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A Generalized Two-Step Modulus-Based Matrix Splitting Iteration Method for Implicit Complementarity Problems of H_+-Matrices

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Abstract. In this paper, a generalized two-step modulus-based matrix splitting iteration method for solving the implicit complementarity problems has been presented. The convergent analysis with the system matrix being H_+-matrices are also discussed. Numerical experiments illustrate that our method is advantageous to the existing methods.

1. Introduction

The complementarity problem is a hot topic, which has several kinds of forms, such as linear complementarity problem [1, 24–41], nonlinear complementarity problem [2–5, 12], implicit complementarity problem [13–15], cone complementarity problem [16–18], etc.. The implicit complementarity problem is a special case in complementarity theory. It is frequently applied to stochastic optimal control problems[22, 23]. Therefore, it is significant to study this problem.

The implicit complementarity problem (ICP) is to find a pair (u,w) of real vectors which satisfy

$$u - m(u) \geq 0,\ w := Au + q \geq 0,\ (u - m(u))^T w = 0, \tag{1}$$

where A and q are a known matrix in $\mathbb{R}^{n\times n}$ and a known vector in $\mathbb{R}^n$, respectively. The mapping $m(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is known invertible, and $u - m(u)$ is supposed to be a invertible mapping.

When $m(u) = 0$, the implicit complementarity problem reduces to the linear complementarity problem [24, 25], abbreviated as LCP(A, q). There are numerous existing methods to solve the LCP; see, for example, the projected iteration methods [6], the chaotic iteration methods [7], and the modulus-based multigrid methods [8]. In 2010, Bai first provided the modulus-based matrix splitting (MMS) iteration method [26] to solve this kind of problems. This method has attracted much attention, and most scholars modified the MMS iteration method by constructing different preconditioners [27–30] or generalized it by putting another parameterized diagonal matrix to the modulus equation of LCP [31–33] and so on [34–36]. And some researchers also give the synchronous multisplitting iteration methods [19, 20, 37–43]. For other iteration methods such as the matrix multisplitting iteration methods, parallel chaotic multisplitting

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iteration methods, and the damped Newton methods for solving the LCP, we refer to [1, 9–11, 13, 21, 25, 44] and the references therein.

When $m(u)$ is a general function, it is called as the implicit complementarity problem. In general, many studied ICP by variational or quasi-variational inequalities [45, 46]. Recently, the modulus-based matrix splitting has been widely used to derive numerical methods for the linear complementarity problems. For instance, Hong et al. raised the modulus-based matrix splitting iteration method [47] and Wang et al. consummated its details [48]. In order to improve the convergence rate, Cao et al. proposed the two-step modulus-based matrix splitting iteration method [49].

In this paper, in view of putting another parameterized diagonal matrix to the modulus equation and splitting the system matrix into two kinds of matrix splittings, we propose a general two-step modulus-based modulus matrix splitting iteration method, which is the generalization of the two-step modulus matrix splitting iteration method [49], to solve the ICP and its variants with different matrix splittings. We discuss its convergent theorems whose system matrix is an H_+-matrix. Finally, numerical experiments are exhibited to indicate the efficiency of our presented method.

The paper is organized as follows. In Sec. 2, we give some symbolic representations and essential lemmas. The generalized two-step modulus-based matrix splitting method, which is the generalization of the two-step modulus-based matrix splitting method and its convergence theorems are given in Sec. 3 and Sec. 4, respectively. Numerical experiments illustrate that our proposed method is advantageous to some existing methods.

2. Preliminaries

In this section, we review some fundamental notations and indispensable lemmas.

For any two vectors $u = (u_1, u_2, \cdots, u_n)^T$ and $v = (v_1, v_2, \cdots, v_n)^T$, we denote $u \geq v$ ($u > v$) provided that the corresponding elements satisfy $u_i \geq v_i$ ($u_i > v_i$). $\max(a, b)$ is the bigger one of a and b. $|u|$ means the absolute value of the vector u, and u^T is its transpose. The notations of matrix is the same as the aforementioned. Denote $A = (a_{ij})$, then we review some special matrices as below:

- The matrix A is a Z-matrix iff $a_{ij} \leq 0$ for any $i \neq j$.
- The Z-matrix A is an M-matrix iff $A^{-1} \geq 0$.
- The matrix A is an H-matrix iff its comparison matrix $\langle A \rangle = \langle a_{ij} \rangle$ is an M-matrix, where

$$\langle a_{ij} \rangle = \begin{cases} |a_{ij}|, \ for\ i = j, \\ -|a_{ij}|, \ for\ i \neq j, \end{cases} \quad i, j = 1, 2, \cdots, n.$$

- The matrix A is an H_+-matrix iff A is an H-matrix with the diagonal elements being positive; see [1].

Furthermore, $A = E - F$ is a splitting of the matrix A iff E is nonsingular, and it is an H-compatible splitting iff it holds that $\langle A \rangle = \langle E \rangle - |F|$. In the following sections, α and β are parameters. D, $-L$ and U are the diagonal, the strictly lower-triangular and the strictly upper-triangular matrices of the matrix A, respectively. Finally, we list six lemmas which are going to be used in subsequent convergent theorems.

Lemma 2.1. *[50] Let A be an H-matrix, then* $|A^{-1}| \leq \langle A \rangle^{-1}$.

Lemma 2.2. *[51] Let* $B \in \mathbb{R}^{n\times n}$ *be a strictly diagonal dominant matrix. Then*

$$\|B^{-1}C\| \leq \max_{1\leq i\leq n} \frac{(|C|e)_i}{(\langle B \rangle e)_i}$$

holds for arbitrary matrix $C \in \mathbb{R}^{n\times n}$, *where* $e = (1, 1, \cdots, 1)^T$.

Lemma 2.3. *[52] Let* $A \in \mathbb{R}^{n\times n}$, *then* $\rho(A) < 1$ *iff* $\lim_{n\to\infty} A^n = 0$.

Lemma 2.4. *[53] Let* $A \in \mathbb{R}^{n\times n}$ *be an M-matrix and* $B \in \mathbb{R}^{n\times n}$ *be a Z-matrix. If* $A \leq B$, *then B is an M-matrix.*

Lemma 2.5. *[54] Let A be a Z-matrix, then the following statements are equivalent:*
(i) A is an M-matrix;
(ii) There exists a positive vector x, such that $Ax > 0$;
(iii) Let $A = M - N$ be a splitting of A and $M^{-1} \geq 0$, $N \geq 0$, then $\rho(M^{-1}N) < 1$.

