-t^2)^{r-s}q_{s-1+\epsilon_2}(t^2),$$

where p_r and $q_{s-1+\epsilon_2}$ are polynomials of degree r and $s-1+\epsilon_2$ respectively. If g has $r+s+1+\epsilon_2$ roots, taking multiplicity into account, in $(0,1)$, then $p_r=q_{s-1+\epsilon_2}=0$.

Proof. The proof strongly relies on that given in the proof of Proposition 2.4 in [10]. We will prove the case $\epsilon_1=1$ and $\epsilon_2=0$. The remaining cases are similar. By performing the change of variable $t\mapsto t^2$, we need to show that if

$$g^*(t)=p_r(t)+\sqrt{t}(1-t)^{r-s}q_{s-1}(t)$$

has $r+s+1$ roots, taking multiplicity into account, in $(0,1)$, then $p_r=q_{s-1}=0$. Using the generalized Rolle's theorem repeatedly, we see that the function

$$h(t):=t^{r+1/2}\frac{d^{r+1}}{dt^{r+1}}g^*(t)=t^{r+1/2}\frac{d^{r+1}}{dt^{r+1}}\Big(\sqrt{t}(1-t)^{r-s}q_{s-1}(t)\Big)$$

has s roots, taking multiplicity into account, in $(0,1)$. Let $T^0=\{(t_1^0,\nu_1),\dots,(t_\lambda^0,\nu_\lambda)\}$ be the set of s roots of $h(t)$, $\nu_1+\cdots+\nu_\lambda=s$, where (t_i^0,ν_i) means that the root t_i^0 has the multiplicity ν_i. Here we identify T^0 with the tuple $\Big((t_1^0,\nu_1),\dots,(t_\lambda^0,\nu_\lambda)\Big)$. We write $q_{s-1}(t)=b_0+b_1t+\cdots+b_{s-1}t^{s-1}$. From the computations in [10, p. 161], we get

$$h(t)=\sum_{k=0}^{s-1}b_k^*h_k(t),$$

where

$$b_k^*=b_k\frac{(-1)^{r-k}}{2^{r+1}}\prod_{i=0}^{k-1}(2i+1)\prod_{i=1}^{s-k-1}(2i-1)\quad\text{and}\quad h_k(t)=\sum_{j=0}^{r-s}a_{k,j}t^{k+j},$$

in which the $a_{k,j}$'s are given by

$$a_{k,j}=\binom{r-s}{j}\prod_{i=0}^{j}(2k+2i+1)\prod_{i=j}^{r-s}\Big(2(r-k-i)-1\Big)>0.$$

By the definition of the multiple root, we have

$$0=h^{(i)}(t_j^0)=\sum_{k=0}^{s-1}b_k^*h_k^{(i)}(t_j^0),\quad 1\le j\le\lambda,\ 0\le i\le\nu_j-1. \tag{6}$$

The determinant of the coefficient matrix corresponding to the system of homogeneous linear equations in (6) is given by

$$\mathrm{VDM}(\mathcal{F}; T^0), \quad \mathcal{F} = \{h_0, h_1, \ldots, h_{s-1}\}.$$

In order to prove the proposition it suffices to check that $\mathrm{VDM}(\mathcal{F}; T^0) \neq 0$ since, in this case, (6) gives $b_k^* = 0$ for $k = 0, \ldots, s-1$. Therefore $b_k = 0$ for $k = 0, \ldots, s-1$. This forces $q_{s-1} = 0$, and hence, $p_r = 0$.

Let $T^N = (t_1^N, \ldots, t_s^N)$ be a tuple of pairwise distinct numbers in $(0,1)$ such that t_j^N converges to the j-the element of T^0 when $N \to \infty$ for $j = 1, \ldots, s$. From equation (2.12) in [10, p. 163], we get

$$\frac{\mathrm{VDM}(\mathcal{F}; T^N)}{\prod_{1\le i<j\le s}(t_j^N - t_i^N)} = \sum_{j_0=0}^{r-s} \sum_{j_1=1}^{r-s+1} \cdots \sum_{j_{s-1}=s-1}^{r-1} a_{0,j_0} a_{1,j_1-1} \cdots a_{s-1,j_{s-1}-(s-1)} s_{j_0,j_1,\ldots,j_{s-1}}(T^N),$$

where $s_{j_0,j_1,\ldots,j_{s-1}}$ is a certain symmetric polynomial. Note that $s_{j_0,j_1,\ldots,j_{s-1}}$ is identical with the Schur polynomial when $j_i = \mu_i + s - i - 1$ with $\mu_0 \ge \mu_1 \ge \cdots \ge \mu_{s-1} \ge 0$ (see [10, p. 162]). In any case $s_{j_0,j_1,\ldots,j_{s-1}}(t_1, \ldots, t_s)$ is a symmetric polynomial in $t_1, \ldots, t_s$ which is of constant sign or equal to 0 when $t_i > 0$ for $i = 1, \ldots, s$. Lemma 2.5 now yields

$$\begin{aligned} \frac{\mathrm{VDM}(\mathcal{F}; T^0)}{\left(\prod_{k=1}^{\lambda} \prod_{i=0}^{\nu_k - 1} i!\right) \prod_{1\le i<j\le \lambda} (t_j^0 - t_i^0)^{\nu_i \nu_j}} &= \lim_{N\to\infty} \frac{\mathrm{VDM}(\mathcal{F}; T^N)}{\prod_{1\le i<j\le s}(t_j^N - t_i^N)} \\ &= \sum_{j_0=0}^{r-s} \sum_{j_1=1}^{r-s+1} \cdots \sum_{j_{s-1}=s-1}^{r-1} a_{0,j_0} a_{1,j_1-1} \cdots a_{s-1,j_{s-1}-(s-1)} s_{j_0,j_1,\ldots,j_{s-1}}(T^0). \end{aligned}$$

Analysis similar to that in [10, p. 163] in which $\mathbf{t}$ is replaced by T^0 shows that the last sum is positive. It follows that $\mathrm{VDM}(\mathcal{F}; T^0) \neq 0$, and the proof is complete. □

Let $p(t) = \sum_{i=0}^{N} a_i t^i$ be a polynomial of degree N. Let us denote by p^{even} and p^{odd} the even part and odd part of p respectively, that is,

$$p^{\mathrm{even}}(t) = \sum_{0\le 2i\le N} a_{2i} t^{2i} \quad \text{and} \quad p^{\mathrm{odd}}(t) = \sum_{1\le 2i+1\le N} a_{2i+1} t^{2i+1}.$$

Using Proposition 3.3 and the same method as in the proof of [10, Proposition 2.5], we obtain the following result. Here we use the Leibniz rule to get the fact that the roots of functions in $(0,1)$ are unchanged when they are multiplied by $1/t$.

