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Hermite-Fejér and Grünwald Interpolation at Generalized Laguerre Zeros

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Abstract. We introduce special Hermite-Fejér and Grünwald operators at the zeros of the generalized Laguerre polynomials. We will prove that these interpolation processes are uniformly convergent in suitable weighted function spaces.

1. Introduction

The Hermite-Fejér and Grünwald operators based at Jacobi zeros have been extensively studied. We recall among the others [1, 2, 4, 8, 16, 18]. By contrast, in particular the Grünwald operator based at the zeros of orthonormal polynomials w.r.t. exponential weights has received few attention in literature [3, 13, 17].

In this paper we introduce a special Grünwald operator and a related Hermite-Fejér operator based at the zeros of orthonormal polynomials w.r.t. a weight of the following kind

$$w(x) = x^\alpha e^{-Q(x)}, \quad x > 0, \ \alpha > -1,$$

where Q satisfies suitable conditions. As a main result we will prove the convergence of the above interpolation processes in suitable function spaces equipped with weighted uniform norm. We will also give some error estimate.

The paper is organized as follows. In Section 2 some notations and basic results are collected and the Hermite-Fejér and Grünwald operators are introduced. In Section 3 we first define the function spaces where the operators are studied and then state our main results. Section 4 contains the proofs of the main results.

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2. Preliminaries and basic facts

We will say that w is a generalized Laguerre weight if it can be written as follows

$$w(x) = x^{\alpha} e^{-Q(x)}, \quad x > 0, \ \alpha > -1,$$

where, letting $Q^*(x) := Q(x^2)$, $Q^* : \mathbb{R} \to \mathbb{R}$ is even and continuous, $Q^{*''}(x)$ is continuous in $(0, +\infty)$, $Q^{*'}(x) > 0$ in $(0, +\infty)$, and, for some $A, B > 1$, we have

$$A \le \frac{(xQ^{*'}(x))'}{Q^{*'}(x)} \le B, \quad x \ge 1.$$

By the latter relation it follows that the function $Q'(x)$ has an algebraic increasing behaviour [7, Lemma 4.1 (a)].

If $\{p_m(w)\}_m$ is the sequence of the orthonormal polynomials w.r.t. w having positive leading coefficients, then the zeros $x_{m,k}, k = 1, \ldots, m$, of $p_m(w)$ satisfy the following bounds

$$0 < C\frac{a_m}{m^2} < x_{m,1} < \ldots < x_{m,m} < a_m\left(1 - \frac{C}{m^{2/3}}\right),$$

where here and in the sequel C is a positive constant which may assume different values in different formulas.

Concerning the so-called Mhaskar-Rahmanov-Saff number (M-R-S number) $a_m = a_m(\sqrt{w})$, we note that if b_m is the Freud number of the weight $|x|^{\alpha+1} e^{-\frac{Q(x^2)}{2}}$, then $a_m = b_m^2$. Moreover, in the sequel we will use the relation

$$2\, a_m Q'(a_m) = m, \tag{1}$$

which follows from the analogous one for the Freud weights. In fact, denoting by $\bar{a}_m := \bar{a}_m(\bar{w})$ the M-R-S number related to the generalized Freud weight $\bar{w}(x) = |x|^{2\alpha+1} e^{-Q^*(x)}$, we have

$$\bar{a}_m Q^{*'}(\bar{a}_m) = m,$$

see [6, p. 184]. Since $Q^*(x) = Q(x^2)$, we have $Q^{*'}(x) = 2xQ'(x^2)$ and then the above equality becomes

$$2\bar{a}_m^2 Q'(\bar{a}_m^2) = m.$$

On the other hand, taking into account that $a_m \sim \bar{a}_m^2$, we get (1).

