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Modulus Hyperinvariant Ideals for a Finitely Quasinilpotent Operators

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Abstract. Let X be a Banach space with an unconditional basis, and let $C \neq \{0\}$ be a collection of continuous linear operators with modulus on X that is finitely modulus-quasinilpotent at a non-zero positive vector. Then C and its right modulus sub-commutant C'_{m} have a common non-trivial invariant closed ideal.

1. Introduction and Preliminary

Let X be a Banach lattice. If $\mathcal{F}$ is a collection of continuous linear operators with modulus on X, we define $|\mathcal{F}| = \{|T|; T \in \mathcal{F}\}$, where $|T|$ denotes the modulus of the operator T. A collection C of continuous linear operators with modulus on X is said to be finitely modulus-quasinilpotent at a vector $x_0 \in X$ if $\lim_{n\to\infty} \||\mathcal{F}|^n x_0\|^{1/n} = 0$ for every finite subset $\mathcal{F}$ of C.

If A and B are continuous linear operators on X with B positive, then A is said to be dominated by B whenever $|Ax| \leq B(|x|)$ holds for all $x \in X$.

Let C be a collection of continuous positive operators on X, then C'_+ denotes the set of all continuous positive operators S on X such that $TS = ST$ for all $T \in C$. We say that C'_+ is the positive commutant of C.

Let C be a collection of continuous linear operators with modulus on X , then C'_{m} denotes the set of all continuous linear operators S with modulus on X such that $|T||S| \leq |S||T|$ for all $T \in C$. We say that C'_{m} is the right modulus sub-commutant of C.

It is easy to see that if C is a collection of positive operators on a Banach lattice, then $C'_+ \subset C'_{\mathrm{m}}$, and "finitely quasinilpotent" and "finitely modulus-quasinilpotent " are equivalent.

A vector subspace $\mathcal{I}$ of X is said to be an (order) ideal whenever $|x| \leq |y|$ and $y \in \mathcal{I}$ imply that $x \in \mathcal{I}$. The ideal generated by a non-empty subset F of X is defined by $\mathcal{I}_F = \{x \in X;$ there are $x_1, \cdots, x_n \in F$ and $\lambda_1, \cdots, \lambda_n > 0$ such that $|x| \leq \sum_{k=1}^n \lambda_k |x_k|\}$. In particular, the ideal generated by a singleton $\{x\}$ is given by $\mathcal{I}_x = \{y \in X;$ there is $\lambda > 0$ such that $|y| \leq \lambda |x|\}$.

It is well known that if X is a Banach space X with an unconditional basis, then X may be regarded as a Banach lattice whenever one is looking for invariant subspaces and invariant ideals. For each given positive integer n, define the functional f_n by $f_n(x) = \alpha_n$ for every $x = \sum_{k=1}^{\infty} \alpha_k e_k$. Then f_n is a continuous linear functional on X.

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In 1954, N. Aronszajn and K. T. Smith [1] showed that every compact operator on a Banach space has a non-trivial invariant closed subspace.

But it was not until 1986 that people solved the invariant closed ideal problem for a special class of compact operators. To be more specific, B. de Pagter [7] proved that every positive quasinilpotent compact operator on a Banach lattice has a non-trivial invariant closed ideal. It is well known that Pagter's result is an affirmative answer of a long standing open question (cf. [1], [2], [5] and [7]).

In 2007, M. Liu [3] showed that if $C \neq \{0\}$ is a collection of continuous positive operators on a Banach space with a Schauder basis that is finitely quasinilpotent at a non-zero positive vector, then C and its positive commutant C'_+ have a common non-trivial invariant closed subspace.

In this paper, we shall extend the result in [3] from the invariant closed subspace for collections of positive operators to the invariant closed ideal for collections of operators with modulus (may be non-positive operators). This paper can be seen as a sequel to [3].

It is well known that the non-trivial invariant closed ideal for any operator is necessarily its non-trivial invariant closed subspace and that each positive has the modulus, but their converses are not necessarily true. Moreover, in section 3, we will give a collection of nonpositive continuous linear operators that satisfies the condition of our main result.