Proposition 3.4. *Let M and k be integers such that $1 \le k \le M$. Let p_{2M-k-1} and q_{k-1} be polynomials of degree $2M-k-1$ and $k-1$ respectively. If*

$$p_{2M-k-1}^{\mathrm{even}}(t) + q_{k-1}^{\mathrm{odd}}(t)(1-t^2)^{M-k} \quad \text{and} \quad p_{2M-k-1}^{\mathrm{odd}}(t) + q_{k-1}^{\mathrm{even}}(t)(1-t^2)^{M-k}$$

has M common roots, taking multiplicity into account, in $(0,1)$, then $p_{2M-k-1} = q_{k-1} = 0$.

For a positive integer m and $\alpha \ge 0$, we set

$$\Phi_{\alpha,m} = \left\{\phi_i^\alpha : \phi_i^\alpha = \frac{(2i+\alpha)\pi}{2m},\ i = 0, 1, \ldots, 2m-1\right\}.$$

Theorem 3.5. *Let m and l be positive integers and $\alpha \in [0,2)$. Let $\theta_1, \ldots, \theta_l$ be l distinct numbers in $(0, \pi/2)$ and $\theta_{2l+1-i} = \pi - \theta_i$ for $i = 1, \ldots, l$. Let $\mu_1, \ldots, \mu_{2l}$ be positive integers such that $\mu_1 + \cdots + \mu_l = m$ and $\mu_{2l+1-i} = \mu_i$ for $i = 1, \ldots, l$. Then, for suitably defined function f on the sphere, there exists a unique polynomial $p \in \mathcal{P}_{2m-1}(S)$ such that*

$$\frac{\partial^i}{\partial\theta^i} \widetilde{p}(\theta,\phi)\Big|_{\theta=\theta_j} = \frac{\partial^i}{\partial\theta^i} \widetilde{f}(\theta,\phi)\Big|_{\theta=\theta_j}, \quad 1 \le j \le l,\ 0 \le i \le \mu_j - 1,\ \phi \in \Phi_{\alpha,m}$$

and

$$\frac{\partial^i}{\partial\theta^i}\widetilde{p}(\theta,\phi)\Big|_{\theta=\theta_j} = \frac{\partial^i}{\partial\theta^i}\widetilde{f}(\theta,\phi)\Big|_{\theta=\theta_j}, \quad l+1 \le j \le 2l,\ 0 \le i \le \mu_j - 1,\ \phi \in \Phi_{\alpha+1,m}.$$

Proof. The number of interpolation conditions is equal to $(2m)^2$ that matches the dimension of $\mathcal{P}_{2m-1}(S)$. Hence it suffices to show that if $p \in \mathcal{P}_{2m-1}(S)$ satisfying the following conditions

$$\frac{\partial^i}{\partial\theta^i}\widetilde{p}(\theta,\phi)\Big|_{\theta=\theta_j} = 0, \quad 1 \le j \le l,\ 0 \le i \le \mu_j - 1,\ \phi \in \Phi_{\alpha,m} \tag{7}$$

and

$$\frac{\partial^i}{\partial\theta^i}\widetilde{p}(\theta,\phi)\Big|_{\theta=\theta_j} = 0, \quad l+1 \le j \le 2l,\ 0 \le i \le \mu_j - 1,\ \phi \in \Phi_{\alpha+1,m}, \tag{8}$$

then $p = 0$. From relation (2.1) in [19], we can write

$$\widetilde{p}(\theta,\phi) = a_0(\cos\theta) + \sum_{k=1}^{2m-1}\Big[(\sin\theta)^k a_k(\cos\theta)\cos k\phi + (\sin\theta)^k b_k(\cos\theta)\sin k\phi\Big],$$

where $a_k(t)$ and $b_k(t)$ are polynomials of degree $2m-1-k$. By [10, Lemma 2.1], for $\phi \in \Phi_{\beta,m}$ with $\beta \in \{\alpha, \alpha+1\}$, we can write

$$\begin{aligned}\widetilde{p}(\theta,\phi) &= a_0(\cos\theta) + \sum_{k=1}^{m-1}\Big[\Big(a_k(\cos\theta)(\sin\theta)^k + u^\beta_{2m-k}(\cos\theta)(\sin\theta)^{2m-k}\Big)\cos(k\phi)\\ &+ \Big(b_k(\cos\theta)(\sin\theta)^k + v^\beta_{2m-k}(\cos\theta)(\sin\theta)^{2m-k}\Big)\sin(k\phi)\Big]\\ &+ \Big(a_m(\cos\theta)\cos\frac{\beta\pi}{2} - b_m(\cos\theta)\sin\frac{\beta\pi}{2}\Big)(\sin\theta)^m\cos(m\phi - \frac{\beta\pi}{2}),\end{aligned}$$

where, for $k = 1, \dots, m-1$,

$$u^\beta_{2m-k}(t) = a_{2m-k}(t)\cos(\beta\pi) + b_{2m-k}(t)\sin(\beta\pi),$$

$$v^\beta_{2m-k}(t) = a_{2m-k}(t)\sin(\beta\pi) - b_{2m-k}(t)\cos(\beta\pi)$$

are polynomials of degree $k-1$. Taking derivatives of the function $\theta \mapsto \tilde{p}(\theta,\phi)$ to order $0, 1, \dots, \mu_j - 1$ at θ_j and using relations (7) and (8), we obtain a trigonometric polynomial of the form

$$c_0 + \sum_{k=1}^{m-1}(c_k\cos(k\phi) + d_k\sin(k\phi)) + c_m\cos(m\phi - \frac{\beta\pi}{2})$$

that vanishes at $2m$ points in $\Phi_{\beta,m}$. As in the proof of [10, Lemma 2.2], the uniqueness of the trigonometric interpolation follows that, for $1 \le j \le 2l$, $0 \le i \le \mu_j - 1$ and $\beta \in \{\alpha, \alpha+1\}$,

$$\frac{d^i}{d\theta^i}a_0(\cos\theta)\Big|_{\theta=\theta_j} = 0, \tag{9}$$

$$\frac{d^i}{d\theta^i}\Big(\Big(a_m(\cos\theta)\cos\frac{\beta\pi}{2} - b_m(\cos\theta)\sin\frac{\beta\pi}{2}\Big)(\sin\theta)^m\Big)\Big|_{\theta=\theta_j} = 0, \tag{10}$$

and

$$\frac{d^i}{d\theta^i}\Big((\sin\theta)^k g^\beta_k(\cos\theta)\Big)\Big|_{\theta=\theta_j} = \frac{d^i}{d\theta^i}\Big((\sin\theta)^k h^\beta_k(\cos\theta)\Big)\Big|_{\theta=\theta_j} = 0, \tag{11}$$

where $g_k^\beta(t) = a_k(t) + u_{2m-k}^\beta(t)(1-t^2)^{m-k}$ and $h_k^\beta(t) = b_k(t) + v_{2m-k}^\beta(t)(1-t^2)^{m-k}$ for $k = 1, \dots, m-1$. Here, in (10) and (11), $\beta = \alpha$ when $1 \le j \le l$ and $\beta = \alpha + 1$ when $l+1 \le j \le 2l$.

