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Spectral Properties of Square Hyponormal Operators

Muneo Chō[a], Dijana Mosić[b], Biljana Načevska Nastovska[c], Taiga Saito[a]

[a]*Department of Mathematics, Kanagawa University, Hiratsuka 259-1293, Japan*
[b]*Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia*
[c]*Faculty of Electrical Engineering and Information Technologies, "Ss Cyril and Methodius" University in Skopje, Macedonia*

Abstract. In this paper, we introduce a square hyponormal operator as a bounded linear operator T on a complex Hilbert space $\mathcal{H}$ such that T^2 is a hyponormal operator, and we investigate some basic properties of this operator. Under the hypothesis $\sigma(T) \cap (-\sigma(T)) \subset \{0\}$, we study spectral properties of a square hyponormal operator. In particular, we show that if z and w are distinct eigen-values of T and $x, y \in \mathcal{H}$ are corresponding eigen-vectors, respectively, then $\langle x, y \rangle = 0$. Also, we define nth hyponormal operators and present some properties of this kind of operators.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space, and let $B(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, we denote by T^*, $\ker(T), R(T), \sigma(T), \sigma_a(T), \sigma_r(T)$, respectively, the adjoint, the null space, the range, the spectrum, the approximate point spectrum and the residual spectrum of T. It is well-known that $\sigma(T) = \sigma_a(T) \cup \sigma_r(T)$.

An operator $T \in B(\mathcal{H})$ is *self-adjoint* if $T = T^*$. An operator $T \in B(\mathcal{H})$ is *normal* and 2-*normal* if $T^*T = TT^*$ and $T^*T^2 = T^2T^*$, respectively. By Fuglede-Putnam Theorem, it is easily to see that T is 2-normal if and only if T^2 is normal (see [4]). An operator $T \in B(\mathcal{H})$ is *positive* (denoted by $T \geq 0$) if $\langle Tx, x\rangle = 0$, for all $x \in \mathcal{H}$. For self-adjoint operators $T, S \in B(\mathcal{H})$, $T \geq S$ means $T - S \geq 0$.

For an operator $T \in B(\mathcal{H})$, let $|T| = (T^*T)^{\frac{1}{2}}$ and $|T^*| = (TT^*)^{\frac{1}{2}}$. For $0 < p \leq 1$, T is said to be *p-hyponormal* if $|T|^{2p} \geq |T^*|^{2p}$. When $p = 1$ and $p = \dfrac{1}{2}$, T is said to be *hyponormal* and *semi-hyponormal*, respectively. Notice that T is hyponormal if and only if $\|T^*x\| \leq \|Tx\|$, for all $x \in \mathcal{H}$. By Corollary 1 of [3], in general, if T is p-hyponormal $(0 < p \leq 1)$, then T^n is $\dfrac{p}{n}$–hyponormal. An operator $T \in B(\mathcal{H})$ is said to be *paranormal* if $\|Tx\|^2 \leq \|T^2x\| \cdot \|x\|$, for all $x \in \mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be *algebraically hyponormal* and *algebraically paranormal* if $p(T)$ is hyponormal and paranormal, for some nonconstant complex polynomial p, respectively.

In [7, 8], the authors showed that if T is algebraically hyponormal and algebraically paranormal, then T is isoloid and Weyl's Theorem holds, respectively.

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Email addresses: chiyom01@kanagawa-u.ac.jp (Muneo Chō), dijana@pmf.ni.ac.rs (Dijana Mosić), bibanmath@gmail.com (Biljana Načevska Nastovska), taiga1996227@outlook.jp (Taiga Saito)

The aim of this paper is to study a bounded linear operator T on a complex Hilbert space such that T^2 is a hyponormal operator. Firstly, notice that there exists an operator T such that T^2 is hyponormal and T is not hyponormal.

Let $\mathcal{H} = \ell^2$ and T be the unilateral shift with the weights $\{a_n \geq 0\}$ such that

$$Tx := (0, a_1x_1.a_2x_2, ...) \quad \text{for} \quad x = (x_1, x_2, ...) \in \mathcal{H}.$$

Then T is hyponormal if and only if $a_j \leq a_{j+1}$ $(j = 1, 2, ...)$, i.e., $\{a_j\}$ is a monotone increasing sequence, for $a_j = 1$ $(j \neq 2)$ and $a_2 = \dfrac{1}{2}$. Since the sequence $\{a_n\}$ is not increasing, the operator T is not hyponormal. But since

$$T^2x = (0, 0, a_1a_2x_1, a_2a_3x_2, ...) \quad \text{and} \quad T^{2*}x = (a_1a_2x_3, a_2a_3x_4, ...),$$

T^2 is hyponormal if and only if $a_ja_{j+1} \leq a_{j+2}a_{j+3}$ for $j = 1, 2, ...$ Hence, by this weights $a_j = 1$ $(j \neq 2)$ and $a_2 = \dfrac{1}{2}$, the operator T^2 is hyponormal and T is not hyponormal.

In [4–6], the authors have studied spectral properties of n-normal operator, that is, an operator T such that T^n is normal, in the cases that $\sigma(T) \bigcap (-\sigma(T)) = \emptyset$ or $\sigma(T) \cap (-\sigma(T)) \subset \{0\}$. Since an operator T such that T^2 is hyponormal is algebraically hyponormal, T is isoloid and Weyl's Theorem holds. Hence, we study other spectral properties of such an operator T in this paper.

2. Basic properties

In the beginning, we introduce a square hyponormal operator and investigate some basic properties of this operator.