2. The Main Result

Now we are in a position to give the main result.

Theorem 1. Let X be a Banach space with an unconditional basis $\{e_n\}$, and let $C \neq \{0\}$ be a collection of continuous linear operators with modulus on X that is finitely modulus-quasinilpotent at a non-zero positive vector x_0. Then C and its right modulus sub-commutant C'_{m} have a common non-trivial invariant closed ideal.

Proof. As in [3], it follows from $x_0 > 0$ that there are an appropriate scalar $\lambda > 0$ and a positive integer n_0 such that $\lambda x_0 \geq e_{n_0} > 0$. It is clear that C is finitely modulus-quasinilpotent at λx_0. Let $\mathcal{G}$ be the multiplicative semigroup generated by $|C|$ (i. e. $\mathcal{G} = \bigcup_{n=1}^{\infty} |C|^n$), and let $\mathcal{A}$ be the algebra of all continuous linear operators on X such that each $A \in \mathcal{A}$ is dominated by some operator of the form $\sum_{j=1}^{n} |S_j| G_j$ with $S_j \in C'_{\mathrm{m}}$ and $G_j \in \mathcal{G}$.

We consider two cases separately.

Case 1. If there is an operator $A_0 \in \mathcal{A}$ such that $A_0 e_{n_0} \neq 0$, then the ideal $\mathcal{I}_{\mathcal{A}e_{n_0}}$ generated by $\mathcal{A}e_{n_0}$ is a non-zero ideal in X, where $\mathcal{A}e_{n_0} := \{Ae_{n_0}; A \in \mathcal{A}\}$.

First we show that $\overline{\mathcal{I}_{\mathcal{A}e_{n_0}}} \neq X$. As in [3], let P denote the natural projection from X onto the vector subspace generated by e_{n_0}. By a modification of the corresponding part of [3], we can prove that

$$P|S|Ge_{n_0} = 0 \tag{1}$$

for all $S \in C'_{\mathrm{m}}$ and all $G \in \mathcal{G}$. (Indeed, it suffices to replace $S, T_l, \mathcal{F}$ and C by $|S|, |T_l|, |\mathcal{F}|$ and $|C|$ respectively.) For every $x \in \mathcal{I}_{\mathcal{A}e_{n_0}}$, the definition of $\mathcal{I}_{\mathcal{A}e_{n_0}}$ implies that there are operators $A_1, A_2, \cdots, A_m \in \mathcal{A}$ such that $|x| \leq \sum_{i=1}^{m} |A_i e_{n_0}|$, and so $x^+ \leq \sum_{i=1}^{m} |A_i e_{n_0}|$ and $x^- \leq \sum_{i=1}^{m} |A_i e_{n_0}|$. For each $i = 1, 2, \cdots, m$, by the definition of $\mathcal{A}$ there are operators $S_{ij} \in C'_{\mathrm{m}}$, $G_{ij} \in \mathcal{G}$ $(j = 1, 2, \cdots, n(i))$ such that $|A_i e_{n_0}| \leq \sum_{j=1}^{n(i)} |S_{ij}| G_{ij} e_{n_0}$. Thus we have

$$x^+ \leq \sum_{i=1}^{m} |A_i e_{n_0}| \leq \sum_{i=1}^{m} \sum_{j=1}^{n(i)} |S_{ij}| G_{ij} e_{n_0}. \tag{2}$$

Thus by (1) we obtain $P(x^+) \leq \sum_{i=1}^{m} \sum_{j=1}^{n(i)} P|S_{ij}| G_{ij} e_{n_0} = 0$, and so $P(x^+) = 0$. Hence it is easy to obtain that $f_{n_0}(x^+) = f_{n_0}(Px^+) = 0$. Similarly, $f_{n_0}(x^-) = 0$. Thus we have $f_{n_0}(x) = 0$ for every $x = x^+ - x^- \in \mathcal{I}_{\mathcal{A}e_{n_0}}$. (For the complex space X, we can obtain $f_{n_0}((\mathrm{Re}x)^+) = f_{n_0}((\mathrm{Re}x)^-) = f_{n_0}((\mathrm{Im}x)^+) = f_{n_0}((\mathrm{Im}x)^-) = 0$, thus we have $f_{n_0}(x) = 0$ for every $x = (\mathrm{Re}x)^+ - (\mathrm{Re}x)^- + i[(\mathrm{Im}x)^+ - (\mathrm{Im}x)^-] \in \mathcal{I}_{\mathcal{A}e_{n_0}}$.) Consequently $f_{n_0}(x) = 0$ for every $x \in \overline{\mathcal{I}_{\mathcal{A}e_{n_0}}}$. Thus by $f_{n_0}(e_{n_0}) = 1$, we obtain $\overline{\mathcal{I}_{\mathcal{A}e_{n_0}}} \neq X$.