Applying Lemma 2.7 in (9) we get $\dfrac{d^i}{dt^i}a_0(t)\Big|_{t=\cos\theta_j} = 0$ for $1 \le j \le 2l$, $0 \le i \le \mu_j - 1$. This forces $a_0 = 0$ since a_0 is a polynomial of degree $2m-1$. Applying Lemma 2.6 and then Lemma 2.7 in (10) we obtain

$$\frac{d^i}{dt^i}(a_m(t))\Big|_{t=\cos\theta_j}\cos\frac{\alpha\pi}{2} - \frac{d^i}{dt^i}(b_m(t))\Big|_{t=\cos\theta_j}\sin\frac{\alpha\pi}{2} = 0, \quad 1 \le j \le l,\ 0 \le i \le \mu_j - 1$$

and

$$\frac{d^i}{dt^i}(a_m(t))\Big|_{t=\cos\theta_j}\cos\frac{(\alpha+1)\pi}{2} - \frac{d^i}{dt^i}(b_m(t))\Big|_{t=\cos\theta_j}\sin\frac{(\alpha+1)\pi}{2} = 0,$$

$l+1 \le j \le 2l,\ 0 \le i \le \mu_j - 1$. Since $a_m(t), b_m(t)$ are univariate polynomials of degree $m-1$, and $\mu_1 + \cdots + \mu_l = \mu_{l+1} + \cdots + \mu_{2l} = m$, the last two relations along with the uniqueness of Hermite interpolation give

$$a_m(t)\cos\frac{\alpha\pi}{2} - b_m(t)\sin\frac{\alpha\pi}{2} = a_m(t)\sin\frac{\alpha\pi}{2} + b_m(t)\cos\frac{\alpha\pi}{2} = 0.$$

It follows that $a_m = b_m = 0$. Similarly, looking at Lemmas 2.6 and 2.7, we conclude from (11) that, for $1 \le k \le m-1$,

$$\begin{cases} \frac{d^i}{dt^i}\Big(a_k(t) + u_{2m-k}^\alpha(t)(1-t^2)^{m-k}\Big)\Big|_{t=\cos\theta_j} = 0 \\ \frac{d^i}{dt^i}\Big(b_k(t) + v_{2m-k}^\alpha(t)(1-t^2)^{m-k}\Big)\Big|_{t=\cos\theta_j} = 0 \end{cases} \quad 1 \le j \le l,\ 0 \le i \le \mu_j - 1, \tag{12}$$

and

$$\begin{cases} \frac{d^i}{dt^i}\Big(a_k(t) + u_{2m-k}^{\alpha+1}(t)(1-t^2)^{m-k}\Big)\Big|_{t=\cos\theta_j} = 0 \\ \frac{d^i}{dt^i}\Big(b_k(t) + v_{2m-k}^{\alpha+1}(t)(1-t^2)^{m-k}\Big)\Big|_{t=\cos\theta_j} = 0 \end{cases} \quad l+1 \le j \le 2l,\ 0 \le i \le \mu_j - 1. \tag{13}$$

Since $\cos\theta_{2l+1-j} = -\cos\theta_j$ and $\mu_{2l+1-j} = \mu_j$ for $1 \le j \le l$, relation (13) is equivalent to

$$\begin{cases} \frac{d^i}{dt^i}\Big(a_k(-t) - u_{2m-k}^\alpha(-t)(1-t^2)^{m-k}\Big)\Big|_{t=\cos\theta_j} = 0 \\ \frac{d^i}{dt^i}\Big(b_k(-t) - v_{2m-k}^\alpha(-t)(1-t^2)^{m-k}\Big)\Big|_{t=\cos\theta_j} = 0 \end{cases} \quad 1 \le j \le l,\ 0 \le i \le \mu_j - 1. \tag{14}$$

Combining (12) and (14), we obtain, for $1 \le j \le l,\ 0 \le i \le \mu_j - 1,\ k = 1, \dots, m-1$,

$$\begin{cases} \frac{d^i}{dt^i}\Big(p_{2m-k-1}^{\text{even}}(t) + q_{k-1}^{\text{odd}}(t)(1-t^2)^{m-k}\Big)\Big|_{t=\cos\theta_j} = 0 \\ \frac{d^i}{dt^i}\Big(p_{2m-k-1}^{\text{odd}}(t) + q_{k-1}^{\text{even}}(t)(1-t^2)^{m-k}\Big)\Big|_{t=\cos\theta_j} = 0, \end{cases}$$

where either $p_{2m-k-1} = a_k$ and $q_{k-1} = u_{2m-k}^\alpha$, or $p_{2m-k-1} = b_k$ and $q_{k-1} = v_{2m-k}^\alpha$. Note that p_{2m-k-1} and q_{k-1} are of degree $2m-k-1$ and $k-1$ respectively. It follows from Proposition 3.4 that $p_{2m-k-1} = q_{k-1} = 0$ for $k = 1, \dots, m-1$, and hence $a_k = b_k = 0$ for $1 \le k \le 2m-1$ and $k \ne m$. Consequently, $\widetilde{p} = 0$, and the proof is complete. □

The polynomial p defined in Theorem 3.5 is called a Hermite-type interpolation polynomial of f. Remark that p depends on $\{(\theta_1, \mu_1), \dots, (\theta_l, \mu_l); \alpha\}$ and f. It will be denoted by $\mathbb{H}^{\text{even}}[\{(\theta_1, \mu_1), \dots, (\theta_l, \mu_l); \alpha\}; f]$.

If $\vartheta_1, \vartheta_2, \dots, \vartheta_m \in (0, \pi/2)$ are not assumed to be distinct, then we can write

$$\{\vartheta_1, \vartheta_2, \dots, \vartheta_m\} = \{(\theta_1, \mu_1), \dots, (\theta_l, \mu_l)\},$$

where $\mu_1 + \cdots + \mu_l = m$. Hence, we can identify the interpolation polynomial $\mathbb{H}^{\text{even}}[\{\vartheta_1, \vartheta_2, \dots, \vartheta_m; \alpha\}; f]$ with $\mathbb{H}^{\text{even}}[\{(\theta_1, \mu_1), \dots, (\theta_l, \mu_l); \alpha\}; f]$. Otherwise, the interpolation polynomial becomes the Lagrange interpolation polynomial $\mathbb{L}^{\text{even}}[\{\vartheta_1, \vartheta_2, \dots, \vartheta_m; \alpha\}; f]$.