Definition 2.1. *For an operator $T \in B(\mathcal{H})$, T is said to be square hyponormal if T^2 is hyponormal.*

The following result follows from the definition of square hyponormal operators.

Theorem 2.2. *Let $T \in B(\mathcal{H})$ be square hyponormal. Then the following statements hold.*
(1) If T is invertible, then so is T^{-1}.
(2) For an even number $n = 2k \in \mathbb{N}$, T^n is $\dfrac{1}{k}$-hyponormal.
(3) If $S \in B(\mathcal{H})$ is unitary equivalent to T, then S is square hyponormal.
(4) If $T - t$ is square hyponormal for all $t > 0$, then T is hyponormal.

Proof. (1) is clear.
(2) Since T^2 is hyponormal, by Corollary 1 of [3], $T^n = T^{2k} = (T^2)^k$ is $\dfrac{1}{k}$-hyponormal.
(3) is clear.
(4) Since

$$0 \leq (T - t)^{2*}(T - t)^2 - (T - t)^2(T - t)^{2*} = T^{2*}T^2 - T^2T^{2*}$$
$$-2t\Big(T^{2*}T + T^*T^2 - TT^{2*} - T^2T^*\Big) + 4t^2(T^*T - TT^*),$$

we obtain that

$$0 \leq \frac{1}{4t^2}\Big((T - t)^{2*}(T - t)^2 - (T - t)^2(T - t)^{2*}\Big) = \frac{1}{4t^2}\Big(T^{2*}T^2 - T^2T^{2*}\Big)$$
$$-\frac{1}{2t}\Big(T^{2*}T + T^*T^2 - TT^{2*} - T^2T^*\Big) + (T^*T - TT^*).$$

Letting $t \to \infty$, we have $T^*T - TT^* \geq 0$. □

We now consider the restriction of a square hyponormal operator to an invariant closed subspace.

Theorem 2.3. *Let $T \in B(\mathcal{H})$ be square hyponormal and M be an invariant closed subspace for T. Then $T_{|M}$ is square hyponormal.*

Proof. Since M is an invariant closed subspace for T, we observe that

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix} : \begin{bmatrix} M \\ M^{\perp} \end{bmatrix} \to \begin{bmatrix} M \\ M^{\perp} \end{bmatrix}.$$

Therefore, for $D = T_1T_2 + T_2T_3$, since

$$T^2 = \begin{bmatrix} T_1^2 & D \\ 0 & T_3^2 \end{bmatrix} \quad \text{and} \quad (T^2)^* = \begin{bmatrix} (T_1^2)^* & 0 \\ D^* & (T_3^2)^* \end{bmatrix},$$

we have

$$(T^2)^*T^2 - T^2(T^2)^* = \begin{bmatrix} (T_1^2)^*T_1^2 - T_1^2(T_1^2)^* - DD^* & (T_1^2)^*D - D(T_3^2)^* \\ D^*T_1^2 - T_3^2D^* & D^*D + (T_3^2)^*T_3^2 - T_3^2(T_3^2)^* \end{bmatrix} \geq 0.$$

Hence we deduce that $(T_1^2)^*T_1^2 - T_1^2(T_1^2)^* - DD^* \geq 0$ and so $(T_1^2)^*T_1^2 - T_1^2(T_1^2)^* \geq 0$. Therefore, $T_{|M}$ is square hyponormal. □

3. Spectral property

Under some additional assumptions, we study spectral properties of a square hyponormal operator in this section. Firstly, we show the following theorem.

Theorem 3.1. *Let $T \in B(\mathcal{H})$ be square hyponormal. If $\mu(\sigma(T)) = 0$, then T^2 is normal, where μ is the planar Lebesgue measure.*

Proof. Since $\mu(\sigma(T)) = 0$, we have that $\mu(\sigma(T^2)) = 0$ by the spectral mapping theorem. By T^2 is hyponormal and Putnam's Theorem, it holds

$$||T^{2*}T^2 - T^2T^{2*}|| \leq \frac{1}{\pi}\mu(\sigma(T^2)) = 0.$$

Hence, T^2 is normal. □

Remark 3.2. If T is p-hyponormal and square hyponormal with $\mu(\sigma(T)) = 0$, then, by Corollary 2 of [3], T is normal. But let $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$. Then S is square hyponormal with $\mu(\sigma(S)) = 0$ and S is not normal.

If T is compact, then $\mu(\sigma(T)) = 0$. Hence, we have the following corollary.

Corollary 3.3. *If $T \in B(\mathcal{H})$ is compact square hyponormal, then T^2 is normal.*

An operator $T \in B(\mathcal{H})$ is said to have SVEP (single-valued extension property) if for every open subset G of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function f on G such that $(T - z)f(z) \equiv 0$ on G, then $f(z) \equiv 0$ on G. It is well known that:

(1) If $\ker(T - z) \perp \ker(T - w)$ for any distinct nonzero eigenvalues z and w, then
T has SVEP.
(2) Let p be polynomial. If $p(T)$ has SVEP, then T has SVEP.

See details in [2, 11, 12]. Since it is clear that a hyponormal operator has SVEP, we have the next corollary by (2).

Corollary 3.4. *Let $T \in B(\mathcal{H})$ be square hyponormal. Then T has* SVEP.