We now prove that $\mathcal{I}_{\mathcal{A}e_{n_0}}$ is invariant under $\mathcal{C}$ and $\mathcal{C}'_{\mathrm{m}}$. To this end, take $x \in \mathcal{I}_{\mathcal{A}e_{n_0}}$, $T \in \mathcal{C}$ and $S \in \mathcal{C}'_{\mathrm{m}}$. Then by (2) we obtain

$$|Tx^+| \leq |T|(x^+) \leq \sum_{i=1}^{m} \sum_{j=1}^{n(i)} |T||S_{ij}|G_{ij}e_{n_0} \leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n(i)} |S_{ij}||T|G_{ij} \right) e_{n_0}. \tag{3}$$

It is easy to see that $|T|G_{ij} \in \mathcal{G}$, and so $\sum_{j=1}^{n(i)} |S_{ij}||T|G_{ij} \in \mathcal{A}$. Thus by (3) we obtain $Tx^+ \in \mathcal{I}_{\mathcal{A}e_{n_0}}$. Similarly, $Tx^- \in \mathcal{I}_{\mathcal{A}e_{n_0}}$, and so $Tx \in \mathcal{I}_{\mathcal{A}e_{n_0}}$. Again, by (2) we obtain

$$|Sx^+| \leq |S|(x^+) \leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n(i)} |S||S_{ij}|G_{ij} \right) e_{n_0}. \tag{4}$$

Since $|S||S_{ij}| \in \mathcal{C}'_{\mathrm{m}}$, we have $\sum_{j=1}^{n(i)} |S||S_{ij}|G_{ij} \in \mathcal{A}$. Thus by (4) we obtain $Sx^+ \in \mathcal{I}_{\mathcal{A}e_{n_0}}$. Similarly, $Sx^- \in \mathcal{I}_{\mathcal{A}e_{n_0}}$, and so $Sx \in \mathcal{I}_{\mathcal{A}e_{n_0}}$.

From the above we see that $\overline{\mathcal{I}_{\mathcal{A}e_{n_0}}}$ is a common non-trivial invariant closed ideal for $\mathcal{C}$ and $\mathcal{C}'_{\mathrm{m}}$.

Case 2. If $Ae_{n_0} = 0$ for all $A \in \mathcal{A}$, then Ker$\mathcal{A} = \{x; A|x| = 0 \text{ for all } A \in \mathcal{A}\}$ is a non-zero closed ideal in X. Since the identity operator $I \in \mathcal{C}'_{\mathrm{m}}$, it follows that $\{0\} \neq \mathcal{C} \subset \mathcal{G} \subset \mathcal{A}$, and so Ker$\mathcal{A} \neq X$.

It only remains to show that Ker$\mathcal{A}$ is invariant under $\mathcal{C}$ and $\mathcal{C}'_{\mathrm{m}}$. To this end, take $x \in$ Ker$\mathcal{A}$, $T \in \mathcal{C}$ and $S \in \mathcal{C}'_{\mathrm{m}}$. For any $A \in \mathcal{A}$, it follows from the definition of $\mathcal{A}$ that there are operators $S_1, S_2, \cdots, S_n \in \mathcal{C}'_{\mathrm{m}}$,and $G_1, G_2, \cdots, G_n \in \mathcal{G}$ such that $|Ay| \leq \sum_{j=1}^{n} |S_j|G_j(|y|)$ for all $y \in X$. Thus we have

$$|A(|Tx|)| \leq \sum_{j=1}^{n} |S_j|G_j(|Tx|) \leq \sum_{j=1}^{n} |S_j|G_j|T|(|x|). \tag{5}$$