Now we introduce our Hermite-Fejér and Grünwald interpolation processes. Consider the points

$$x_1, x_2, \ldots, x_m, x_{m+1},$$

with $x_k := x_{m,k}$ and $x_{m+1} = a_m$. Letting

$$\ell_k(x) = \frac{P(x)}{P'(x_k)(x - x_k)}, \quad P(x) = p_m(w, x)(a_m - x), \quad k = 1, \ldots, m + 1,$$

and

$$v_k(x) = 1 - 2\, \ell_k'(x_k)(x - x_k), \quad k = 1, \ldots, m + 1, \tag{2}$$

for every continuous function f on $\mathbb{R}^+$ ($f \in C^0(\mathbb{R}^+)$), we define the Hermite-Fejér and Grünwald operators as follows

$$F_m(w, f, x) = \sum_{x_1 \le x_k \le \theta a_m} \ell_k^2(x) v_k(x) f(x_k) \tag{3}$$

and

$$G_m(w, f, x) = \sum_{x_1 \le x_k \le \theta a_m} \ell_k^2(x) f(x_k), \tag{4}$$

respectively, for a fixed parameter $0 < \theta < 1$.

In the sequel we will give a theoretical justification for the previous definitions.

3. Function spaces and main results

With $u(x) = x^\gamma e^{-Q(x)}$, $\gamma \geq 0$, we introduce the function space

$$C_u = \left\{ f \in C^0(\mathbb{R}^+) \ : \ \lim_{\substack{x\to+\infty \\ x\to 0}} f(x)u(x) = 0 \right\}, \tag{5}$$

endowed with the norm

$$\|f\|_{C_u} = \sup_{x\geq 0} |f(x)|u(x) = \|fu\|. \tag{6}$$

We will write $\|f\|_A := \sup_{x\in A} |f(x)|$, $A \subset \mathbb{R}^+$. An important property of the weight function u is the following: for every polynomial P_m of degree at most m ($P_m \in \mathbb{P}_m$) the inequalities [9–12]:

$$\|P_m u\| \leq C\|P_m u\|_{\mathcal{I}_m}, \quad \mathcal{I}_m = \left[\frac{a_m}{m^2}, a_m\right] \tag{7}$$

and

$$\|P_m u\|_{\{x\,:\,x>(1+\delta)a_m\}} \leq C e^{-\mathcal{A}m} \|P_m u\| \tag{8}$$

hold true, where $\delta > 0$ and $C, \mathcal{A}$ are independent of m and P_m.

Concerning a_m, here $a_m = a_m(u)$ is the square of the Freud number of the weight $u^*(x) = |x|^{2\gamma+1}e^{-Q(x^2)}$. Since $a_m(w) \sim a_m(\sqrt{w}) \sim a_m(u)$, with a slight abuse of notation, we used and we will use in the sequel, the symbol a_m.

Now we are able to motivate the definitions (3) and (4) of the operators F_m and G_m, respectively.

As a consequence of (8), for all $f \in C_u$ and $0 < \theta < 1$ fixed, the weight function u satisfies the following property:

$$\|fu\| \leq C\left[\|fu\|_{[0,\theta a_m]} + E_M(f)_u\right], \tag{9}$$

where $E_M(f)_u$ is the error of best approximation of f in C_u by means of polynomials of degree at most $M = \lfloor \frac{\theta m}{1+\theta} \rfloor \sim m$ and C is independent of f. Therefore $\|fu\|_{[0,\theta a_m]}$ is the dominant part of $\|fu\|$. This fact suggests that one can approximate only the finite section χf of f, being χ the characteristic function of the interval $[0, \theta a_m]$, $0 < \theta < 1$.

Now we introduce another weight function. The weight $\bar{u}(x) = u(x)\log^\lambda(2 + x)$, $\lambda \geq 1$. We define the function space $C_{\bar{u}}$ as C_u. Obviously $C_{\bar{u}} \subset C_u$.