Let $\mathcal{K}(\mathcal{H})$ be the set of all compact operators on $\mathcal{H}$. Then, for $T \in B(\mathcal{H})$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined as follows:

$$\sigma_w(T) = \bigcap_{K \in \mathcal{K}(\mathcal{H})} \sigma(T+K) \ \text{ and } \ \sigma_b(T) = \bigcap_{K \in \mathcal{K}(\mathcal{H});\, TK=KT} \sigma(T+K).$$

If T has SVEP, then $\sigma_w(T) = \sigma_b(T)$ by Corollary 3.53 of [2]. Let $\mathcal{H}(\sigma(T))$ denote the set of all analytic function defined on an open set containing $\sigma(T)$. Then, by Corollary 3.72 of [2], we have the following result.

Corollary 3.5. *Let $T \in B(\mathcal{H})$ be square hyponormal. Then, for $f \in \mathcal{H}(\sigma(T))$,*

$$\sigma_w(f(T)) = \sigma_b(f(T)) = f(\sigma_w(T)) = f(\sigma_b(T)).$$

Next for $T \in B(\mathcal{H})$, we set the following property:

$$(*) \qquad \sigma(T) \bigcap (-\sigma(T)) \subset \{0\}.$$

Then we begin with the following result.

Theorem 3.6. *Let $T \in B(\mathcal{H})$ be square hyponormal with $(*)$ and M be an invariant subspace for T. If $\sigma(T_{|M}) = \{z\}$, then the following assertions hold.*
(1) If $z = 0$, then $(T_{|M})^2 = 0$.
(2) If $z \neq 0$, then $T_{|M} = z$.

Proof. (1) By Theorem 2.3, $T_{|M}$ is square hyponormal. Since $\sigma((T_{|M})^2) = \{0\}$, we have $(T_{|M})^2 = 0$ by Putnam's theorem.
(2) Similarly, from $\sigma((T_{|M})^2) = \{z^2\}$, we get $(T_{|M})^2 = z^2$ and hence

$$0 = (T_{|M})^2 - z^2 = (T_{|M} + z)(T_{|M} - z).$$

By the assumption $(*)$, $-z \notin \sigma(T)$ and there exists $(T_{|M} + z)^{-1}$. Hence, it holds $T_{|M} - z = 0$. □

Theorem 3.7. *Let $T \in B(\mathcal{H})$ be a square hyponormal operator. If T satisfies $(*)$, then $\sigma(T) = \{\overline{z} : z \in \sigma_a(T^*)\}$.*

Proof. Since $\sigma(T) = \sigma_a(T) \cup \sigma_r(T)$, we may show $\sigma_a(T) \subset \{\overline{z} : z \in \sigma_a(T^*)\}$.
(1) If $0 \in \sigma_a(T)$, then $0 \in \sigma_a(T^2)$ and T^2 is hyponormal. Hence, it is easy to see $0 \in \sigma_a(T^*)$.
(2) Let $z \in \sigma_a(T)$ and $z \neq 0$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $(T - z)x_n \to 0$ as $n \to \infty$. Thus, $(T^2 - z^2)x_n \to 0$ as $n \to \infty$. Because T^2 is hyponormal, we have $(T^2 - z^2)^* x_n \to 0$ and $(T^* + \overline{z})(T^* - \overline{z})x_n \to 0$ as $n \to \infty$. By the assumption $(*)$, $-\overline{z} \notin \sigma(T^*)$ which gives $(T^* - \overline{z})x_n \to 0$ as $n \to \infty$ and therefore $\overline{z} \in \sigma_a(T^*)$. It completes the proof. □

Theorem 3.8. *Let $T \in B(\mathcal{H})$ be square hyponormal and satisfy $(*)$.*
(1) If z and w are distinct eigen-values of T and $x, y \in \mathcal{H}$ are corresponding eigen-vectors, respectively, then $\langle x, y \rangle = 0$.
(2) If z, w are distinct values of $\sigma_a(T)$ and $\{x_n\}$, $\{y_n\}$ are the sequences of unit vectors in $\mathcal{H}$ such that $(T - z)x_n \to 0$ and $(T - w)y_n \to 0$ $(n \to \infty)$, then $\lim_{n\to\infty} \langle x_n, y_n \rangle = 0$.

Proof. (1) follows from (2). So, we show (2). Since $(T - z)x_n \to 0$ and $(T - w)y_n \to 0$ $(n \to \infty)$, it holds that $(T^2 - z^2)x_n \to 0$ and $(T^2 - w^2)y_n \to 0$. Because T^2 is hyponormal, we get $(T^{*2} - \overline{w}^2)y_n \to 0$. Hence,

$$\lim_{n\to\infty} z^2 \langle x_n, y_n \rangle = \lim_{n\to\infty} \langle z^2 x_n, y_n \rangle = \lim_{n\to\infty} \langle T^2 x_n, y_n \rangle = \lim_{n\to\infty} \langle x_n, T^{*2} y_n \rangle = \lim_{n\to\infty} w^2 \langle x_n, y_n \rangle.$$

If $z^2 = w^2$, then $(z + w)(z - w) = 0$. Since $z \neq w$, we have $z = -w$. By $(*)$, this implies $z = w = 0$. Therefore, $z^2 \neq w^2$, and so $\lim_{n\to\infty} \langle x_n, y_n \rangle = 0$. □

Thus, we have the following corollary.

Corollary 3.9. *Let $T \in B(\mathcal{H})$ be square hyponormal and satisfy* $(*)$. *If z and w are distinct eigen-values of T, then* $\ker(T - z) \perp \ker(T - w)$.