Lemma 2.6. *[47] Let $A = E - F$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$, γ be a positive constant, Ω be a positive diagonal matrix and $g(u) = u - m(u)$ be an invertible mapping. For the ICP (1), the following statements hold true:*
(i) if (u, w) is a solution of the ICP(1). then $x = \frac{\gamma}{2}(u - \Omega^{-1}w - m(u))$ satisfies the implicit fixed-point equation

$$(\Omega + E)x = Fx + (\Omega - A)|x| - \gamma Am[g^{-1}(\frac{1}{\gamma}(|x| + x))] - \gamma q. \tag{2}$$

(ii) if x satisfies the implicit fixed-pointed equation (2), then

$$u = \frac{1}{\gamma}(|x| + x) + m(u) \quad \text{and} \quad w = \frac{1}{\gamma}\Omega(|x| - x) \tag{3}$$

is a solution of the ICP(1).

3. The general two-step modulus-based matrix splitting

Based on (2) and (3), Hong and Li provided the MMS iteration method[47]. However, they did not point out the inner-outer iteration, and some parameters of their method were indefinite. Then, in [48], they got rid of these advantages and presented the more complete version as below.

Method 3.1. [48] (The MMS iteration method for the ICP(1))

Step 1. Given $\varepsilon > 0$, $u_0 \in Z$, set $k := 0$;

Step 2. Find the solution $u^{(k+1)}$:
(1) Compute the initial vector

$$w^{(k)} = Au^{(k)} + q,$$
$$x^{(k,0)} = \frac{\gamma}{2}(u^{(k)} - m(u^{(k)}) - \Omega^{-1}w^{(k)}).$$

Set $j := 0$.
(2) Iteratively calculate $x^{(k,j+1)} \in \mathbb{R}^n$ by solving the equations

$$(\Omega + E)x^{(k,j+\frac{1}{2})} = Fx^{(k,j)} + (\Omega - A)|x^{(k,j)}| + \gamma Am(u^{(k)}) - \gamma q. \tag{4}$$

(3) Compute

$$u^{(k+1)} = \frac{1}{\gamma}(|x^{(k,j+1)}| + x^{(k,j+1)}) + m(u^{(k)}). \tag{5}$$

Step 3. If $RES = |(Au^{(k+1)} + q)^T(u^{(k+1)} - m(u^{(k+1)}))| < \varepsilon$, then terminate. Otherwise, set $k := k + 1$ and return to Step 2.

Remark 3.2. *We can utilize some special iterative schemes by choosing diverse matrix splittings to this method. For instance, set*

$$E = \frac{1}{\alpha}(D - \beta L),\ F = \frac{1}{\alpha}((1 - \alpha)D + (\alpha - \beta)L + \alpha U).$$

Based on the aforementioned matrix splitting, we can obtain the modulus-based accelerated over-relaxation (MAOR) iteration method. With different values of the parameters, we can get the other methods such as the modulus-based successive over-relaxation (MSOR) iteration method ($\alpha = \beta$), the modulus-based Gauss-Seidel (MGS) iteration method ($\alpha = \beta = 1$) and the modulus-based Jacobi (MJ) iteration method ($\alpha = 1, \beta = 0$) [48].

In [49], Cao and Wang raised the TMMS iteration method. Define a set:

$$Z = \{z|Az + q \geq 0,\ z - m(z) \geq 0\}.$$

They need two splittings of the matrix A, i.e., $A = E_1 - F_1 = E_2 - F_2$. We show this method as the following is more standard:

Method 3.3. [49] (The TMMS iteration method for the ICP(1))

Step 1. Given $\varepsilon > 0$, $u_0 \in Z$, set $k := 0$;

Step 2. Find the solution $u^{(k+1)}$:
(1) Compute the initial vector

$$w^{(k)} = Au^{(k)} + q,$$
$$x^{(k,0)} = \frac{\gamma}{2}(u^{(k)} - m(u^{(k)}) - \Omega^{-1}w^{(k)}).$$

Set $j := 0$.
(2) Iteratively calculate $x^{(k,j+1)} \in \mathbb{R}^n$ by solving the equations

$$\begin{cases} (\Omega + E_1)x^{(k,j+\frac{1}{2})} = F_1x^{(k,j)} + (\Omega - A)|x^{(k,j)}| + \gamma Am(u^{(k)}) - \gamma q, \\ (\Omega + E_2)x^{(k,j+1)} = F_2x^{(k,j+\frac{1}{2}))} + (\Omega - A)|x^{(k,j+\frac{1}{2}))}| + \gamma Am(u^{(k)}) - \gamma q. \end{cases} \tag{6}$$

(3) Compute

$$u^{(k+1)} = \frac{1}{\gamma}(|x^{(k,j+1)}| + x^{(k,j+1)}) + m(u^{(k)}). \tag{7}$$

Step 3. If $RES = |(Au^{(k+1)} + q)^T(u^{(k+1)} - m(u^{(k+1)}))| < \varepsilon$, then terminate. Otherwise, set $k := k + 1$ and return to Step 2.

Remark 3.4. *We can utilize some special iterative schemes by choosing diverse matrix splittings to this method. For instance, set*

$$\begin{cases} E_1 = \frac{1}{\alpha}(D - \beta L),\ F_1 = \frac{1}{\alpha}((1-\alpha)D + (\alpha - \beta)L + \alpha U), \\ E_2 = \frac{1}{\alpha}(D - \beta U),\ F_2 = \frac{1}{\alpha}((1-\alpha)D + (\alpha - \beta)U + \alpha L). \end{cases}$$

Based on the aforementioned matrix splitting, we can obtain the two-step modulus-based accelerated over-relaxation (TMAOR) iteration method. With different values of the parameters, we can get the other methods such as the two-step modulus-based successive over-relaxation (TMSOR) iteration method ($\alpha = \beta$), the two-step modulus-based Gauss-Seidel (TMGS) iteration method ($\alpha = \beta = 1$) and the two-step modulus-based Jacobi (TMJ) iteration method ($\alpha = 1, \beta = 0$) [49].

In order to improve computing efficiency, on account of the general modulus-based matrix splitting iteration method [31], we get the the general two-step modulus-based matrix splitting (GTMMS) iteration method for solving the ICP(1) as below. There is a slight difference. We need two matrix splittings of the matrix $A\Omega_1$, instead of the matrix A. Hence, let $A\Omega_1 = E'_{\Omega_1} - F'_{\Omega_1} = E''_{\Omega_1} - F''_{\Omega_1}$.