With τ^* defined by $\tau Q(\tau^{*2}) = 1$, let $t^* = \tau^{*2}$ (For example, if $u^*(x) = e^{-x^\alpha}, x \in \mathbb{R}^+$, then $\tau^* = \frac{1}{\tau^{1/(2\alpha)}}$ and $t^* = \frac{1}{\tau^{1/\alpha}}$). With this notation, we define a suitable modulus of smoothness as follows

$$\omega_\varphi(f,t)_{\bar{u}} = \Omega_\varphi(f,t)_{\bar{u}} + \inf_{P\in\mathbb{P}_0} \|(f-P)\bar{u}\|_{[0,At^2]} + \inf_{P\in\mathbb{P}_0} \|(f-P)\bar{u}\|_{[At^*,+\infty)},$$

being $\varphi(x) = \sqrt{x}$ and

$$\Omega_\varphi(f,t)_{\bar{u}} = \sup_{0<h\leq t} \|(\overrightarrow{\Delta}_h f)\bar{u}\|_{[Ah^2, Ah^*]},$$

with

$$\overrightarrow{\Delta}_h f(x) = f\left(x + \frac{h}{2}\varphi(x)\right) - f(x).$$

The following inequality

$$E_m(f)_{\bar{u}} = \inf_{P\in\mathbb{P}_m} \|(f-P)\bar{u}\| \leq C\omega_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{\bar{u}} \tag{10}$$

holds true and

$$\lim_{m\to\infty} \omega_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{\bar{u}} = 0.$$

The above relations are not available in the literature but they can be easily deduced following [5, 6, 14]. We omit the proof.

Next lemma shows that $F_m(w) : C_{\bar{u}} \to C_u$ is a bounded map.

Lemma 3.1. *Assume that the parameters α, γ and λ of the weights w and $\bar{u}$ satisfy the condition*

$$\alpha > -1, \quad \gamma \geq 0, \quad 0 \leq \gamma - \alpha - \frac{1}{2} < 1, \quad \lambda = 1, \tag{11}$$

then, for any $f \in C_{\bar{u}}$,

$$\|F_m(w, f)u\| \leq C\|f\bar{u}\|_{[0,\theta a_m]}, \tag{12}$$

where C is independent of m and f.

Now, the following theorem states the convergence of the operator $F_m(w, f)$ in $C_{\bar{u}}$.

Theorem 3.2. *Under the assumptions of Lemma 3.1 on the parameters α, γ and λ, for any $f \in C_{\bar{u}}$ we get*

$$\|(f - F_m(w, f))u\| \leq C\left[\omega_\varphi\left(f, \frac{\sqrt{a_m}}{m}\log m\right)_{\bar{u}} + e^{-\mathcal{A}m}\|f\bar{u}\|\right],$$

where C and $\mathcal{A}$ are independent of m and f.

As a consequence of Theorem 3.2, we are able to prove the following theorem dealing with the convergence of the Grünwald polynomial.

Theorem 3.3. *Assume that the parameters α, γ and λ of the weights w and $\bar{u}$ satisfy the conditions*

$$\alpha > -1, \quad \gamma \geq 0, \quad 0 \leq \gamma - \alpha - \frac{1}{2} < 1, \quad \lambda > 1.$$

Then, for any $f \in C_{\bar{u}}$ we have

$$\lim_m \|(f - G_m(w, f))u\| = 0.$$

4. Proofs

The following inequalities, useful in the sequel, can be easily deduced following from [5, 6, 14]:

$$\left| p_m^2(w, x)w(x)\varphi(x)\sqrt{a_m - x + \frac{a_m}{m^{2/3}}} \right| \leq C, \quad x \in \mathcal{I}_m, \tag{13}$$

$$\frac{1}{|p_m'^2(w, x_k)|w(x_k)} \sim \Delta^2 x_k \varphi(x_k)\sqrt{a_m - x_k}, \quad x_1 \leq x_k \leq \theta a_m, \tag{14}$$

and

$$\Delta x_k \sim \frac{\sqrt{a_m}}{m}\sqrt{x_k} \sim \frac{a_m}{m}, \quad x_1 \leq x_k \leq \theta a_m. \tag{15}$$