Let M be a subspace of $\mathcal{H}$. M is said to be a *reducing subspace* for T if $T(M) \subset M$ and $T^*(M) \subset M$, that is, M is an invariant subspace for T and T^*. Then we have a following result.

Theorem 3.10. *Let $T \in B(\mathcal{H})$ be square hyponormal and satisfy* $(*)$. *If z is a non-zero eigen-value of T, then* $\ker(T - z) = \ker(T^2 - z^2) \subset \ker(T^{*2} - \overline{z}^2) = \ker(T^* - \overline{z})$ *and hence* $\ker(T - z)$ *is a reducing subspace for T.*

Proof. Firstly, we show that $\ker(T-z) = \ker(T^2-z^2)$. Because it is clear that $\ker(T-z) \subset \ker(T^2-z^2)$, we will verify that $\ker(T^2-z^2) \subset \ker(T-z)$. Let $x \in \ker(T^2-z^2)$, i.e., $(T^2-z^2)x = 0$. Then $(T+z)(T-z)x = 0$. Since $z \neq 0$, by the assumption $(*)$, we have $-z \notin \sigma(T)$. Hence, it follows $(T-z)x = 0$ and $x \in \ker(T-z)$. Therefore, $\ker(T^2-z^2) \subset \ker(T-z)$ and $\ker(T-z) = \ker(T^2-z^2)$. Since T^2 is hyponormal, $\ker(T^2-z^2) \subset \ker(T^{*2}-\overline{z}^2)$. Evidently, $\ker(T^* - \overline{z}) \subset \ker(T^{*2} - \overline{z}^2)$. Let $x \in \ker(T^{*2} - \overline{z}^2)$. Because $(T^* + \overline{z})(T^* - \overline{z})x = 0$ and $T^* + \overline{z}$ is invertible by the assumption $(*)$, we obtain that $x \in \ker(T^* - \overline{z})$. Hence, $\ker(T^{*2} - \overline{z}^2) = \ker(T^* - \overline{z})$. Finally, by the above results, it is clear that $\ker(T - z)$ is a reducing subspace for T. $\square$

The following remark is same with the corresponding in the paper of [5].

Remark 3.11. *In general,* $\ker(T)$ *is not a reducing subspace for a square hyponormal operator T.*
(1) Let T be as in Example 2.3 of [1], that is, let $\mathcal{H} = \ell^2$, $\{e_j\}_{j=1}^{\infty}$ be the standard orthonormal basis of ℓ^2 and T be defined by

$$Te_j = \begin{cases} e_1 & (j = 1) \\ e_{j+1} & (j = 2k) \\ 0 & (j = 2k+1). \end{cases}$$

Then T is a square hyponormal operator and satisfies $(*)$. Since $e_3 \in \ker(T)$ and $TT^*e_3 = e_3 \neq 0$, $\ker(T)$ does not reduce T. Let P be the orthogonal projection to the first coordinate. Since $T^2 = P$, it is clear that $\ker(T) \subsetneqq \ker(T^2) = \ker(P)$.
(2) We give an easy example. Let $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$. Since $S^2 = 0$ and $\sigma(S) = \{0\}$, S is square hyponormal and satisfies $(*)$. Let $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $x \in \ker(S)$ and $SS^*x = x \neq 0$. Hence, $\ker(S)$ does not reduce S and $\ker(S) \subsetneqq \ker(S^2) = \mathbb{C}^2$.

For an isolated point λ of $\sigma(T)$, the Riesz idempotent for λ is defined by

$$E_T(\{\lambda\}) = \frac{1}{2\pi i} \int_{\partial D} (z - T)^{-1}\, dz,$$

where D is a closed disk centered at λ which contains no other points of $\sigma(T)$. For an operator $T \in \mathcal{L}(\mathcal{H})$, the *quasinilpotent part* of T is defined by

$$\mathcal{H}_0(T) := \{x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Then $\mathcal{H}_0(T)$ is a linear (not necessarily closed) subspace of $\mathcal{H}$. It is known that if T has SVEP, then

$$\mathcal{H}_0(T - \lambda) = \{x \in \mathcal{H} : \lim_{n\to\infty} \|(T - \lambda)^n x\|^{\frac{1}{n}} = 0\} = E_T(\{\lambda\})\mathcal{H}$$

for all $\lambda \in \mathbb{C}$. In general, $\ker(T-\lambda)^m \subset \mathcal{H}_0(T-\lambda)$ and $\mathcal{H}_0(T-\lambda)$ is not closed. However, if λ is an isolated point of $\sigma(T)$, then $E_T(\{\lambda\})\mathcal{H} = \mathcal{H}_0(T-\lambda)$ and $\mathcal{H}_0(T-\lambda)$ is closed. Also, if T is normal and $T = \int_{\sigma(T)} \lambda dF(\lambda)$ is the spectral decomposition of T, then

$$\mathcal{H}_0(T-\lambda) = E_T(\{\lambda\})\mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*.$$

In 2012. J. T. Yuan and G. X. Ji ([12, Lemma 5.2]) proved following Lemma.

Lemma 3.12. *Let $T \in B(\mathcal{H})$, m be a positive integer and λ be an isolated point of $\sigma(T)$.*
(i) *The following assertions are equivalent:*
(a) $E_T(\{\lambda\})\mathcal{H} = \ker(T-\lambda)^m$.
(b) $\ker(E_T(\{\lambda\})) = (T-\lambda)^m\mathcal{H}$.
In this case, λ is a pole of the resolvent of T and the order of λ is not greater than m.
(ii) *If λ is a pole of the resolvent of T and the order of λ is m, then the following assertions are equivalent:*
(a) *$E_T(\{\lambda\})$ is self-adjoint.*
(b) $\ker((T-\lambda)^m) \subset \ker((T-\lambda)^{*m})$.
(c) $\ker((T-\lambda)^m) = \ker((T-\lambda)^{*m})$.