Method 3.5. *(The GTMMS iteration method for the ICP(1))*

Step 1. *Given $\varepsilon > 0$, $u_0 \in Z$, set $k := 0$;*

Step 2. *Find the solution $u^{(k+1)}$:*
(1) Compute the initial vector

$$w^{(k)} = Au^{(k)} + q,$$
$$x^{(k,0)} = \frac{1}{2}(\Omega_1^{-1}u^{(k)} - \Omega_1^{-1}m(u^{(k)}) - \Omega_2^{-1}w^{(k)}).$$

Set $j := 0$.
(2) Iteratively calculate $x^{(k,j+1)} \in \mathbb{R}^n$ by solving the equations

$$\begin{cases} (\Omega_2 + E'_{\Omega_1})x^{(k,j+\frac{1}{2})} = F'_{\Omega_1}x^{(k,j)} + (\Omega_2 - A\Omega_1)|x^{(k,j)}| + Am(u^{(k)}) - q, \\ (\Omega_2 + E''_{\Omega_1})x^{(k,j+1)} = F''_{\Omega_1}x^{(k,j+\frac{1}{2})} + (\Omega_2 - A\Omega_1)|x^{(k,j+\frac{1}{2})}| + Am(u^{(k)}) - q. \end{cases} \tag{8}$$

(3) Compute

$$u^{(k+1)} = \Omega_1(|x^{(k,j+1)}| + x^{(k,j+1)}) + m(u^{(k)}). \tag{9}$$

Step 3. *If $RES = |(Au^{(k+1)} + q)^T(u^{(k+1)} - m(u^{(k+1)}))| < \varepsilon$, then terminate. Otherwise, set $k := k+1$ and return to Step 2.*

Remark 3.6. *Similar to the TMMS iteration method, we can also acquire analogous methods by appropriate choices of the parameter and the matrix splitting. Set*

$$\begin{cases} E'_{\Omega_1} = \frac{1}{\alpha}(\hat{D} - \beta\hat{L}),\ F'_{\Omega_1} = \frac{1}{\alpha}((1-\alpha)\hat{D} + (\alpha-\beta)\hat{L} + \alpha\hat{U}), \\ E''_{\Omega_1} = \frac{1}{\alpha}(\hat{D} - \beta\hat{U}),\ F''_{\Omega_1} = \frac{1}{\alpha}((1-\alpha)\hat{D} + (\alpha-\beta)\hat{U} + \alpha\hat{L}), \end{cases}$$

where $\hat{D}$, $-\hat{L}$ and $\hat{U}$ are the diagonal, the strictly lower-triangular and the strictly upper-triangular matrices of the matrix $A\Omega_1$, respectively. Based on the aforementioned matrix splitting, we can obtain the general two-step modulus-based accelerated over-relaxation (GTMAOR) iteration method. With different values of the parameters, we can get the other methods such as the general two-step modulus-based successive over-relaxation (GTMSOR) iteration method ($\alpha = \beta$), the general two-step modulus-based Gauss-Seidel (GTMGS) iteration method ($\alpha = \beta = 1$) and the general two-step modulus-based Jacobi (GTMJ) iteration method ($\alpha = 1, \beta = 0$).

4. Convergence theorems

In this section, we are going to discuss convergence properties of Method 3.5 when the system matrix A of the ICP(1) is an H_+-matrix.

Theorem 4.1. *Let A be an H_+-matrix in $\mathbb{R}^{n\times n}$, and Ω_1 and Ω_2 be known positive diagonal matrices. $A\Omega_1 = E'_{\Omega_1} - F'_{\Omega_1} = E''_{\Omega_1} - F''_{\Omega_1}$ are H-compatible splittings. $m(\cdot)$ is a Lipschitz continuous function, i.e., it holds that*

$$|m(a) - m(b)| \le l|a-b|,\ \forall a, b \in \mathbb{R}^n,$$

wherein l is the Lipschitz constant. Set $\zeta_1 = (\Omega_2 + \langle E'_{\Omega_1}\rangle)^{-1}(|\Omega_2 - A\Omega_1| + |F'_{\Omega_1}|)$, $\zeta_2 = |(\Omega_2 + E'_{\Omega_1})^{-1}A|$, $\zeta_3 = (\Omega_2 + \langle E''_{\Omega_1}\rangle)^{-1}(|\Omega_2 - A\Omega_1| + |F''_{\Omega_1}|)$ and $\zeta_4 = |(\Omega_2 + E''_{\Omega_1})^{-1}A|$. If $l(\frac{2\|\zeta_2\zeta_3+\zeta_4\|+1}{1-\|\zeta_1\zeta_3\|}) < 1$ and the parameter matrices Ω_1 and Ω_2 satisfy

$$\Omega_2 e > \max(D\Omega_1 e - T^{-1}(\langle E'_{\Omega_1}\rangle - |F'_{\Omega_1}|)Te, D\Omega_1 e - T^{-1}(\langle E''_{\Omega_1}\rangle - |F''_{\Omega_1}|)Te),$$

where D is the diagonal matrix of A and T is a positive diagonal matrix such that $(\langle E'_{\Omega_1}\rangle - |F'_{\Omega_1}|)T$ and $(\langle E''_{\Omega_1}\rangle - |F''_{\Omega_1}|)T$ are s.d.d. matrix. Then for any initial vector $u^{(0)} \in Z$, the iteration sequence $\{u^{(k)}\}_{k=0}^{\infty}$ resulted from Method 3.5 converges to the unique solution u^* of the ICP (1).

Proof. Assume that $u^* \in Z$ is the solution of the ICP(1), then $w^* = Au^* + q$. According to Method 3.5, it holds that

$$x^* = \frac{1}{2}(\Omega_1^{-1}u^* - \Omega_2^{-1}w^* - \Omega_1^{-1}m(u^*)),\ |x^*| = \frac{1}{2}(\Omega_1^{-1}u^* + \Omega_2^{-1}w^* - \Omega_1^{-1}m(u^*))$$

are the solutions of the implicit fixed-point modulus equations

$$\begin{cases} (\Omega_2 + E'_{\Omega_1})x^* = F'_{\Omega_1}x^* + (\Omega_2 - A\Omega_1)|x^*| + Am(u^*) - q, \\ (\Omega_2 + E''_{\Omega_1})x^* = F''_{\Omega_1}x^* + (\Omega_2 - A\Omega_1)|x^*| + Am(u^*) - q. \end{cases} \tag{10}$$

By Lemma 2.6, we have

$$u^* = \Omega_1(|x^*| + x^*) + m(u^*). \tag{11}$$

Subtracting (9) from (11) and taking the absolute values on two sides, it is easy to get

$$\begin{aligned} |u^{(k+1)} - u^*| &= |\Omega_1(|x^{(k,j+1)}| + x^{(k,j+1)}) + m(u^{(k)}) - \Omega_1(|x^*| + x^*) - m(u^*)| \\ &\leq |m(u^{(k)}) - m(u^*)| + \Omega_1(||x^{(k,j+1)}| - |x^*|| + |x^{(k,j+1)} - x^*|) \\ &\leq l|u^{(k)} - u^*| + 2\Omega_1|x^{(k,j+1)} - x^*|. \end{aligned} \tag{12}$$

According to $A\Omega_1 = E'_{\Omega_1} - F'_{\Omega_1} = E''_{\Omega_1} - F''_{\Omega_1}$ being H-compatible splittings, i.e., $\langle A\Omega_1\rangle = \langle E'_{\Omega_1}\rangle - |F'_{\Omega_1}| = \langle E''_{\Omega_1}\rangle - |F''_{\Omega_1}|$, it holds that

$$\langle A\Omega_1\rangle \leq \langle E'_{\Omega_1}\rangle \leq \mathrm{diag}(E'_{\Omega_1}) \text{ and } \langle A\Omega_1\rangle \leq \langle E''_{\Omega_1}\rangle \leq \mathrm{diag}(E''_{\Omega_1}).$$