4. Some properties of integral means of Hermite interpolation polynomials

4.1. Integral means of Hermite interpolation of the first kind

Fix $\alpha \in [0, 2)$. For $\theta \in (0, \pi)$, let us define

$$\mathbb{M}^{\text{odd}}(\{(\theta_1, \mu_1), \dots, (\theta_l, \mu_l)\}; f)(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \widetilde{\mathbb{H}}^{\text{odd}}(\theta, \phi) d\phi,$$

where

$$\widetilde{\mathbb{H}}^{\text{odd}}(\theta, \phi) = \mathbb{H}^{\text{odd}}[\{(\theta_1, \mu_1), \dots, (\theta_l, \mu_l); \Theta_{\alpha,m}\}; f](\sin\theta \sin\phi, \sin\theta \cos\phi, \cos\theta).$$

Evidently, $\mathbb{M}^{\text{odd}}(\{(\theta_1, \mu_1), \dots, (\theta_l, \mu_l)\}; f)(\theta)$ is a trigonometric polynomial of degree at most $2m$ in θ.

For a function f defined on S, we set

$$\widehat{f_i}(t) = f(\sqrt{1-t^2}\sin\varphi_i^\alpha, \sqrt{1-t^2}\cos\varphi_i^\alpha, t), \quad t \in [-1, 1],$$

where $\varphi_i^\alpha = \dfrac{(2i+\alpha)\pi}{2m+1} \in \Theta_{\alpha,m}$ for $i = 0, \dots, 2m$. The following lemma plays an important role in this subsection.

Lemma 4.1. *Under the assumptions of Theorem 3.2 we have*

$$\mathbb{M}^{\text{odd}}(\{(\theta_1, \mu_1), \dots, (\theta_l, \mu_l)\}; f)(\theta) = \frac{1}{2m+1} \sum_{i=0}^{2m} \mathbf{H}[\{(\cos\theta_1, \mu_1), \dots, (\cos\theta_l, \mu_l)\}; \widehat{f_i}](\cos\theta).$$

Proof. We write

$$\widetilde{\mathbb{H}}^{\text{odd}}(\theta, \phi) = a_0(\cos\theta) + \sum_{k=1}^{2m} \Big[a_k(\cos\theta)(\sin\theta)^k \cos k\phi + b_k(\cos\theta)(\sin\theta)^k \sin k\phi\Big],$$

where $a_k(t)$ and $b_k(t)$ are polynomials of degree $2n - k$. Integrating both sides, we get

$$\mathbb{M}^{\text{odd}}(\{(\theta_1, \mu_1), \dots, (\theta_l, \mu_l)\}; f)(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \widetilde{\mathbb{H}}^{\text{odd}}(\theta, \phi) d\phi = a_0(\cos\theta). \tag{15}$$

It is a trigonometric polynomial of degree $2m$ in θ. It follows from (15) and the quadrature formula for trigonometric polynomials in [20, Vol 2., p. 8] or [19, p. 762] that

$$a_0(\cos\theta) = \frac{1}{2\pi} \int_0^{2\pi} \widetilde{\mathbb{H}}^{\text{odd}}(\theta, \phi) d\phi = \frac{1}{2m+1} \sum_{i=0}^{2m} \widetilde{\mathbb{H}}^{\text{odd}}(\theta, \varphi_i^\alpha).$$

From the interpolation conditions for Hermite interpolation, for $1 \le j \le l$, $0 \le k \le \mu_j - 1$, we have

$$\frac{d^k}{d\theta^k} a_0(\cos\theta)\Big|_{\theta=\theta_j} = \frac{1}{2m+1} \sum_{i=0}^{2m} \frac{d^k}{d\theta^k} \widetilde{\mathbb{H}}^{\text{odd}}(\theta, \varphi_i^\alpha)\Big|_{\theta=\theta_j} = \frac{1}{2m+1} \sum_{i=0}^{2m} \frac{d^k}{d\theta^k} \widetilde{f_i}(\theta, \varphi_i^\alpha)\Big|_{\theta=\theta_j}.$$

Corollary 2.8 now leads to

$$a_0^{(k)}(\cos\theta_j) = \frac{1}{2m+1} \sum_{i=0}^{2m} \widehat{f_i}^{(k)}(\cos\theta_j) = \Big(\frac{1}{2m+1} \sum_{i=0}^{2m} \widehat{f_i}\Big)^{(k)}(\cos\theta_j), \quad 1 \le j \le l,\ 0 \le k \le \mu_j - 1.$$

Above relation consists of $\mu_1 + \cdots + \mu_l = 2m + 1$ equations. Since $a_0 \in \mathcal{P}_{2m}(\mathbb{R})$, we conclude that

$$\begin{aligned} a_0(t) &= \mathbf{H}[\{(\cos\theta_1, \mu_1), \ldots, (\cos\theta_l, \mu_l)\}; a_0](t) \\ &= \mathbf{H}\Big[\{(\cos\theta_1, \mu_1), \ldots, (\cos\theta_l, \mu_l)\}; \frac{1}{2m+1}\sum_{i=0}^{2m} \widehat{f_i}\Big](t) \\ &= \frac{1}{2m+1}\sum_{i=0}^{2m} \mathbf{H}[\{(\cos\theta_1, \mu_1), \ldots, (\cos\theta_l, \mu_l)\}; \widehat{f_i}](t), \quad t \in \mathbb{R}. \end{aligned} \tag{16}$$

Combining (15) and (16), we get the desired relation. The proof is complete. □

For $0 < \rho_1 < \rho_2 < \pi$, we denote by $Z(\rho_1, \rho_2)$ the following spherical zone:

$$Z(\rho_1, \rho_2) = \{(\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta) : \rho_1 \le \theta \le \rho_2, 0 \le \phi \le 2\pi\}.$$

Theorem 4.2. *Let m be a positive integer, $\alpha \in [0, 2)$ and $0 < \rho_1 < \rho_2 < \pi$. Let f be a function defined on $Z(\rho_1, \rho_2)$ such that $\widehat{f_i} \in C^{2m}([\cos\rho_2, \cos\rho_1])$ for $i = 0, \ldots, 2m$. Then the following two maps are continuous:*

$$(\vartheta_1, \ldots, \vartheta_{2m+1}) \in [\rho_1, \rho_2]^{2m+1} \longmapsto \mathbb{M}^{\text{odd}}(\{\vartheta_1, \ldots, \vartheta_{2m+1}\}; f)$$

and

$$(\vartheta_1, \ldots, \vartheta_{2m+1}) \in [\rho_1, \rho_2]^{2m+1} \longmapsto \int\limits_{Z(\rho_1,\rho_2)} \mathbb{H}^{\text{odd}}[\{\vartheta_1, \ldots, \vartheta_{2m+1}; \Theta_{\alpha,m}\}; f](x, y, z)d\omega,$$

where ω is the surface area measure on S. Here, in the first map, the topology on the space of all trigonometric polynomials of degree at most $2m$ is induced by any norm.