By this lemma, we prove the following theorem.

Theorem 3.13. *Let $T \in B(\mathcal{H})$ be square hyponormal and satisfy $(*)$. Let λ be an isolated point of spectrum of T. Then the following statements hold.*
(i) *If $\lambda = 0$, then $\mathcal{H}_0(T) = \ker(T^2) = \ker(T^{*2})$, $E_T(\{0\})$ is self-adjoint and the order of pole λ is not greater than* 2.
(ii) *If $\lambda \neq 0$, then $\mathcal{H}_0(T-\lambda) = \ker(T-\lambda) = \ker((T-\lambda)^*)$, $E_T(\{\lambda\})$ is self-adjoint and the order of pole λ is* 1.

Proof. (i) Assume that $\lambda = 0$. Since $\sigma(T^2) = \{z^2 : z \in \sigma(T)\}$, it follows that 0 is an isolated point of spectrum of T^2. We prove that $\mathcal{H}_0(T) = \mathcal{H}_0(T^2)$. Let $x \in \mathcal{H}_0(T)$. Then $\|T^n x\|^{\frac{1}{n}} \longrightarrow 0$ and thus $\|T^{2n}x\|^{\frac{1}{2n}} = \left(\|T^{2n}x\|^{\frac{1}{n}}\right)^{\frac{1}{2}} \longrightarrow 0$ and $\|T^{2n}x\|^{\frac{1}{n}} \longrightarrow 0$. Hence, $x \in \mathcal{H}_0(T^2)$. Conversely, let $x \in \mathcal{H}_0(T^2)$. Then $\|T^{2n}x\|^{\frac{1}{n}} \longrightarrow 0$ and so $\|T^{2n}x\|^{\frac{1}{2n}} = \left(\|T^{2n}x\|^{\frac{1}{n}}\right)^{\frac{1}{2}} \longrightarrow 0$. From

$$\begin{aligned}\|T^{2n+1}x\|^{\frac{1}{2n+1}} &\leq \left(\|T\|\|T^{2n}x\|\right)^{\frac{1}{2n+1}} \\ &\leq \|T\|^{\frac{1}{2n+1}} \left(\|T^{2n}x\|^{\frac{1}{2n}}\right)^{\frac{2n}{2n+1}} \longrightarrow 0 \ \ (n \to \infty),\end{aligned}$$

it follows that $x \in \mathcal{H}_0(T)$. Therefore, $\mathcal{H}_0(T) = \mathcal{H}_0(T^2)$. Since T^2 is hyponormal, we observe that $E_{T^2}(\{0\})\mathcal{H} = \mathcal{H}_0(T^2) = \ker(T^2) = \ker(T^{*2})$ by Stampfli [10]. So,

$$E_T(\{0\})\mathcal{H} = \mathcal{H}_0(T) = \mathcal{H}_0(T^2) = E_{T^2}(\{0\})\mathcal{H} = \ker(T^2) = \ker(T^{*2}).$$

Now, 0 is a pole of the resolvent of T, the order of 0 is not greater than 2 and $E_T(\{0\})$ is self-adjoint by Lemma 3.12.
(ii) Next we assume that $\lambda \neq 0$. Then λ^2 is an isolated point of $\sigma(T^2)$ by Lemma 2.1 of [5]. We will prove $\mathcal{H}_0(T-\lambda) = \mathcal{H}_0(T^2-\lambda^2)$. Let $x \in \mathcal{H}_0(T-\lambda)$. Then $\|(T-\lambda)^n x\|^{\frac{1}{n}} \to 0$ and

$$\begin{aligned}\|(T^2-\lambda^2)^n x\|^{\frac{1}{n}} &\leq \|(T+\lambda)^n\|^{\frac{1}{n}}\|(T-\lambda)^n x\|^{\frac{1}{n}} \\ &\leq \|T+\lambda\|\|(T-\lambda)^n x\|^{\frac{1}{n}} \longrightarrow 0,\end{aligned}$$

which implies $\mathcal{H}_0(T-\lambda) \subset \mathcal{H}_0(T^2-\lambda^2)$. Conversely, let $x \in \mathcal{H}_0(T^2-\lambda^2)$. Since $T+\lambda$ is invertible by the assumption $(*)$, we have

$$\begin{aligned}\|(T-\lambda)^n x\|^{\frac{1}{n}} &= \|(T+\lambda)^{-n}(T+\lambda)^n(T-\lambda)^n x\|^{\frac{1}{n}} \\ &\leq \|\left\{(T+\lambda)^{-1}\right\}^n\|^{\frac{1}{n}}\|(T^2-\lambda^2)^n x\|^{\frac{1}{n}} \\ &\leq \|(T+\lambda)^{-1}\|\|(T^2-\lambda^2)^n x\|^{\frac{1}{n}} \longrightarrow 0.\end{aligned}$$