In the light of Lemma 2.4, it is obvious that E'_{Ω_1} and E''_{Ω_1} are H_+-matrices. Hence, based on Lemma 2.1, we have

$$|(\Omega_2 + E'_{\Omega_1})^{-1}| \leq (\Omega_2 + \langle E'_{\Omega_1}\rangle)^{-1},\ |(\Omega_2 + E''_{\Omega_1})^{-1}| \leq (\Omega_2 + \langle E''_{\Omega_1}\rangle)^{-1}.$$

By subtracting (10) from (8), we can acquire the following equations

$$\begin{cases} x^{(k,j+\frac{1}{2})} - x^* = (\Omega_2 + E'_{\Omega_1})^{-1}[F'_{\Omega_1}(x^{(k,j)} - x^*) + (\Omega_2 - A\Omega_1)(|x^{(k,j)}| - |x^*|) + A(m(u^{(k)}) - m(u^*)], \\ x^{(k,j+1)} - x^* = (\Omega_2 + E''_{\Omega_1})^{-1}[F''_{\Omega_1}(x^{(k,j+\frac{1}{2})} - x^*) + (\Omega_2 - A\Omega_1)(|x^{(k,j+\frac{1}{2})}| - |x^*|) + A(m(u^{(k)}) - m(u^*)]. \end{cases} \tag{13}$$

By taking the absolute values on both sides, we acquire

$$\begin{aligned} |x^{(k,j+\frac{1}{2})} - x^*| &= |(\Omega_2 + E'_{\Omega_1})^{-1}[F'_{\Omega_1}(x^{(k,j)} - x^*)(\Omega_2 - A\Omega_1)(|x^{(k,j)}| - |x^*|) + A(m(u^{(k)}) - m(u^*)]| \\ &\leq (|(\Omega_2 + E'_{\Omega_1})^{-1}(\Omega_2 - A\Omega_1)| + |(\Omega_2 + E'_{\Omega_1})^{-1}F'_{\Omega_1}|)|x^{(k,j)} - x^*| + |(\Omega_2 + E'_{\Omega_1})^{-1}A||m(u^{(k)}) - m(u^*)| \\ &\leq (\Omega_2 + \langle E'_{\Omega_1}\rangle)^{-1}(|\Omega_2 - A\Omega_1| + |F'_{\Omega_1}|)|x^{(k,j)} - x^*| + l|(\Omega_2 + E'_{\Omega_1})^{-1}A||u^{(k)} - u^*| \\ &= \zeta_1|x^{(k,j)} - x^*| + l\zeta_2|u^{(k)} - u^*|, \end{aligned} \tag{14}$$

where $\zeta_1 = (\Omega_2 + \langle E'_{\Omega_1}\rangle)^{-1}(|\Omega_2 - A\Omega_1| + |F'_{\Omega_1}|)$ and $\zeta_2 = |(\Omega_2 + E'_{\Omega_1})^{-1}A|$. Similar to analysis of the first equation, we acquire

$$\begin{aligned} |x^{(k,j+1)} - x^*| &= |(\Omega_2 + E''_{\Omega_1})^{-1}[F''_{\Omega_1}(x^{(k,j+\frac{1}{2})} - x^*) + (\Omega_2 - A\Omega_1)(|x^{(k,j+\frac{1}{2})}| - |x^*|) + A(m(u^{(k)}) - m(u^*)]| \\ &\leq (|(\Omega_2 + E''_{\Omega_1})^{-1}(\Omega_2 - A\Omega_1)| + |(\Omega_2 + E''_{\Omega_1})^{-1}F''_{\Omega_1}|)|x^{(k,j+\frac{1}{2})} - x^*| + |(\Omega_2 + E''_{\Omega_1})^{-1}A||m(u^{(k)}) - m(u^*)| \\ &\leq (\Omega_2 + \langle E''_{\Omega_1}\rangle)^{-1}(|\Omega_2 - A\Omega_1| + |F''_{\Omega_1}|)|x^{(k,j+\frac{1}{2})} - x^*| + l|(\Omega_2 + E''_{\Omega_1})^{-1}A||u^{(k)} - u^*| \\ &= \zeta_3|x^{(k,j+\frac{1}{2})} - x^*| + l\zeta_4|u^{(k)} - u^*|, \end{aligned} \tag{15}$$

where $\zeta_3 = (\Omega_2 + \langle E''_{\Omega_1}\rangle)^{-1}(|\Omega_2 - A\Omega_1| + |F''_{\Omega_1}|)$ and $\zeta_4 = |(\Omega_2 + E''_{\Omega_1})^{-1}A|$

Combining (14) and (15), it is appreciable that

$$|x^{(k,j+1)} - x^*| \leq \zeta_3\zeta_1|x^{(k,j)} - x^*| + l(\zeta_3\zeta_2 + \zeta_4)|u^{(k)} - u^*|.$$

It is obvious that

$$|x^{(k,0)} - x^*| \leq \tfrac{1}{2}(\Omega_1^{-1} + l\Omega_1^{-1} + |\Omega_2^{-1}A|)|u^{(k)} - u^*|.$$

According to (12) and the above discussion, we obtain

$$\begin{aligned}
|u^{(k+1)} - u^*| &\leq l|u^{(k)} - u^*| + 2\Omega_1[\zeta_3\zeta_1|x^{(k,j)} - x^*| + l(\zeta_3\zeta_2 + \zeta_4)|u^{(k)} - u^*|] \\
&= 2\Omega_1\zeta_3\zeta_1|x^{(k,j)} - x^*| + l(2\Omega_1(\zeta_3\zeta_2 + \zeta_4) + I)|u^{(k)} - u^*| \\
&\leq 2\Omega_1(\zeta_3\zeta_1)^{j+1}|x^{(k,0)} - x^*| \\
&\quad + l(2\Omega_1(\zeta_3\zeta_1)^j(\zeta_3\zeta_2 + \zeta_4) + \cdots + 2\Omega_1(\zeta_3\zeta_2 + \zeta_4) + I)|u^{(k)} - u^*| \\
&\leq [\Omega_1(\zeta_3\zeta_1)^{j+1}(\Omega_1^{-1} + l\Omega_1^{-1} + |\Omega_2^{-1}A|) + l\zeta_5]|u^{(k)} - u^*| \\
&= \tilde{\Gamma}|u^{(k)} - u^*|,
\end{aligned}$$

where $\tilde{\Gamma} = \Omega_1(\zeta_3\zeta_1)^{j+1}(\Omega_1^{-1} + l\Omega_1^{-1} + |\Omega_2^{-1}A|) + l\zeta_5$ and $\zeta_5 = 2\Omega_1 \sum_{m=0}^{j} (\zeta_3\zeta_1)^m(\zeta_3\zeta_2 + \zeta_4) + I$.