Proof. Using Lemma 4.1, we can write

$$\mathbb{M}^{\text{odd}}(\{\vartheta_1, , \ldots, \vartheta_{2m+1}\}; f)(\theta) = \frac{1}{2m+1}\sum_{i=0}^{2m} \mathbf{H}[\{\cos\vartheta_1, , \ldots, \cos\vartheta_{2m+1}\}; \widehat{f_i}](\cos\theta).$$

By hypothesis that the function $\widehat{f_i}$ is of class $C^{2m}([\cos\rho_2, \cos\rho_1])$, we can use Theorem 2.3 to get the continuity property of the map

$$(\vartheta_1, \ldots, \vartheta_{2m+1}) \in [\rho_1, \rho_2]^{2m+1} \longmapsto \mathbf{H}[\{\cos\vartheta_1, \ldots, \cos\vartheta_{2m+1}\}; \widehat{f_i}], \quad 0 \le i \le 2m,$$

which proves the first assertion.

Now, let $\{\vartheta_i^k\}_{k=1}^{\infty}$ be a sequence in $[\rho_1, \rho_2]$ such that $\lim_{k\to\infty} \vartheta_i^k = \vartheta_i$ for $i = 1, \ldots, 2m + 1$. By the above,

$$\lim_{k\to\infty} \mathbb{M}^{\text{odd}}(\{\vartheta_1^k, \ldots, \vartheta_{2m+1}^k\}; f) = \mathbb{M}^{\text{odd}}(\{\vartheta_1, \ldots, \vartheta_{2m+1}\}; f).$$

Above relation can be understood as the convergence under the sup-norm over $[\rho_1, \rho_2]$. It follows that

$$\lim_{k\to\infty} \int\limits_{\rho_1}^{\rho_2} \mathbb{M}^{\text{odd}}(\{\vartheta_1^k, \ldots, \vartheta_{2m+1}^k\}; f)(\theta)\sin\theta d\theta = \int\limits_{\rho_1}^{\rho_2} \mathbb{M}^{\text{odd}}(\{\vartheta_1, \ldots, \vartheta_{2m+1}\}; f)(\theta)\sin\theta d\theta.$$

Using the change of variable formula

$$\int\limits_{Z(\rho_1,\rho_2)} g(x, y, z)d\omega = \int\limits_{\rho_1}^{\rho_2}\int\limits_{0}^{2\pi} \widetilde{g}(\theta, \phi)\sin\theta d\theta d\phi \tag{17}$$

we obtain

$$\lim_{k\to\infty}\int\limits_{Z(\rho_1,\rho_2)} \mathbb{H}^{\rm odd}[\{\vartheta_1^k,\dots,\vartheta_{2m+1}^k;\Theta_{\alpha,m}\};f](x,y,z)d\omega = \int\limits_{Z(\rho_1,\rho_2)} \mathbb{H}^{\rm odd}[\{\vartheta_1,\dots,\vartheta_{2m+1};\Theta_{\alpha,m}\};f](x,y,z)d\omega,$$

which proves the continuity property of the second map. □

For simplicity, we only give a convergence theorem relating to the Lagrange interpolation on the sphere.

Theorem 4.3. *Let M and N be positive integers with $M \geq N \geq 1$. Let $0 < \rho_1 < \rho_2 < \pi$ and $f \in C^M(Z(\rho_1,\rho_2))$. Let $A_m = \{\vartheta_1^m,\dots,\vartheta_{2m+1}^m\}$ be sets of distinct points in $[\rho_1,\rho_2]$ such that $\Delta(\{\cos\vartheta_1^m,\dots,\cos\vartheta_{2m+1}^m\},[\cos\rho_2,\cos\rho_1])$ grows at most like a polynomial of degree N in m. Then*

$$\sup_{\theta\in[\rho_1,\rho_2]}\Big|\mathbb{M}^{\rm odd}(A_m;f)(\theta) - \frac{1}{2\pi}\int_0^{2\pi}\widetilde{f}(\theta,\phi)d\phi\Big| = o\Big(\frac{1}{m^{M-N}}\Big)$$

and

$$\Big|\int\limits_{Z(\rho_3,\rho_4)}\mathbb{L}^{\rm odd}[\{A_m,\Theta_{\alpha,m}\};f](x,y,z)d\omega - \int\limits_{Z(\rho_3,\rho_4)} f(x,y,z)d\omega\Big| = o\Big(\frac{1}{m^{M-N}}\Big),\quad \rho_1\leq\rho_3<\rho_4\leq\rho_2.$$

Proof. The idea of the proof is inspired by [16, Theorem 4.1]. Let us define two types of modulus of continuity

$$\zeta\Big(\frac{1}{m}\Big) = \sup\Big\{\Big|\frac{\partial^M}{\partial\phi^M}\widetilde{f}(\theta,\phi)\Big|_{\phi=\phi_1} - \frac{\partial^M}{\partial\phi^M}\widetilde{f}(\theta,\phi)\Big|_{\phi=\phi_2}\Big| : \phi_1,\phi_2\in[0,2\pi],\quad |\phi_1-\phi_2|\leq\frac{1}{2m},\quad \rho_1\leq\theta\leq\rho_2\Big\}$$

and

$$\eta\Big(\frac{1}{m}\Big) = \sup\Big\{\Big|\frac{\partial^M}{\partial t^M}F(t,\phi)\Big|_{t=t_1} - \frac{\partial^M}{\partial t^M}F(t,\phi)\Big|_{t=t_2}\Big| : t_1,t_2\in[\cos\rho_2,\cos\rho_1],\quad |t_1-t_2|\leq\frac{1}{2m},\quad \phi\in[0,2\pi]\Big\},$$

where $F(t,\phi) = f(\sqrt{1-t^2}\sin\phi,\ \sqrt{1-t^2}\cos\phi,t)$ for $\cos\rho_2\leq t\leq\cos\rho_1$, $0\leq\phi\leq2\pi$. Note that ζ and η can be written in term of classical modulus of continuity

$$\zeta(\frac{1}{m}) = \sup_{\theta\in[\rho_1,\rho_2]}\omega\Big(\frac{\partial^M}{\partial\phi^M}\widetilde{f}(\theta,\cdot);\frac{1}{2m}\Big),\quad \eta(\frac{1}{m}) = \sup_{\phi\in[0,2\pi]}\omega\Big(\frac{\partial^M}{\partial t^M}F(\cdot,\phi);\frac{1}{2m}\Big).$$