Hence, $\mathcal{H}_0(T-\lambda) \supset \mathcal{H}_0(T^2-\lambda^2)$ and $\mathcal{H}_0(T-\lambda) = \mathcal{H}_0(T^2-\lambda^2)$. Because T^2 is hyponormal, it follows that

$$E_{T^2}(\{\lambda^2\})\mathcal{H} = \mathcal{H}_0(T^2-\lambda^2) = \ker(T^2-\lambda^2) = \ker(T^{*2}-\overline{\lambda}^2)$$

by Stampfli [10]. Hence

$$E_T(\{\lambda\})\mathcal{H} = \mathcal{H}_0(T-\lambda) = \mathcal{H}_0(T^2-\lambda^2) = E_{T^2}(\{\lambda\})\mathcal{H} = \ker(T^2-\lambda^2) = \ker(T^{*2}-\overline{\lambda}^2).$$

Since $(T+\lambda)^*$ is invertible, we get

$$E_T(\{\lambda\})\mathcal{H} = \ker(T-\lambda) = \ker\left((T-\lambda)^*\right).$$

Thus, λ is a pole of the resolvent of T, the order of λ is not greater than 2 and $E_T(\{\lambda\})$ is self-adjoint by Lemma 3.12.
□

Let D be a bounded open subset of $\mathbb{C}$ and $L^2(D,\mathcal{H})$ be the Hilbert space of measurable function $f : D \longrightarrow \mathcal{H}$ such that

$$\|f\| = \left(\int_D \|f(z)\|^2\, d\mu(z)\right)^{\frac{1}{2}} < \infty,$$

where μ is the planar Lebesgue measure. Let $W^2(D,\mathcal{H})$ be the Sobolev space with respect to $\overline{\partial}$ and of order 2 whose derivatives $\overline{\partial} f$ and $\overline{\partial}^2 f$ in the sense of distributions belong to $L^2(D,\mathcal{H})$. The norm $\|f\|_{W^2}$ is given by

$$\|f\|_{W^2} = \left(\|f\|^2 + \|\overline{\partial} f\|^2 + \|\overline{\partial}^2 f\|^2\right)^{\frac{1}{2}} \quad \text{for } f \in L^2(D,\mathcal{H}).$$

In [4], Alzuraiqi and Patel proved the following.

Proposition 3.14. (Alzraiqi and Patel [4], Theorem 2.37) Let D be an arbitrary bounded disk in $\mathbb{C}$. If $T \in B(\mathcal{H})$ is 2-normal with the assumption $\sigma(T) \bigcap (-\sigma(T)) = \emptyset$, then the operator

$$z - T \,:\, W^2(D,\mathcal{H}) \longrightarrow L^2(D,\mathcal{H})$$

is one to one for every $z \in \mathbb{C}$.

We would like to prove this result as follows.

Theorem 3.15. *Let D be an arbitrary bounded disk in $\mathbb{C}$ and $T \in B(\mathcal{H})$ be square hyponormal with $(*)$. Then the operator*

$$z - T \,:\, W^2(D,\mathcal{H}) \longrightarrow L^2(D,\mathcal{H})$$

is one to one for every $z \in \mathbb{C}$.

Proof. Let $f \in W^2(D,\mathcal{H})$, $S = z - T$ and $Sf = 0$. We show $f = 0$. Then

$$\begin{aligned}\|f\|^2_{W^2} &= \|f\|^2_{2,D} + \|\overline{\partial} f\|^2_{2,D} + \|\overline{\partial}^2 f\|^2_{2,D}\\ &= \int_D \|f(z)\|^2 d\mu(z) + \int_D \|\overline{\partial} f(z)\|^2 d\mu(z) + \int_D \|\overline{\partial}^2 f(z)\|^2 d\mu(z) < \infty,\end{aligned}$$

and

$$\begin{aligned}\|Sf\|^2_{W^2} &= \|(z-T)f\|^2_{W^2}\\ &= \|(z-T)f\|^2_{2,D} + \|\overline{\partial}((z-T)f)\|^2_{2,D} + \|\overline{\partial}^2((z-T)f)\|^2_{2,D}\\ &= \|(z-T)f\|^2_{2,D} + \|(z-T)\overline{\partial} f\|^2_{2,D} + \|(z-T)\overline{\partial}^2 f\|^2_{2,D} = 0.\end{aligned}$$

Hence,

$$\|(z-T)\overline{\partial}^i f\|^2_{2,D} = \int_D \|(z-T)\overline{\partial}^i f(z)\|^2 d\mu(z) = 0 \ \ (i = 0, 1, 2).$$

Let i be $i = 0, 1, 2$. Since $(z-T)\overline{\partial}^i f(z) = 0$ for $z \in D$, if $z \in D \setminus \sigma(T)$, then $\overline{\partial}^i f(z) = 0$ because $z - T$ is invertible. This implies

$$\|(z-T)^*\overline{\partial}^i f\|^2_{2,D\setminus\sigma(T)} = \int_{D\setminus\sigma(T)} \|(z-T)^*\overline{\partial}^i f(z)\|^2 d\mu(z) = 0.$$

Since

$$\begin{aligned}\|(z^2-T^2)\overline{\partial}^i f\|^2_{2,D} &= \int_D \|(z^2-T^2)\overline{\partial}^i f(z)\|^2 d\mu(z) \\ &\le \left(\sup_{z\in D}\|z+T\|\right)^2 \int_D \|(z-T)\overline{\partial}^i f(z)\|^2 d\mu(z) \\ &= \left(\sup_{z\in D}\|z+T\|\right)^2 \|(z-T)\overline{\partial}^i f\|^2_{2,D} = 0,\end{aligned}$$

we have $(z^2-T^2)\overline{\partial}^i f(z) = 0$ for $z \in D$. Because T^2 is hyponormal, then

$$\int_D \|(z^2-T^2)^*\overline{\partial}^i f(z)\|^2 d\mu(z) = \|(z^2-T^2)^*\overline{\partial}^i f\|^2_{2,D} \le \|(z^2-T^2)\overline{\partial}^i f\|^2_{2,D} = 0.$$