Then, we are going to research the conditions that ensure the convergence of Method 3.5, i.e.,

$$\rho(\tilde{\Gamma}) < 1. \tag{16}$$

On the basis of the definition of $\tilde{\Gamma}$, it is appreciable that

$$\begin{aligned}
\rho(\tilde{\Gamma}) &= \rho(\Omega_1(\zeta_3\zeta_1)^{j+1}(\Omega_1^{-1} + l\Omega_1^{-1} + |\Omega_2^{-1}A|) + l\zeta_5) \\
&\leq ||\Omega_1(\zeta_3\zeta_1)^{j+1}(\Omega_1^{-1} + l\Omega_1^{-1} + |\Omega_2^{-1}A|) + l\zeta_5|| \\
&\leq ||\Omega_1|| \cdot ||(\zeta_3\zeta_1)^{j+1}|| \cdot ||\Omega_1^{-1} + l\Omega_1^{-1} + |\Omega_2^{-1}A||| + l||\Omega_1||(||2\sum_{m=0}^{j}(\zeta_3\zeta_1)^m(\zeta_3\zeta_2 + \zeta_4)|| + 1).
\end{aligned}$$

Analogous to analysis of Theorem 5 [31], it holds that $\rho(\zeta_3\zeta_1) < 1$ if the parameter matrices Ω_1 and Ω_2 satisfy $\Omega_2 e > \max(D\Omega_1 e - T^{-1}(\langle E'_{\Omega_1} - |F'_{\Omega_1}|)Te, D\Omega_1 e - T^{-1}(\langle E''_{\Omega_1}\rangle - |F''_{\Omega_1}|)Te)$ for any positive diagonal matrix T such that $(\langle E'_{\Omega_1}\rangle - |F'_{\Omega_1}|)T$ and $(\langle E''_{\Omega_1}\rangle - |F''_{\Omega_1}|)T$ are s.d.d. matrix. Based on Lemma 2.3, we have $(\zeta_3\zeta_1)^{j+1} = 0,\ j \to \infty$. $||\Omega_1||$ and $||\Omega_1^{-1} + l\Omega_1^{-1} + |\Omega_2^{-1}A|||$ are constant. Hence, for arbitrary positive number ϵ, there is $N \in \mathbb{N}_+$ such that

$$||\Omega_1||||(\zeta_3\zeta_1)^{j+1}||||\Omega_1^{-1} + l\Omega_2^{-1} + |\Omega_1^{-1}A||| \leq \epsilon,\ \forall j \geq N.$$

Thus, for $\forall j \geq N$, it holds that

$$\begin{aligned}
\rho(\tilde{\Gamma}) &\leq \epsilon + l||\Omega_1||(||2\sum_{m=0}^{j}(\zeta_3\zeta_1)^m(\zeta_3\zeta_2 + \zeta_4)|| + 1) \\
&\leq \epsilon + l||\Omega_1||(\frac{2||\zeta_3\zeta_2 + \zeta_4||}{1 - ||\zeta_3\zeta_1||} + 1) \\
&\leq \epsilon + l||\Omega_1||(\frac{2||\zeta_3\zeta_2 + \zeta_4|| + 1}{1 - ||\zeta_3\zeta_1||}).
\end{aligned}$$

Owing to l satisfying the condition $l||\Omega_1||(\frac{2||\zeta_3\zeta_2+\zeta_4||+1}{1-||\zeta_3\zeta_1||}) < 1$, we obtain $\rho(\tilde{\Gamma}) < 1$. Therefore, $\lim_{k\to\infty} u^{(k)} = u^*$. □

Remark 4.2. *In the above discussions, we extend the convergent theoretics of Method 3.3 to the general case. When $\Omega_1 = I$ and $\Omega_2 = \Omega$, Theorem 4.1 reduces to Theorem 3.3 in [49] with $\Omega \geq \max(\mathrm{diag}(E_1), \mathrm{diag}(E_2))$.*

The succeeding part is the convergent theorem of GTMAOR iteration method.

Theorem 4.3. *Let A be an H_+-matrix in $\mathbb{R}^{n\times n}$, and Ω_1 and Ω_2 be known positive diagonal matrices such that $A\Omega_1$ is an H_+-matrix. $A\Omega_1 = \hat{D} - \hat{L} - \hat{U} := \hat{D} - \hat{B}$ satisfies*

$$(\hat{D} - 2|\bar{B}|)Ye > 0 \text{ and } \Omega_2 \geq \max(\hat{D}, \tfrac{1}{\alpha}\hat{D}),$$

wherein $\hat{D}$, $\hat{B}$, $\hat{L}$ and $\hat{U}$ are the diagonal, the non-diagonal, the strictly lower triangular and the strictly upper triangular parts of the matrix $A\Omega_1$, respectively. $m(\cdot)$ is a Lipschitz continuous function, i.e., it holds that

$$|m(a) - m(b)| \leq l|a - b|,\ \forall a, b \in \mathbb{R}^n,$$

wherein l is the Lipschitz constant. Set $\Psi_1 = \|(\Omega_2 + \langle E'_{\Omega_1}\rangle)^{-1}(|\Omega_2 - A\Omega_1| + |F'_{\Omega_1}|)\|$, $\Psi_2 = \|(\Omega_2 + E'_{\Omega_1})^{-1}A\|$, $\Psi_3 = \|(\Omega_2 + \langle E''_{\Omega_1}\rangle)^{-1}(|\Omega_2 - A\Omega_1| + |F''_{\Omega_1}|)\|$ and $\Psi_4 = \|(\Omega_2 + E''_{\Omega_1})^{-1}A\|$. If l satisfies $l < \frac{1-\Psi_1\Psi_3}{\|\Omega_1\|(2\Psi_2\Psi_3+\Psi_4+1)}$. Then, for arbitrary initial vector, the GTMAOR iteration method is convergent for

$$0 < \beta \leq \alpha \leq 2 \text{ and } \alpha < \tfrac{1}{\rho(\hat{D}^{-1}|\bar{B}|)}.$$

Proof. Based on Theorem 4.1, we only need to justify the condition (16). Set

$$E'_{\Omega_1} = \tfrac{1}{\alpha}(\hat{D} - \beta\hat{L}),\ F'_{\Omega_1} = \tfrac{1}{\alpha}[(1-\alpha)\hat{D} + (\alpha-\beta)\hat{L} + \alpha\hat{U}]$$

and

$$E''_{\Omega_1} = \tfrac{1}{\alpha}(\hat{D} - \beta\hat{U}),\ F''_{\Omega_1} = \tfrac{1}{\alpha}[(1-\alpha)\hat{D} + (\alpha-\beta)\hat{U} + \alpha\hat{L}].$$