Proof. [Proof of Lemma 3.1] We first note that, for $x_1 \le x_k \le \theta a_m$,

$$\ell_k(x) = \frac{P(x)}{P'(x_k)(x-x_k)} = l_k(x)\frac{a_m - x}{a_m - x_k}, \tag{16}$$

where

$$l_k(x) = \frac{p_m(w,x)}{p'_m(w,x_k)(x-x_k)}.$$

Using (13) and (14), for $\frac{a_m}{m^2} \le x \le a_m$ and $k = 1, \dots, j$, we get

$$\frac{\ell_k^2(x)u(x)}{\bar{u}(x_k)} \le C\left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac{1}{2}} \frac{\Delta^2 x_k}{\log(2+x_k)(x-x_k)^2}. \tag{17}$$

Moreover, since by definition (16), we have

$$\ell'_k(x_k) = -\frac{1}{a_m - x_k} + \frac{p''_m(w,x_k)}{p'_m(w,x_k)},$$

by (2) we get

$$v_k(x) = 1 + 2\left[\frac{1}{a_m - x_k} - \frac{p''_m(w,x_k)}{p'_m(w,x_k)}\right](x - x_k) =: 1 - \bar{v}_k(x). \tag{18}$$

Let us consider the sequence $\{q_m(\bar{w})\}_m$ of orthonormal polynomials w.r.t. the generalized Freud weight $\bar{w}(x) = |x|^{2\alpha+1}e^{-Q(x^2)}$ and let us denote by $y_k, k = 1, \dots, m$, the zeros of $q_m(\bar{w})$. We have $q_{2m}(\bar{w},x) = p_m(w,x^2)$ and then

$$\frac{q''_{2m}(\bar{w},x)}{q'_{2m}(\bar{w},x)} = \frac{1}{x} + 2x\frac{p''_m(w,x^2)}{p'_m(w,x^2)},$$

i.e.,

$$\frac{p''_m(w,x^2)}{p'_m(w,x^2)} = \frac{1}{2x}\frac{q''_{2m}(\bar{w},x)}{q'_{2m}(\bar{w},x)} - \frac{1}{2x^2}. \tag{19}$$

Now, using [9, Theorem 3.6, p. 42], we get

$$\left|\frac{q''_{2m}(\bar{w},y_k)}{q'_{2m}(\bar{w},y_k)}\right| \le C\left[\frac{|y_k|}{a_m^2(\sqrt{\bar{w}})} + |y_k|Q'(y_k^2) + \frac{1}{|y_k|}\right]$$

and, therefore, by (19)

$$\left|\frac{p''_m(w,x_k^2)}{p'_m(w,x_k^2)}\right| \le C\left[\frac{1}{a_m^2(\sqrt{\bar{w}})} + Q'(y_k^2) + \frac{1}{y_k^2}\right].$$

Consequently,

$$\left|\frac{p''_m(w,x_k)}{p'_m(w,x_k)}\right| \le C\left[1 + Q'(x_k) + \frac{1}{x_k}\right] \tag{20}$$

and then

$$|v_k(x)| \le C\left[1 + |x-x_k| + Q'(x_k)|x-x_k| + \frac{|x-x_k|}{x_k}\right]. \tag{21}$$

Moreover, in virtue of (15), (1) and (21), it is easy to verify that

$$\frac{\Delta x_k}{x_k} \le C, \quad \frac{a_m}{m^2} < x \le a_m, \tag{22}$$

$$\frac{Q'(x_k)\Delta x_k}{\log(2+x_k)} \le C\frac{Q'(a_m)\Delta x_k}{\log(2+a_m)} \le C\frac{m}{a_m \log(2+a_m)}\frac{a_m}{m} \le \frac{C}{\log m}, \quad x_1 \le x_k \le \theta a_m, \tag{23}$$

and

$$\frac{\ell_d^2(x)u(x)}{u(x_d)}|v_d(x)| \le C\frac{|v_d(x)|}{\log(x_d+2)} \le C, \tag{24}$$

x_d being a zero closest to x and $\frac{Q'(x)}{\log(2+x)}$ an increasing function.