Observing $G_j|T| \in \mathcal{G}$, we see that $\sum_{j=1}^{n} |S_j|G_j|T| \in \mathcal{A}$. Since $x \in$ Ker$\mathcal{A}$, it follows that $\sum_{j=1}^{n} |S_j|G_j|T|(|x|) = 0$. Thus by (5) we have $|A(|Tx|)| = 0$, and so $A|Tx| = 0$ for all $A \in \mathcal{A}$. Consequently $Tx \in$Ker$\mathcal{A}$. Similarly, we have

$$|A(|Sx|)| \leq \sum_{j=1}^{n} |S_j|G_j(|Sx|) \leq \sum_{j=1}^{n} |S_j|G_j|S|(|x|).$$

Since $G_j \in \mathcal{G}, G_j$ is an operator of the form $|T_{j_1}||T_{j_2}|\cdots|T_{j_k}|$, where $T_{j_1}, T_{j_2}, \cdots, T_{j_k} \in \mathcal{C}$. Thus we obtained

$$\begin{aligned} |A(|Sx|)| &\leq \sum_{j=1}^{n} |S_j||T_{j_1}||T_{j_2}|\cdots|T_{j_k}||S|(|x|) \\ &\leq \sum_{j=1}^{n} |S_j||S||T_{j_1}||T_{j_2}|\cdots|T_{j_k}|(|x|) = \sum_{j=1}^{n} |S_j||S|G_j(|x|). \end{aligned} \tag{6}$$

Since $|S_j||S| \in \mathcal{C}'_{\mathrm{m}}$, it follows that $\sum_{j=1}^{n} |S_j||S|G_j \in \mathcal{A}$. Thus by $x \in$ Ker$\mathcal{A}$ we obtain $\sum_{j=1}^{n} |S_j||S|G_j(|x|) = 0$. Thus by (6) we have $|A(|Sx|)| = 0$, and so $A|Sx| = 0$ for all $A \in \mathcal{A}$. Consequently $Sx \in$Ker$\mathcal{A}$.

From the above we conclude that Ker$\mathcal{A}$ is a common non-trivial invariant closed ideal for $\mathcal{C}$ and $\mathcal{C}'_{\mathrm{m}}$, and this completes the proof.

3. An example

We conclude this paper with the following example for a non-commutative finitely modulus-quasinilpotent collection $\mathcal{C}$ of continuous non-positive operators that satisfies the conditions of Theorem 1.

Let T_a, S_a and B_a be operators on the sequence space l^p $(1 \leq p < \infty)$ with matrix respectively

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2}a_2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3}a_3 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{4}a_4 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{5}a_5 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2}a_2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3}a_3 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{4}a_4 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{5}a_5 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & a_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_4 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $a = (a_0, a_1, a_2, \cdots)$, $a_k = 1$ or $a_k = -1$. Set $C = \{T_a, S_a; a = (a_0, a_1, a_2, \cdots), a_k = 1 \text{ or } a_k = -1\}$. Then the collection C of operators satisfies our demands. Indeed, as in [3], we can show that $T_aS_ae_1 \neq S_aT_ae_1$ and $\lim_{n\to\infty} \||\mathcal{F}|^n e_2\|^{1/n} = 0$ for every subset $\mathcal{F}$ of C, where e_n denotes the vector in l^p whose n-th component is one and every other zero.

Moreover, it is clear that $\{B_a; a = (a_0, a_1, a_2, \cdots),\ a_k = 1 \text{ or } a_k = -1\} \subset C'_{\mathrm{m}}$. Thus by Theorem 1 all operators in the collection $\{T_a, S_a, B_a; a = (a_0, a_1, a_2, \cdots), a_k = 1 \text{ or } a_k = -1\}$ of non-positive operators have a common non-trivial invariant closed ideal.

It should be noticed that operators in the main result of [3] are positive, while operators in Example 1 and the main result of this paper may be non-positive.

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