Denote $A\Omega_1 := G = (g_{ij}) \in \mathbb{R}^{n\times n}$, and construct an irreducible matrix $\bar{G}$ as

$$\bar{g}_{ij} = \begin{cases} \epsilon, & g_{ij} = 0, \\ g_{ij}, & g_{ij} \neq 0, \end{cases} \quad i, j = 1, 2, \ldots, n.$$

Since the diagonal elements of the matrix A are positive and the matrix Ω_1 is positive diagonal, the diagonal elements of the matrix G is not zero, which means they are the same as the diagonal elements of the matrix $\bar{G}$. Let $\bar{G} = \hat{D} - \bar{L} - \bar{U} = \hat{D} - \bar{B}$ with $\bar{B}$, $\bar{L}$ and $\bar{U}$ being the non-diagonal, the strictly lower-triangular and the strictly upper-triangular matrices of the matrix $\bar{G}$, respectively. G is an H_+-matrix, then $\rho(\hat{D}^{-1}|\hat{B}|) < 1$. For sufficiently small $\epsilon > 0$, based on the continuity of the spectral radius, it holds that $\rho(\hat{D}^{-1}|\bar{B}|) < 1$, which means that $\langle\bar{G}\rangle$ is an M-matrix, i.e., $\bar{G}$ is an H_+-matrix. Furthermore, $\bar{G}$ is irreducible means that $\hat{D}^{-1}|\bar{B}|$ is nonnegative irreducible. According to Perron-Frobenius theorem, there is a vector $y = (y_1, y_2, \ldots, y_n)^T > 0$ such that

$$\hat{D}^{-1}|\bar{B}|y = \rho(\hat{D}^{-1}|\bar{B}|)y,$$

i.e.,

$$|\bar{B}|y = \rho(\hat{D}^{-1}|\bar{B}|)\hat{D}y,$$

which means $\rho(\hat{D}^{-1}|\bar{B}|) > 0$. Based on the above discussion, it is appreciate that $0 < \rho(\hat{D}^{-1}|\bar{B}|) < 1$. Similarly, set

$$\bar{E}'_{\Omega_1} = \tfrac{1}{\alpha}(\hat{D} - \beta\bar{L}),\ \bar{F}'_{\Omega_1} = \tfrac{1}{\alpha}[(1-\alpha)\hat{D} + (\alpha-\beta)\bar{L} + \alpha\bar{U}]$$

and

$$\bar{E}''_{\Omega_1} = \tfrac{1}{\alpha}(\hat{D} - \beta\bar{U}),\ \bar{F}''_{\Omega_1} = \tfrac{1}{\alpha}[(1-\alpha)\hat{D} + (\alpha-\beta)\bar{U} + \alpha\bar{L}].$$

Then, we will prove that $\Omega_2 + \bar{E}'_{\Omega_1}$ is an H_+-matrix. We only need to prove $\langle\Omega_2 + \bar{E}'_{\Omega_1}\rangle$ is an M-matrix. Via the direct calculation and the conditions $0 < \beta \leq \alpha < 2$, we have

$$\langle\Omega_2 + \bar{E}'_{\Omega_1}\rangle = \Omega_2 + \tfrac{1}{\alpha}\hat{D} - \tfrac{\beta}{\alpha}|\bar{L}| \geq \tfrac{1}{\alpha}\hat{D} - |\bar{B}|.$$

From $\alpha < \frac{1}{\rho(\hat{D}^{-1}|\bar{B}|)}$, we can obtain that $\frac{1}{\alpha}\hat{D} - |\bar{B}|$ is an M-matrix, which means $\langle\Omega_2 + \bar{E}'_{\Omega_1}\rangle$ is an M-matrix. Based on Lemma 2.1, we have

$$|(\Omega_2 + \bar{E}'_{\Omega_1})^{-1}| \le \langle \Omega_2 + \bar{E}'_{\Omega_1} \rangle^{-1} = (\Omega_2 + \langle \bar{E}'_{\Omega_1} \rangle)^{-1}.$$

Denote $Y = \mathrm{diag}(y_1, y_2, \dots, y_n)$. Then

$$Y^{-1}\mathcal{L}(\bar{E}'_{\Omega_1}, \bar{F}'_{\Omega_1}, \Omega_2)Y = (\langle \Omega_2 + \bar{E}'_{\Omega_1} \rangle Y)^{-1}(|\bar{F}'_{\Omega_1}| + |\Omega_2 - \bar{G}|)Y.$$

Let $\Omega_1 = \mathrm{diag}(\omega'_1, \omega'_2, \dots, \omega'_n)$ and $\Omega_2 = \mathrm{diag}(\omega''_1, \omega''_2, \dots, \omega''_n)$. For sufficiently small $\varepsilon > 0$, when $0 < \beta \le \alpha \le 2$ and $\Omega_2 \ge \max(\hat{D}, \frac{1}{\alpha}\hat{D})$, $\langle \Omega_2 + \bar{E}'_{\Omega_1} \rangle Ye$ satisfies

$$\begin{aligned}
\langle \Omega_2 + \bar{E}'_{\Omega_1} \rangle Ye &= \omega''_i y_i + \frac{1}{\alpha} g_{ii} y_i - \frac{\beta}{\alpha}\sum_{j=1}^{i-1} |\bar{g}_{ij}| y_j \\
&\ge \frac{2}{\alpha} g_{ii} y_i - \frac{\beta}{\alpha}\sum_{j=1}^{i-1} |\bar{g}_{ij}| y_j \\
&= \frac{2}{\alpha} g_{ii} y_i - \frac{\beta}{\alpha}\sum_{j\neq i} |\bar{g}_{ij}| y_j + \frac{\beta}{\alpha}\sum_{j\neq i} |\bar{g}_{ij}| y_j - \frac{\beta}{\alpha}\sum_{j=1}^{i-1} |\bar{g}_{ij}| y_j \\
&= \frac{2 - \beta\cdot\rho(\hat{D}^{-1}|\bar{B}|)}{\alpha} g_{ii} y_i + \frac{\beta}{\alpha}\sum_{j=i+1}^{n} |\bar{g}_{ij}| y_j \\
&\ge \frac{2 - \beta\cdot\rho(\hat{D}^{-1}|\bar{B}|)}{\alpha} g_{ii} y_i > 0.
\end{aligned}$$