Since $f\in C^M(Z(\rho_1,\rho_2))$, the function $\dfrac{\partial^M}{\partial\phi^M}\widetilde{f}(\theta,\phi)$ is continuous on $[\theta_1,\theta_2]\times[0,2\pi]$. It follows that $\zeta(\dfrac{1}{m})$ tends to 0 when $m\to\infty$. Similarly, $\lim_{m\to\infty}\eta(\dfrac{1}{m}) = 0$. As $\widehat{f_i}\in C^M([\cos\rho_2,\cos\rho_1])$ the Jackson theorem in (3) shows that there exists a constant C_0 depending only on ρ_1,ρ_2 and M such that

$$\mathrm{dist}_I(\widehat{f_i},\mathcal{P}_{2m}(\mathbb{R}))\leq\frac{C_0}{(2m)^M}\omega(D^M\widehat{f_i};\frac{1}{2m})\leq\frac{C_0}{2^Mm^M}\eta\Big(\frac{1}{m}\Big),\quad 0\leq i\leq 2m,$$

where $\omega(g;\dfrac{1}{d})$ is the ordinary modulus of continuity, $I=[\cos\rho_2,\cos\rho_1]$. Combining the above estimates with the Lebesgue inequality (2) and the hypothesis on the Lebesgue constant, we obtain

$$\sup_{t\in I}\big|\widehat{f_i}(t) - \mathbf{L}[\{\cos\vartheta_1^m,\dots,\cos\vartheta_{2m+1}^m\};\widehat{f_i}](t)\big|\leq\frac{C_1}{m^{M-N}}\eta\Big(\frac{1}{m}\Big),\quad 0\leq i\leq 2m,$$

where C_1 is a constant independent of m. Lemma 4.1 now yields

$$\sup_{\theta\in[\rho_1,\rho_2]} \Big|\frac{1}{2m+1}\sum_{i=0}^{2m} \widehat{f_i}(\cos\theta) - \mathbb{M}^{\text{odd}}(A_d; f)(\theta)\Big| \le \frac{C_1}{m^{M-N}}\eta\Big(\frac{1}{m}\Big). \tag{18}$$

For fixed $\theta \in [\rho_1, \rho_2]$, from the first Jackson theorem [11, Theorem 3, p. 57], we can find a constant $C_2 = C_2(M)$ depending only on M and a trigonometric polynomial T_{2m} of degree at most $2m$ (depending on θ) such that

$$\sup_{\phi\in[0,2\pi]} |\widetilde{f}(\theta,\phi) - T_{2m}(\phi)| \le \frac{C_2}{(2m)^M}\omega\Big(\frac{\partial^M}{\partial\phi^M}\widetilde{f}(\theta,\phi); \frac{1}{2m}\Big) \le \frac{C_2}{2^M m^M}\zeta\Big(\frac{1}{m}\Big).$$

It follows that

$$\Big|\frac{1}{2m+1}\sum_{i=0}^{2m} \widetilde{f}(\theta,\varphi_i^\alpha) - \frac{1}{2m+1}\sum_{i=0}^{2m} T_{2m}(\varphi_i^\alpha)\Big| \le \frac{C_2}{2^M m^M}\zeta\Big(\frac{1}{m}\Big),$$

and

$$\Big|\frac{1}{2\pi}\int_0^{2\pi} \widetilde{f}(\theta,\phi)d\phi - \frac{1}{2\pi}\int_0^{2\pi} T_{2m}(\phi)d\phi\Big| \le \frac{C_2}{2^M m^M}\zeta\Big(\frac{1}{m}\Big).$$

On the other hand, using the quadrature formula for T_{2m}, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} T_{2m}(\phi)d\phi = \frac{1}{2m+1}\sum_{i=0}^{2m} T_{2m}(\varphi_i^\alpha).$$

From what has already been proved in the last three relations, we deduce that

$$\Big|\frac{1}{2m+1}\sum_{i=0}^{2m} \widetilde{f}(\theta,\varphi_i^\alpha) - \frac{1}{2\pi}\int_0^{2\pi} \widetilde{f}(\theta,\phi)d\phi\Big| \le \frac{C_2}{2^{M-1}m^M}\zeta\Big(\frac{1}{m}\Big), \quad \rho_1 \le \theta \le \rho_2. \tag{19}$$

Combining (18) and (19), we obtain the following uniform estimate on $[\rho_1, \rho_2]$

$$\begin{aligned}\sup_{\theta\in[\rho_1,\rho_2]} \Big|\mathbb{M}^{\text{odd}}(A_m; f)(\theta) - \frac{1}{2\pi}\int_0^{2\pi} \widetilde{f}(\theta,\phi)d\phi\Big| &\le \frac{C_1}{m^{M-N}}\eta\Big(\frac{1}{m}\Big) + \frac{C_2}{2^{M-1}m^M}\zeta\Big(\frac{1}{m}\Big)\\ &= o\Big(\frac{1}{m^{M-N}}\Big).\end{aligned}$$

Using (17) and the first assertion, we easily prove the estimate for the surface integral over $Z(\rho_1, \rho_2)$. The details are left to the reader. □

Corollary 4.4. *Under the hypotheses of Theorem 4.3 except for the Lebesgue constant, if the Lebesgue constant grows like* $\log m$*, then the same estimates in Theorem 4.3 hold in which* $o(\frac{1}{m^{M-N}})$ *is replaced by* $o(\frac{\log m}{m^M})$.

Proof. We reuse the notations and estimates presented in the proof of Theorem 4.3. The Lebesgue inequality and the hypothesis that the Lebesgue constant grows like $\log m$ enable us to find $C_3 > 0$ such that

$$\sup_{t\in I} \big|\widehat{f_i}(t) - \mathbf{L}[\{\cos\vartheta_1^m, \ldots, \cos\vartheta_{2m+1}^m\}; \widehat{f_i}](t)\big| \le \frac{C_3\log m}{m^M}\eta\Big(\frac{1}{m}\Big), \quad 0 \le i \le 2m.$$

Lemma 4.1 now leads to

$$\sup_{\theta\in[\rho_1,\rho_2]}\Big|\frac{1}{2m+1}\sum_{i=0}^{2m}\widehat{f_i}(\cos\theta)-\mathbb{M}^{\text{odd}}(A_m;f)(\theta)\Big|\le\frac{C_3\log m}{m^M}\eta\Big(\frac{1}{m}\Big). \tag{20}$$

Combining (19) and (20) we finally obtain

$$\begin{aligned}\sup_{\theta\in[\rho_1,\rho_2]}\Big|\mathbb{M}^{\text{odd}}(A_m;f)(\theta)-\frac{1}{2\pi}\int_0^{2\pi}\widetilde{f}(\theta,\phi)d\phi\Big| &\le \frac{C_3\log m}{m^M}\eta\Big(\frac{1}{m}\Big)+\frac{C_2}{2^{M-1}m^M}\zeta\Big(\frac{1}{m}\Big)\\ &= o\Big(\frac{\log m}{m^M}\Big).\end{aligned}$$

The proof is complete. □

4.2. Integral means of Hermite interpolation of the second kind

For $\theta\in(0,\pi)$, let us set

$$\mathbb{M}^{\text{even}}(\{(\theta_1,\mu_1),\dots,(\theta_l,\mu_l)\};f)(\theta)=\frac{1}{2\pi}\int_0^{2\pi}\widetilde{\mathbb{H}}^{\text{even}}(\theta,\phi)d\phi, \tag{21}$$

where

$$\widetilde{\mathbb{H}}^{\text{even}}(\theta,\phi)=\mathbb{H}^{\text{even}}[\{(\theta_1,\mu_1),\dots,(\theta_l,\mu_l);\alpha\};f](\sin\theta\sin\phi,\sin\theta\cos\phi,\cos\theta).$$

Obviously, $\mathbb{M}^{\text{even}}(\{(\theta_1,\mu_1),\dots,(\theta_l,\mu_l)\};f)(\theta)$ is a trigonometric polynomial of degree at most $2m-1$ in θ. Modifying the proof of Lemma 4.1 slightly, we obtain the following result.