So,

$$0 = (z^2-T^2)^*\overline{\partial}^i f(z) = (z+T)^*(z-T)^*\overline{\partial}^i f(z) \ \text{ for } z \in D.$$

If $z \in D \cap (\sigma(T) \setminus (-\sigma(T)))$, then $z + T$ and $(z+T)^*$ are invertible. Hence, $(z-T)^*\overline{\partial}^i f(z) = 0$ for $z \in D \cap (\sigma(T) \setminus (-\sigma(T)))$. Since D is bounded, $\|\overline{\partial}^i f\|^2_{2,D} < \infty$ and the planar Lebesgue measure of $\sigma(T) \bigcap (-\sigma(T))$ is 0, we have

$$\begin{aligned}\|(z-T)^*\overline{\partial}^i f\|^2_{2,D} &= \int_{D\setminus\sigma(T)} \|(z-T)^*\overline{\partial}^i f(z)\|^2 d\mu(z) \\ &\quad + \int_{D\cap(\sigma(T)\setminus(-\sigma(T)))} \|(z-T)^*\overline{\partial}^i f(z)\|^2 d\mu(z) \\ &\quad + \int_{D\cap\sigma(T)\cap(-\sigma(T))} \|(z-T)^*\overline{\partial}^i f(z)\|^2 d\mu(z) \\ &\le 0 + 0 + \max_{z\in D}\|(z-T)^*\|^2 \int_{D\cap\sigma(T)\cap(-\sigma(T))} \|\overline{\partial}^i f(z)\|^2 d\mu(z) = 0.\end{aligned}$$

By [9, Proposition 2.1], we obtain $\|(I-P)f\|_{2,D} = 0$. Thus, $f(z) = (Pf)(z)$ for $z \in D$. From $Sf = 0$, we have $(Sf)(z) = (z-T)f(z) = (z-T)(Pf)(z) = 0$ for $z \in D$.

Since T has the single-valued extension property by Corollary 3.4 and Pf is analytic, it follows that $0 = (Pf)(z) = f(z)$ for $z \in D$. Hence, $f = 0$ and S is one to one. □ □

An operator $T \in B(\mathcal{H})$ is said to be *polaroid* if every isolated point of the spectrum of T is a pole of the resolvent. In [1], Aiena showed that if T is algebraically paranormal on a Banach space, then the following results hold.
(1) T is polaroid (Theorem 1.3).
(2) If T is quasinilpotent, then T is nilpotent (Lemma 1.2).
Hence, it is clear that if $T \in B(\mathcal{H})$ is square hyponormal, then T is polaroid.

4. nth hyponormal operators

We now introduce and study nth hyponormal operators.

Definition 4.1. *For $n \in \mathbb{N}$ and an operator $T \in B(\mathcal{H})$, T is said to be nth hyponormal if T^n is hyponormal.*

As Theorem 2.3, we can verify the following result.

Theorem 4.2. *Let $n \in \mathbb{N}$, $T \in B(\mathcal{H})$ be nth hyponormal and M be an invariant closed subspace for T. Then $T_{|M}$ is nth hyponormal.*

For an nth hyponormal operator $T \in B(\mathcal{H})$, we consider the following property:

$$(**) \qquad \sigma(T) \bigcap \Big(\bigcup_{j=1}^{n-1} e^{\frac{2j\pi}{n} i} \sigma(T) \Big) \subset \{0\}.$$

Theorem 4.3. *Let $n \in \mathbb{N}$, $T \in B(\mathcal{H})$ be nth hyponormal with $(**)$ and M be an invariant subspace for T. If $\sigma(T_{|M}) = \{z\}$, then the following assertions hold.*
(1) If $z = 0$, then $(T_{|M})^n = 0$.
(2) If $z \neq 0$, then $T_{|M} = z$.

Proof. (1) By Theorem 4.2, $T_{|M}$ is nth hyponormal. Since $\sigma((T_{|M})^n) = \{0\}$, by Putnam's theorem, we conclude that $(T_{|M})^n = 0$.
(2) Because $\sigma((T_{|M})^n) = \{z^n\}$, then $(T_{|M})^n = z^n$ and so

$$0 = (T_{|M})^n - z^n = (T_{|M} - e^{\frac{2\pi}{n} i} z)(T_{|M} - e^{\frac{4\pi}{n} i} z) \cdots (T_{|M} - e^{\frac{2(n-1)\pi}{n} i} z)(T_{|M} - z).$$

From $z \neq 0$ and $(**)$, there exists $(T_{|M} - e^{\frac{2j\pi}{n} i} z)^{-1}$, for every $j = 1, ..., n-1$, and thus $T_{|M} - z = 0$. □

Theorem 4.4. *Let $n \in \mathbb{N}$ and $T \in B(\mathcal{H})$ be an nth hyponormal operator. If T satisfies $(**)$, then $\sigma(T) = \{\overline{z} : z \in \sigma_a(T^*)\}$.*