Hence, $\Omega_2 Y + \langle \bar{E}'_{\Omega_1} \rangle Y$ is a s.d.d. matrix. In addition, since $0 < \beta \le \alpha < 2$, it holds that

$$\begin{aligned}
\|Y^{-1}\mathcal{L}(\bar{E}'_{\Omega_1}, \bar{F}'_{\Omega_1}, \Omega_2)Y\|_\infty =& \|(\langle \Omega_2 + \bar{E}'_{\Omega_1} \rangle Y)^{-1}(|\bar{F}'_{\Omega_1}| + |\Omega_2 - \bar{G}|)Y\|_\infty \\
=& \|(\langle \Omega_2 + \frac{1}{\alpha}\hat{D} - \frac{\beta}{\alpha}\bar{L} \rangle Y)^{-1} \cdot [\frac{1}{\alpha}|(1-\alpha)\hat{D} + (\alpha-\beta)\bar{L} + \alpha\bar{U}|Y + |\Omega_2 - \bar{G}|Y]\|_\infty \\
=& \|[(\Omega_2 + \frac{1}{\alpha}\hat{D} - \frac{\beta}{\alpha}|\bar{L}|)Y]^{-1} \\
&\cdot ((\Omega_2 - \hat{D})Y + \frac{|1-\alpha|}{\alpha}\hat{D}Y + \frac{\alpha-\beta}{\alpha}|\bar{L}|Y + |\bar{U}|Y + |\bar{B}|Y)\|_\infty \\
\le& \max_{1\le i\le n} \frac{[((\Omega_2 - \hat{D})Y + \frac{|1-\alpha|}{\alpha}\hat{D}Y + \frac{\alpha-\beta}{\alpha}|\bar{L}|Y + |\bar{U}|Y + |\bar{B}|Y)e]_i}{[(\Omega_2 + \frac{1}{\alpha}\hat{D} - \frac{\beta}{\alpha}|\bar{L}|)Ye]_i}, \\
\le& \max_{1\le i\le n} \frac{(\omega''_i + \frac{|1-\alpha|-\alpha}{\alpha} g_{ii})y_i + \frac{\alpha-\beta}{\alpha}\sum_{j=1}^{i-1}|\bar{g}_{ij}|y_j + \sum_{j=i+1}^{n}|\bar{g}_{ij}|y_j + \sum_{j\neq i}|\bar{g}_{ij}|y_j}{\omega''_i y_i + \frac{1}{\alpha} g_{ii} y_i - \frac{\beta}{\alpha}\sum_{j=1}^{i-1}|\bar{g}_{ij}|y_j},
\end{aligned}$$

which holds according to Lemma 2.2.

Since $(\hat{D} - 2|\bar{B}|)Ye > 0$, then

$$\begin{aligned}
&(\omega''_i y_i + \frac{1}{\alpha} g_{ii} y_i - \frac{\beta}{\alpha}\sum_{j=1}^{i-1}|\bar{g}_{ij}|y_j) - [(\omega''_i + \frac{|1-\alpha|-\alpha}{\alpha} g_{ii})y_i \\
&\qquad + \frac{\alpha-\beta}{\alpha}\sum_{j=1}^{i-1}|\bar{g}_{ij}|y_j + \sum_{j=i+1}^{n}|\bar{g}_{ij}|y_j + \sum_{j\neq i}|\bar{g}_{ij}|y_j] \\
&> g_{ii} y_i - 2\sum_{j\neq i}|\bar{g}_{ij}|y_j > 0.
\end{aligned}$$

Hence, it is simple that $\|Y^{-1}\mathcal{L}(\bar{E}'_{\Omega_1}, \bar{F}'_{\Omega_1}, \Omega_2)Y\|_\infty < 1$.

Analogously, we can also get $\|Y^{-1}\mathcal{L}(\bar{E}''_{\Omega_1}, \bar{F}''_{\Omega_1}, \Omega_2)Y\|_\infty < 1$. It holds that

$$\rho(\zeta_3\zeta_1) < \|\zeta_3\zeta_1\| = \lim_{\varepsilon\to 0}(\|Y^{-1}\mathcal{L}(\bar{E}''_{\Omega_1}, \bar{F}''_{\Omega_1}, \Omega_2)Y\|_\infty \|Y^{-1}\mathcal{L}(\bar{E}'_{\Omega_1}, \bar{F}'_{\Omega_1}, \Omega_2)Y\|_\infty) < 1,$$

which means $(\zeta_3\zeta_1)^{j+1} \to 0,\ j \to \infty$. Since $\|\Omega_1\|$ and $\|\Omega_1^{-1} + l\Omega_1^{-1} + |\Omega_2^{-1}A|\|$ are constants.

Hence, for arbitrary positive number ϵ, there is $N \in \mathbb{N}_+$ such that

$$\|\Omega_1\|\|(\zeta_3\zeta_1)^{j+1}\|\|\Omega_1^{-1} + l\Omega_2^{-1} + |\Omega_1^{-1}A|\| \le \epsilon,\ \forall j \ge N.$$

Let $\|\zeta_i\| = \Psi_i,\ i = 1,2,3,4$. Thus, for $\forall j \ge N$, it holds that

$$\begin{aligned}\rho(\tilde{\Gamma}) &\le \epsilon + l\|\Omega_1\|(2\sum_{m=0}^{j}(\Psi_1\Psi_3)^m(\Psi_2\Psi_3 + \Psi_4) + 1)\\ &\le \epsilon + l\|\Omega_1\|(\frac{2\Psi_2\Psi_3 + \Psi_4}{1 - \Psi_1\Psi_3} + 1)\\ &\le \epsilon + l\|\Omega_1\|(\frac{2\Psi_2\Psi_3 + \Psi_4 + 1}{1 - \Psi_1\Psi_3}).\end{aligned}$$

Because of l satisfying the condition $l < \frac{1-\Psi_1\Psi_3}{\|\Omega_1\|(2\Psi_2\Psi_3+\Psi_4+1)}$, we obtain $\rho(\tilde{\Gamma}) < 1$. Therefore, $\lim_{k\to\infty} u^{(k)} = u^*$, which prove the GTMAOR iteration method is convergent. □

Remark 4.4. *Set $\Omega_1 = I$ and $\Omega_2 = \Omega$, then Theorem 4.3 is the convergent results of the TMAOR method.*

Remark 4.5. *We can directly obtain that the GTMSOR iteration method is convergent for*

$$0 < \alpha < \min(2, \tfrac{1}{\rho(\hat{D}^{-1}|\hat{B}|)}),$$

and other conditions of Theorem 4.3 remain unchanged.

5. Numerical results

In this section, in order to demonstrate the efficiency of our proposed method, we will do some numerical experiments, which include three aspects: the elapsed CPU time in seconds (CPU), the norm of absolute residual vectors (RES), and the iteration numbers (IT), respectively. All of these numerical results were performed in Matlab (R2017a) on an an Intel(R) Core(TM) i5-4210U CPU @ 1.70GHz, 4.00GB RAM.

In the numerical computations, we choose the initial vector $u^{(0)}$ to be zero vector. 'RES' is defined as

$$\text{RES}(u^{(k)}) := |(Au^{(k)} + q)^T(u^{(k)} - m(u^{(k)}))|,$$

where $u^{(k)}$ is the kth approximate solution to the ICP, and all iterations are terminated either the maximum iteration numbers exceed 2000 or $\text{RES}(u^{(k)}) \le 10^{-5}$. The inner iteration steps j is set to be 3.