Lemma 4.5. *Under the assumptions of Theorem 3.5, we have*

$$\mathbb{M}^{\text{even}}(\{(\theta_1,\mu_1),\dots,(\theta_l,\mu_l);f\})(\theta)=\frac{1}{2m}\sum_{i=0}^{2m-1}\mathbf{H}[\{(\cos\theta_1,\mu_1),\dots,(\cos\theta_{2l},\mu_{2l})\};\overline{f}_i](\cos\theta),$$

where

$$\overline{f}_i(t)=\begin{cases}f(\sqrt{1-t^2}\sin\phi_i^\alpha,\ \sqrt{1-t^2}\cos\phi_i^\alpha,t) & \text{if}\quad \phi_i^\alpha\in\Phi_{\alpha,m},\ 0<t<1\\ f(\sqrt{1-t^2}\sin\phi_i^{\alpha+1},\ \sqrt{1-t^2}\cos\phi_i^{\alpha+1},t) & \text{if}\quad \phi_i^{\alpha+1}\in\Phi_{\alpha+1,m},\ -1<t<0.\end{cases}$$

Moreover, if $\overline{f}_i$ is an even function for $i=0,\dots,2m-1$, then

$$\mathbb{M}^{\text{even}}(\{(\theta_1,\mu_1),\dots,(\theta_l,\mu_l);f\})(\theta)=\frac{1}{2m}\sum_{i=0}^{2m-1}\mathbf{H}[\{(\cos^2\theta_1,\mu_1),\dots,(\cos^2\theta_l,\mu_l)\};\overline{f}_i^*](\cos^2\theta),$$

where $\overline{f}_i^*(t)=\overline{f}_i(\sqrt{t})$ for $0<t<1$ and $i=0,\dots,2m-1$.

Proof. As in the proof of Theorem 3.5, we write

$$\widetilde{\mathbb{H}}^{\text{even}}(\theta,\phi)=a_0(\cos\theta)+\sum_{k=1}^{2m-1}\Big[a_k(\cos\theta)(\sin\theta)^k\cos k\phi+b_k(\cos\theta)(\sin\theta)^k\sin k\phi\Big],$$

where $a_k(t)$ and $b_k(t)$ are polynomials of degree $2n-1-k$. It follows that

$$\mathbb{M}^{\text{even}}(\{(\theta_1,\mu_1),\dots,(\theta_l,\mu_l)\};f)(\theta)=\frac{1}{2\pi}\int_0^{2\pi}\widetilde{\mathbb{H}}^{\text{even}}(\theta,\phi)d\phi=a_0(\cos\theta). \tag{22}$$

It is a trigonometric polynomial of degree at $2m-1$ in θ. It follows from (22) and the quadrature formula for trigonometric polynomials in [20, Vol. 2, p. 8] that

$$a_0(\cos\theta) = \frac{1}{2m}\sum_{i=0}^{2m-1} \widetilde{\mathbb{H}}^{\text{even}}(\theta, \phi_i^{\beta}), \quad \beta \in \{\alpha, \alpha+1\}.$$

From the interpolation conditions for $\widetilde{\mathbb{H}}^{\text{even}}(\cdot)$ we have, for $1 \le j \le l$, $0 \le k \le \mu_j - 1$,

$$\frac{d^k}{d\theta^k} a_0(\cos\theta)\Big|_{\theta=\theta_j} = \frac{1}{2m}\sum_{i=0}^{2m-1} \frac{d^k}{d\theta^k}\widetilde{\mathbb{H}}^{\text{even}}(\theta, \phi_i^{\alpha})\Big|_{\theta=\theta_j} = \frac{1}{2m}\sum_{i=0}^{2m-1} \frac{d^k}{d\theta^k}\widetilde{f_i}(\theta, \phi_i^{\alpha})\Big|_{\theta=\theta_j},$$

and, for $l+1 \le j \le 2l$, $0 \le k \le \mu_j - 1$,

$$\frac{d^k}{d\theta^k} a_0(\cos\theta)\Big|_{\theta=\theta_j} = \frac{1}{2m}\sum_{i=0}^{2m-1} \frac{d^k}{d\theta^k}\widetilde{\mathbb{H}}^{\text{even}}(\theta, \phi_i^{\alpha+1})\Big|_{\theta=\theta_j} = \frac{1}{2m}\sum_{i=0}^{2m-1} \frac{d^k}{d\theta^k}\widetilde{f_i}(\theta, \phi_i^{\alpha+1})\Big|_{\theta=\theta_j}.$$

Using Corollary 2.8, we conclude from the definition of $\overline{f}_i$ that

$$a_0^{(k)}(\cos\theta_j) = \frac{1}{2m}\sum_{i=0}^{2m-1} \overline{f}_i^{(k)}(\cos\theta_j) = \Big(\frac{1}{2m}\sum_{i=0}^{2m-1} \overline{f}_i\Big)^{(k)}(\cos\theta_j), \quad 1 \le j \le 2l,\ 0 \le k \le \mu_j - 1.$$

The number of equations in the last relation is $\mu_1 + \cdots + \mu_{2l} = 2m$. Since $a_0 \in \mathcal{P}_{2m-1}(\mathbb{R})$, it follows that

$$\begin{aligned} a_0(t) &= \mathbf{H}[\{(\cos\theta_1, \mu_1), \ldots, (\cos\theta_{2l}, \mu_{2l})\}; a_0](t) \\ &= \mathbf{H}\Big[\{(\cos\theta_1, \mu_1), \ldots, (\cos\theta_{2l}, \mu_{2l})\}; \frac{1}{2m}\sum_{i=0}^{2m-1} \overline{f}_i\Big](t) \\ &= \frac{1}{2m}\sum_{i=0}^{2m-1} \mathbf{H}[\{(\cos\theta_1, \mu_1), \ldots, (\cos\theta_{2l}, \mu_{2l})\}; \overline{f}_i](t), \quad t \in \mathbb{R}. \end{aligned} \tag{23}$$

Combining (22) and (23), we get the first assertion.