Now, in order to estimate (12), we first note that by (7), we have

$$\|F_m(w,f)u\| \le C\|F_m(w,f)u\|_{\mathcal{I}_m}.$$

Recalling (3) and taking into account (17) and (24), for $\frac{a_m}{m^2} < x \le a_m$, we get

$$u(x)|F_m(w,f,x)| \le C\,\|f\bar{u}\|_{[0,\theta a_m]}\left[\sum_{\substack{x_1 \le x_k \le \theta a_m \\ k \ne d-1,d,d+1}} \left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac{1}{2}} \frac{\Delta^2 x_k}{(x-x_k)^2}\frac{\bar{v}_k(x)}{\log(2+x_k)} + 1\right]. \tag{25}$$

We estimate the sum in (25) only in the case $x > 2$, being the case $0 < x \le 2$ similar. We write

$$\begin{aligned} u(x)|F_m(w,f,x)| &\le C\,\|f\bar{u}\|_{[0,\theta a_m]}\left[\sum_{x_1 \le x_k \le 1} + \sum_{1 < x_k \le \frac{x}{2}} + \sum_{\frac{x}{2} < x_k < x_{d-2}} + \sum_{x_{d+2} \le x_k < \theta a_m} +1\right] \\ &=: C\,\|f\bar{u}\|_{[0,\theta a_m]}\left[\sigma_1(x)+\sigma_2(x)+\sigma_3(x)+\sigma_4(x)+1\right]. \end{aligned}$$

For $x_1 \le x_k \le 1$, (21) becomes

$$|v_k(x)| \le C\frac{x}{x_k} + CQ'(x_k)x.$$

Then, taking into account that $x - x_k > \frac{x}{2}$ and (22), we deduce

$$\sigma_1(x) \le Cx^{\gamma-\alpha-\frac{3}{2}} \sum_{x_1 \le x_k \le 1} x_k^{\alpha-\gamma+\frac{1}{2}}\Delta x_k \le Cx^{\gamma-\alpha-\frac{3}{2}}\int_0^1 t^{\alpha-\gamma+\frac{1}{2}}dt \le C,$$

being $\gamma-\alpha-\frac{1}{2} < 1$ and $\gamma-\alpha-\frac{3}{2} \le 0$. We note that, for $1 < x_k \le \theta a_m$, (21) becomes

$$|\bar{v}_k(x)| \le 1 + CQ'(x_k)|x-x_k|, \tag{26}$$

and, then,

$$\sigma_2(x) \le \sum_{1 < x_k \le \frac{x}{2}} \left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac{1}{2}} \frac{\Delta^2 x_k}{(x-x_k)^2}\left[1 + C\frac{Q'(x_k)}{\log(2+x_k)}|x-x_k|\right].$$

Thus, using $x - x_k > \frac{x}{2}$, (22) and (23), we get

$$\begin{aligned} \sigma_2(x) &\le Cx^{\gamma-\alpha-\frac{5}{2}} \sum_{1 < x_k \le \frac{x}{2}} x_k^{\alpha-\gamma-\frac{1}{2}}\Delta x_k + C\sum_{1 < x_k \le \frac{x}{2}} \left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac{1}{2}} \frac{\Delta x_k}{(x-x_k)} \\ &\le C\frac{a_m}{m}x^{\gamma-\alpha-\frac{5}{2}}\int_1^{\frac{x}{2}} t^{\alpha-\gamma-\frac{1}{2}}dt + C\int_1^{\frac{x}{2}}\left(\frac{x}{t}\right)^{\gamma-\alpha-\frac{1}{2}}\frac{dt}{(x-t)} \\ &\le C + C\int_0^{\frac{1}{2}} y^{\alpha-\gamma+\frac{1}{2}}\frac{dy}{(1-y)} \le C, \end{aligned}$$