We choose three methods such as MSOR, TMSOR and GTMSOR in the following experiments. Furthermore, on the basis of the existing literature, we set the parameters $\alpha = 1.2$, $t = 2.5$ and $\gamma = 2$. We take $\Omega = \gamma t D_A$ in TMMSOR methods, and $\Omega_1 = \frac{t}{\alpha}I$ and $\Omega_2 = tD_A$ in both GMMSOR and TGMMSOR methods, respectively.

Example 5.1. *[28] Let m be a prescribed positive integer and $n = m^2$. Consider the ICP(1), in which $A \in \mathbb{R}^{n\times n}$ is a block tridiagonal matrix*

Table 1: Numerical results for Example 5.1

Algorithm		m=10	m=20	m=30	m=40
	CPU	0.0222	3.5643	85.9189	772.4247
MSOR	RES	1.2815e-06	9.8599e-06	9.9055e-06	9.9520e-06
	IT	25	133	450	956
	CPU	0.0452	4.3827	91.7140	824.1270
TMSOR	RES	1.6585e-06	4.6080e-06	9.6860e-06	9.7956e-06
	IT	20	75	250	529
	CPU	0.0115	1.5031	29.6515	259.7250
GTMSOR	RES	9.4070e-07	3.4515e-06	8.7211e-06	9.5469e-06
	IT	7	23	79	168

$$A = tridiag(-I, S, -I) = \begin{pmatrix} S & -I & 0 & \cdots & 0 & 0 \\ -I & S & -I & \cdots & 0 & 0 \\ 0 & -I & S & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & S & -I \\ 0 & 0 & 0 & \cdots & -I & S \end{pmatrix} \in \mathbb{R}^{n\times n}$$

and the block diagonal matrix S is defined as a tridiagonal matrix

$$S = tridiag(-1, 4, -1) = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 4 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 4 \end{pmatrix} \in \mathbb{R}^{m\times m}.$$

The vector $q \in \mathbb{R}^n$ and the point-to-point mapping $m(u)$ are

$$q = (-1, 1, -1, 1, \cdots, (-1)^{n-1}, (-1)^n)^T \in \mathbb{R}^n \text{ and } m(u) = (u_1^3, u_2^3, \cdots, u_n^3)^T \in \mathbb{R}^n,$$

respectively.

In this example, we set four different sizes, i.e., $n = 100, 400, 900, 1600$. From Table 1, it is simple to find that the GTMSOR method outperforms the MSOR and TMSOR methods in both IT and CPU, which manifests the GTMSOR method has an advantage over the others. The convergent rate becomes faster as the system matrix A size n is increasing. Numerical experiment illustrates the convergence speed is accelerated by the general two-step modulus-based matrix splitting.

Example 5.2. *[28] Let m be a prescribed positive integer and $n = m^2$. Consider the ICP(1), in which $A \in \mathbb{R}^{n\times n}$ is a block tridiagonal matrix*

$$A = tridiag(-I, S, -I) = \begin{pmatrix} S & -0.5I & 0 & \cdots & 0 & 0 \\ -1.5I & S & -0.5I & \cdots & 0 & 0 \\ 0 & -1.5I & S & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & S & -0.5I \\ 0 & 0 & 0 & \cdots & -1.5I & S \end{pmatrix} \in \mathbb{R}^{n\times n}$$

and the block diagonal matrix S is defined as a tridiagonal matrix

$$S = tridiag(-1.5, 4, -0.5) = \begin{pmatrix} 4 & -0.5 & 0 & \cdots & 0 & 0 \\ -1.5 & 4 & -0.5 & \cdots & 0 & 0 \\ 0 & -1.5 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & -0.5 \\ 0 & 0 & 0 & \cdots & -1.5 & 4 \end{pmatrix} \in \mathbb{R}^{m\times m}.$$

The vector $q \in \mathbb{R}^n$ and the point-to-point mapping $m(u)$ are

$$q = (-1, 1, -1, 1, \cdots, (-1)^{n-1}, (-1)^n)^T \in \mathbb{R}^n \text{ and } m(u) = (u_1^3, u_2^3, \cdots, u_n^3)^T \in \mathbb{R}^n,$$

respectively.

Table 2: Numerical results for Example 5.2

Algorithm		m=10	m=20	m=30	m=40
	CPU	0.0607	1.3721	14.2336	80.7220
MSOR	RES	7.6775e-06	7.6661e-06	8.9724e-06	8.6018e-06
	IT	24	45	78	107
	CPU	0.0351	1.5444	13.8348	78.7359
TMSOR	RES	2.4662e-07	1.9021e-06	6.0939e-06	2.5492e-06
	IT	17	26	39	51
	CPU	0.0197	0.5457	5.4097	28.6644
GTMSOR	RES	8.7702e-07	2.8091e-06	6.8682e-07	4.1120e-07
	IT	6	9	15	19

In this example, we set four different sizes, i.e., $n = 100, 400, 1600, 3600$. From the Table 2, we can observe that the CPU of the GTMMS method is half of the other presented methods. Therefore, in terms of computing efficiency, the GTMSOR method precedes the MSOR and the TMSOR method.

Example 5.3. *[47] Let m be a prescribed positive integer and $n = m^2$. Consider the ICP(1), in which $A \in \mathbb{R}^{n\times n}$ is*

$$A = \begin{pmatrix} S & -I & -I & & \\ & S & -I & \ddots & \\ & & S & \ddots & -I \\ & & & \ddots & -I \\ & & & & S \end{pmatrix} \in \mathbb{R}^{n\times n}$$

and the block diagonal matrix S is defined as a tridiagonal matrix

$$S = tridiag(-1, 8, -1) = \begin{pmatrix} 8 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 8 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 8 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 8 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 8 \end{pmatrix} \in \mathbb{R}^{m\times m}.$$

The vector $q \in \mathbb{R}^n$ and the point-to-point mapping $m(u)$ are

$$q = (-1, 1, -1, 1, \cdots, (-1)^{n-1}, (-1)^n)^T \in \mathbb{R}^n \text{ and } m(u) = (u_1^3, u_2^3, \cdots, u_n^3)^T \in \mathbb{R}^n,$$

respectively.

In Table 3, as for the three mentioned aspects, our suggested method is faster in CPU and IT and smaller in RES. Furthermore, we can also gain the conclusion that the GTMSOR method has an advantage over the MSOR and TMSOR methods. These results all prove our proposed method is a better method.

Table 3: Numerical results for Example 5.3

Algorithm		m=10	m=20	m=40	m=60
MSOR	CPU	0.0177	0.4805	13.0513	122.3386
	RES	9.2304e-06	8.3774e-06	5.3743e-06	4.5922e-06
	IT	12	14	16	17
TMSOR	CPU	0.0108	0.4004	12.3242	117.8681
	RES	7.8810e-07	7.2184e-06	1.3861e-06	3.7938e-06
	IT	7	7	8	8
GTMSOR	CPU	0.0254	0.1810	4.6610	40.5694
	RES	1.6485e-06	4.6253e-06	1.4303e-10	4.9490e-10
	IT	2	2	3	3

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