By hypothesis, we see that $\cos\theta_i = -\cos\theta_{2l+1-i}$ and $\mu_i = \mu_{2l+1-i}$ for $i = 1, \ldots, l$. Since any function $\overline{f}_i(t)$ is even, we can use [15, Proposition 1] to get

$$\mathbf{H}[\{(\cos\theta_1, \mu_1), \ldots, (\cos\theta_{2l}, \mu_{2l})\}; \overline{f}_i](t) = \mathbf{H}[\{(\cos^2\theta_1, \mu_1), \ldots, (\cos^2\theta_l, \mu_l)\}; \overline{f}_i^*](t^2).$$

The second assertion now follows directly from the first one, and the proof is complete. □

Theorem 4.6. *Let m be positive integer and $0 < \rho_1 < \rho_2 < \pi/2$. Let f be a function defined on S such that $\overline{f}_i \in C^{m-1}([\cos\rho_2, \cos\rho_1])$ and $\overline{f}_i$ is even in $[-\cos\rho_1, -\cos\rho_2] \cup [\cos\rho_2, \cos\rho_1]$ for $i = 0, \ldots, 2m$. Then the following two maps are continuous:*

$$(\vartheta_1, \ldots, \vartheta_m) \in [\rho_1, \rho_2]^m \longmapsto \mathbb{M}^{\text{even}}(\{\vartheta_1, \ldots, \vartheta_m\}; f)$$

and

$$(\vartheta_1, \ldots, \vartheta_m) \in [\rho_1, \rho_2]^m \longmapsto \int\limits_{Z(\rho_1,\rho_2)} \mathbb{H}^{\text{even}}[\{\vartheta_1, \ldots, \vartheta_m; \alpha\}; f](x, y, z)d\omega,$$

where ω is the surface area measure on S. Here, in the first map, the topology on the space of all trigonometric polynomials of degree at most $2m-1$ is induced by any norm.

Proof. From Lemma 4.5 it follows that

$$\mathbb{M}^{\text{even}}(\{\vartheta_1,\dots,\vartheta_m\};f)(\theta)=\frac{1}{2m}\sum_{i=0}^{2m-1}\mathbf{H}[\{\cos^2\vartheta_1,\dots,\cos^2\vartheta_m\};\overline{f}_i^*](\cos^2\theta).$$

Since $\overline{f}_i$ is of class $C^{m-1}([\cos\rho_2,\cos\rho_1])$ and $\overline{f}_i^*(t)=\overline{f}_i(\sqrt{t})$, Theorem 2.3 leads to the continuity of the map

$$(\vartheta_1,\dots,\vartheta_m)\in[\rho_1,\rho_2]^m\longmapsto\mathbf{H}[\{\cos^2\vartheta_1,\dots,\cos^2\vartheta_m\};\overline{f}_i^*],\quad 0\le i\le 2m-1,$$

which along with the above relation proves the first assertion.

Now, let $\{\vartheta_i^k\}_{k=1}^\infty$ be a sequence in $[\rho_1,\rho_2]$ such that $\lim_{k\to\infty}\vartheta_i^k=\vartheta_i$ for $i=1,\dots,m$. The first assertion gives

$$\lim_{k\to\infty}\mathbb{M}^{\text{even}}(\{\vartheta_1^k,\dots,\vartheta_m^k\};f)=\mathbb{M}^{\text{even}}(\{\vartheta_1,\dots,\vartheta_m\};f).$$

Hence

$$\lim_{k\to\infty}\int_{\rho_1}^{\rho_2}\mathbb{M}^{\text{even}}(\{\vartheta_1^k,\dots,\vartheta_m^k\};f)(\theta)\sin\theta d\theta=\int_{\rho_1}^{\rho_2}\mathbb{M}^{\text{even}}(\{\vartheta_1,\dots,\vartheta_m\};f)(\theta)\sin\theta d\theta.$$

Using the change of variable formula (17), we obtain

$$\lim_{k\to\infty}\int_{Z(\rho_1,\rho_2)}\mathbb{H}^{\text{even}}[\{\vartheta_1^k,\dots,\vartheta_m^k;\alpha\};f](x,y,z)d\omega=\int_{Z(\rho_1,\rho_2)}\mathbb{H}^{\text{even}}[\{\vartheta_1,\dots,\vartheta_m;\alpha\};f](x,y,z)d\omega,$$

which finishes the proof. □

Remark 4.7. *The conclusion in Theorem 4.6 still hold true when we replace the hypothesis* $0<\rho_1<\rho_2<\pi/2$ *by the hypothesis* $\pi/2<\rho_1<\rho_2<\pi$.

We have a convergence theorem which proof is similar to that given in Theorem 4.3. We state the result without proof.

Theorem 4.8. *Let* M *and* N *be positive integers with* $M\ge N\ge 1$ *and let* $0<\rho_1<\rho_2<\pi/2$. *Let* f *be a function defined on* S *such that* $f\in C^M(Z(\rho_1,\rho_2))$ *and* $\overline{f}_i$ *is even in* $[-\cos\rho_1,-\cos\rho_2]\cup[\cos\rho_2,\cos\rho_1]$ *for* $i=0,\dots,2m-1$. *Let* $A_m=\{\vartheta_1^m,\dots,\vartheta_m^m\}$ *be sets of distinct points in* $[\rho_1,\rho_2]$ *such that* $\Delta(\{\cos^2\vartheta_1^m,\dots,\cos^2\vartheta_m^m\},[\cos^2\rho_2,\cos^2\rho_1])$ *grows at most like a polynomial of degree* N *in* m. *Then*

$$\sup_{\theta\in[\rho_1,\rho_2]}\left|\mathbb{M}^{\text{even}}(A_m;f)(\theta)-\frac{1}{2\pi}\int_0^{2\pi}\widetilde{f}(\theta,\phi)d\phi\right|=o\Big(\frac{1}{m^{M-N}}\Big)$$

and

$$\left|\int_{Z(\rho_3,\rho_4)}\mathbb{L}^{\text{even}}[\{A_m,\alpha\};f](x,y,z)d\omega-\int_{Z(\rho_3,\rho_4)}f(x,y,z)d\omega\right|=o\Big(\frac{1}{m^{M-N}}\Big),\quad \rho_1\le\rho_3<\rho_4\le\rho_2.$$

Open problems. We only state some questions for the odd case.

1. Is there an explicit formula or an error formula for $\mathbb{H}^{\text{odd}}$?
2. Let $f\in C^{2m}(Z(\rho_1,\rho_2))$ with $0<\rho_1<\rho_2<\pi$. Is the map
$$(\vartheta_1,\dots,\vartheta_{2m+1})\in[\rho_1,\rho_2]^{2m+1}\longmapsto\mathbb{H}^{\text{odd}}[\{\vartheta_1,\dots,\vartheta_{2m+1},\Theta_{\alpha,m}\};f]\in\mathcal{P}_{2m}(S)$$
continuous?
3. Does exist an array $\{A_m\}\subset[\rho_1,\rho_2]$ such that, for any sufficiently smooth function f, the sequence $\left\{\mathbb{L}^{\text{odd}}[\{A_m;\Theta_{\alpha,m}\};f]\right\}$ converges to f uniformly on $Z(\rho_1,\rho_2)$?
4. Can we replace the estimates in Theorem 4.3 by L^p estimates?

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