being $\alpha - \gamma + \frac{1}{2} > -1$. Moreover, taking into account (26), (23) and $x \sim x_k$, we obtain

$$\begin{aligned}\sigma_3(x) &\leq \sum_{\frac{x}{2} < x_k < x_{d-2}} \frac{\Delta^2 x_k}{(x-x_k)^2}\left[1 + C\frac{Q'(x_k)}{\log(2+x_k)}|x-x_k|\right] \\ &\leq C + \frac{C}{\log m}\sum_{\frac{x}{2} < x_k < x_{d-2}} \frac{\Delta x_k}{|x-x_k|} \leq C.\end{aligned}$$

Finally, proceeding as done for the estimate of $\sigma_3(x)$, we obtain

$$\sigma_4(x) \leq C.$$

Summing up, for $x \geq 2$,

$$\|F_m(w,f)u\| \leq C\|f\bar{u}\|_{[0,\theta a_m]}.$$

□

In order prove Theorem 3.2, we introduce the Hermite polynomial based at the zeros $x_k, k = 1, \dots, m+1$, interpolating a function g which is continuous with its first derivative:

$$\begin{aligned}H_m(w,g,x) &= \sum_{k=1}^{m+1} \ell_k^2(x)v_k(x)g(x_k) + \sum_{k=1}^{m+1} \ell_k^2(x)(x-x_k)g'(x_k) \\ &=: F_m^*(w,g,x) + T_m^*(w,g,x).\end{aligned}$$

Note that $F_m(w,g,x) = F_m^*(w,\chi g,x)$. Letting $T_m(w,g,x) = T_m^*(w,\chi g,x)$, the proposition that follows will be useful to our aims.

Proposition 4.1. *Assuming that the parameters α and γ satisfy* (11)*, then, for every g s.t. $\|g'\varphi u\| < +\infty$, we have*

$$\|T_m(w,g)u\| \leq C\frac{\sqrt{a_m}}{m}\log m\|g'\varphi u\|_{[0,\theta a_m]}, \tag{27}$$

where C is independent of m and f. Moreover, for every polynomial $P_M \in \mathbb{P}_M$, with $M = \left\lfloor \frac{\theta m}{1+\theta} \right\rfloor$, $0 < \theta < 1$, we get

$$\|H_m(w,(1-\chi)P_M)u\| \leq Ce^{-\mathcal{A}m}\|P_M u\|, \tag{28}$$

where C and $\mathcal{A}$ are independent of m and Q_M.

Proof. In order to prove the inequality (27) we recall (17). Then, using (15), we get

$$|T_m(w,g,x)|u(x) \leq C\frac{\sqrt{a_m}}{m}\|g'\varphi u\|_{[0,\theta a_m]} \sum_{x_1 \leq x_k \leq \theta a_m} \left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac{1}{2}} \frac{\Delta x_k}{|x-x_k|}.$$

Now, by similar arguments to those used for the proof of Lemma 3.1, it is possible to prove that

$$\sum_{k=1}^{j} \left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac{1}{2}} \frac{\Delta x_k}{|x-x_k|} \leq C\log m.$$

Then, (27) easily follows.

In order to prove (28) we need to estimate $\|F_m^*(w,(1-\chi)P_M)u\|$ and $\|T_m^*(w,(1-\chi)P_M)u\|$. We give the details only for the bound of the latter norm, since the estimate of the former is similar.