Proof. Because $\sigma(T) = \sigma_a(T) \cup \sigma_r(T)$, we verify that $\sigma_a(T) \subset \{\overline{z} : z \in \sigma_a(T^*)\}$.
(1) If $0 \in \sigma_a(T)$, then $0 \in \sigma_a(T^n)$ and, because T^n is hyponormal, we can get $0 \in \sigma_a(T^*)$.
(2) For $z \in \sigma_a(T)$ and $z \neq 0$, there exists a sequence $\{x_m\}$ of unit vectors such that $(T - z)x_m \to 0$ as $m \to \infty$. We observe that $(T^n - z^n)x_m = (T^{n-1} + T^{n-2}z + \cdots + z^{n-1})(T - z)x_m \to 0$ as $m \to \infty$ and T^n is hyponormal, which gives $(T^n - z^n)^* x_m \to 0$ as $m \to \infty$. By the hypothesis $(**)$ and z is non-zero, all operators $(T^* - e^{\frac{2\pi}{n} i}\overline{z}), (T^* - e^{\frac{4\pi}{n} i}\overline{z}), ..., (T^* - e^{\frac{2(n-1)\pi}{n} i}\overline{z})$ are invertible. Hence, by $T^{*n} - \overline{z}^n = (T^* - e^{\frac{2\pi}{n} i}\overline{z})(T^* - e^{\frac{4\pi}{n} i}\overline{z}) \cdots (T^* - e^{\frac{2(n-1)\pi}{n} i}\overline{z})(T^* - \overline{z})$, we have that $(T^* - \overline{z})x_m \to 0$ as $m \to \infty$, that is, $\overline{z} \in \sigma_a(T^*)$, which completes the proof. □

Theorem 4.5. *Let $n \in \mathbb{N}$ and $T \in B(\mathcal{H})$ be nth hyponormal satisfying $(**)$.*
(1) If z and w are distinct eigen-values of T and $x, y \in \mathcal{H}$ are corresponding eigen-vectors, respectively, then $\langle x, y \rangle = 0$.
(2) If z, w are distinct values of $\sigma_a(T)$ and $\{x_m\}$, $\{y_m\}$ are the sequences of unit vectors in $\mathcal{H}$ such that $(T - z)x_m \to 0$ and $(T - w)y_m \to 0$ $(m \to \infty)$, then $\lim_{m \to \infty} \langle x_m, y_m \rangle = 0$.

Proof. Since (1) follows from (2), we will only prove (2). From $(T - z)x_m \to 0$ and $(T - w)y_m \to 0$ $(m \to \infty)$, we get $(T^n - z^n)x_m \to 0$ and $(T^n - w^n)y_m \to 0$. Further, because T^n is hyponormal, $(T^{*n} - \overline{w}^n)y_m \to 0$. Therefore,

$$\lim_{m\to\infty} z^n \langle x_m, y_m \rangle = \lim_{m\to\infty} \langle z^n x_m, y_m \rangle = \lim_{m\to\infty} \langle T^n x_m, y_m \rangle = \lim_{m\to\infty} \langle x_m, T^{*n} y_m \rangle = \lim_{n\to\infty} w^n \langle x_m, y_m \rangle.$$

In the case that $z^n = w^n$, by $0 = z^n - w^n = (z - w)(z - e^{\frac{2\pi}{n} i} w)(z - e^{\frac{4\pi}{n} i} w) \cdots (z - e^{\frac{2(n-1)\pi}{n} i} w)$, $z \neq w$ and $(**)$, we deduce that $z = w = 0$. So, $z^n \neq w^n$, and $\lim_{m \to \infty} \langle x_m, y_m \rangle = 0$. □

Corollary 4.6. *Let $n \in \mathbb{N}$ and $T \in B(\mathcal{H})$ be nth hyponormal satisfing* $(**)$. *If z and w are distinct eigen-values of T, then* $\ker(T - z) \perp \ker(T - w)$.

Corollary 4.7. *Let $n \in \mathbb{N}$ and $T \in B(\mathcal{H})$ be nth hyponormal satisfing* $(**)$. *Then T has* SVEP.

In a similar manner as Theorem 3.10, we prove the next result.

Theorem 4.8. *Let $n \in \mathbb{N}$ and $T \in B(\mathcal{H})$ be nth hyponormal satisfing* $(**)$. *If z is a non-zero eigen-value of T, then* $\ker(T - z) = \ker(T^n - z^n) \subset \ker(T^{*n} - \overline{z}^n) = \ker(T^* - \overline{z})$ *and hence* $\ker(T - z)$ *is a reducing subspace for T.*

As Theorem 3.13 and Theorem 3.15, we can verify the following theorems.

Theorem 4.9. *Let $n \in \mathbb{N}$ and $T \in B(\mathcal{H})$ be nth hyponormal satisfing* $(**)$. *Let λ be an isolated point of spectrum of T. Then the following statements hold.*
(i) *If $\lambda = 0$, then $\mathcal{H}_0(T) = \ker(T^n) = \ker(T^{*n})$, $E_T(\{0\})$ is self-adjoint and the order of pole λ is not greater than n.*
(ii) *If $\lambda \neq 0$, then $\mathcal{H}_0(T - \lambda) = \ker(T - \lambda) = \ker((T - \lambda)^*)$, $E_T(\{\lambda\})$ is self-adjoint and the order of pole λ is* 1.

Theorem 4.10. *Let D be an arbitrary bounded disk in $\mathbb{C}$, $n \in \mathbb{N}$ and $T \in B(\mathcal{H})$ be nth hyponormal satisfing* $(**)$. *Then the operator*

$$z - T \,:\, W^2(D, \mathcal{H}) \longrightarrow L^2(D, \mathcal{H})$$

is one to one for every $z \in \mathbb{C}$.

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