Using (17) and (15), we get

$$\begin{aligned}|T_m^*(w,(1-\chi)P_M,x)|u(x) &\leq C\frac{\sqrt{a_m}}{m}\|P_M'\varphi u\|_{[\theta a_m,+\infty)}\sum_{\theta a_m<x_k\leq x_m}\left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac{1}{2}}\frac{\Delta x_k}{\log(2+x_k)|x-x_k|}\\ &\leq C\frac{\sqrt{a_m}}{m}m^\tau\|P_M'\varphi u\|_{[\theta a_m,+\infty)},\end{aligned}$$

for some $\tau>0$. Finally, by (8) and the Bernstein inequality [11], we obtain

$$\begin{aligned}\frac{\sqrt{a_m}}{m}m^\tau\|P_M'\varphi u\|_{[\theta a_m,+\infty)} &\leq C\frac{\sqrt{a_m}}{m}m^\tau e^{-\mathcal{A}m}\|P_M'\varphi u\|\leq Cm^\tau e^{-\mathcal{A}m}\|P_M u\|\\ &\leq Ce^{-\mathcal{A}m}\|P_M u\|\end{aligned}$$

and, then

$$\|T_m^*(w,(1-\chi)P_M)u\|\leq Ce^{-\mathcal{A}m}\|P_M u\|$$

easily follows. □

Now we can prove Theorem 3.2.

Proof. [Proof of Theorem 3.2] Denoting by $P_N\in\mathbb{P}_N$, $N=\left\lfloor\frac{M}{\log M}\right\rfloor$, $M=\left\lfloor\frac{\theta m}{1+\theta}\right\rfloor$, the polynomial of best approximation of $f\in C_{\bar{u}}$, we can write

$$\begin{aligned}f-F_m(w,f) &= f-P_N+H_m(w,P_N)-F_m(w,f)\\ &= f-P_N+F_m(w,P_N-f)+T_m(w,P_N)+H_m(w,(1-\chi)P_N),\end{aligned}$$

using Lemma 3.1 and Proposition 4.1, we get

$$\|(f-F_m(w,f))u\|\leq C\left[\|(f-P_N)\bar{u}\|+\frac{\sqrt{a_N}}{N}\|P_N'\varphi\bar{u}\|+e^{-\mathcal{A}m}\|P_N\bar{u}\|\right].$$

Recalling (10) we have

$$\|(f-P_N)\bar{u}\|\leq C\omega_\varphi\left(f,\frac{\sqrt{a_N}}{N}\right)_{\bar{u}}\sim\omega_\varphi\left(f,\frac{\sqrt{a_m}}{m}\log m\right)_{\bar{u}}.$$

Moreover, since (see [15] for a similar argument)

$$\frac{\sqrt{a_N}}{N}\|P_N'\varphi\bar{u}\| \leq C\omega_\varphi\left(f,\frac{\sqrt{a_N}}{N}\right)_{\bar{u}}\sim\omega_\varphi\left(f,\frac{\sqrt{a_m}}{m}\log m\right)_{\bar{u}}$$

and

$$\|P_N\bar{u}\|\leq 2\|f\bar{u}\|,$$

the theorem follows. □

Proof. [Proof of Theorem 3.3] By (3)-(4) and (18) we have

$$f-G_m(w,f)=[f-F_m(w,f)]+\bar{F}_m(w,f),$$

where

$$\bar{F}_m(w,f)=\sum_{x_1\leq x_k\leq\theta a_m}\ell_k^2(x)\bar{v}_k(x)f(x_k).$$

Using (18) and (20), we deduce that $|\bar{v}_k(x)|$ satisfies the same bound of $|v_k(x)|$ (see (21)), i.e.

$$|\bar{v}_k(x)| \leq C\left[1+|x-x_k|+Q'(x_k)|x-x_k|+\frac{|x-x_k|}{x_k}\right]. \tag{29}$$

Then, following step by step the proof of (12) with $\lambda > 1$, we deduce that

$$\|\bar{F}_m(w,f)u\| \leq \frac{C}{\log^{\lambda-1} m}\|f\bar{u}\|.$$

Using the above bound and taking into account Theorem 3.2, we get

$$\|(f-G_m(w,f))u\| \leq C\left[\omega_\varphi\left(f,\frac{\sqrt{a_m}}{m}\log m\right)_{\bar{u}} + e^{-\mathcal{A}m}\|f\bar{u}\| + \frac{\|f\bar{u}\|}{\log^{\lambda-1} m}\right].$$

The proof is then complete. □

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