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Further Results on Hybrid (b, c)-Inverses in Rings

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Abstract. Let R be a ring and $b, c \in R$. In this paper, the absorption law for the hybrid (b, c)-inverse in a ring is considered. Also, by using the Green's preorders and relations, we obtain the reverse order law of the hybrid (b, c)-inverse. As applications, we obtain the related results for the (b, c)-inverse.

1. Introduction

Core inverse, dual core inverse, and Mary inverse, as well as classical generalized inverses, are special types of outer inverses. In [2], Drazin introduced a new class of outer inverse and called it (b, c)-inverse, which encompasses the above-mentioned generalized inverses.

Definition 1.1. *Let R be an associative ring and let $b, c \in R$. An element $a \in R$ is (b, c)-invertible if there exists $y \in R$ such that*

$$y \in (bRy) \cap (yRc), \quad yab = b, \quad cay = c.$$

If such y exists, it is unique and is denoted by $a^{\|(b,c)}$. Drazin [2] also presented an equivalent characterization for the (b, c)-inverse y of a as $yay = y$, $yR = bR$ and $Ry = Rc$.

As generalizations of (b, c)-inverses, hybrid (b, c)-inverses and annihilator (b, c)-inverses were introduced in [2]. The symbols $\mathrm{lann}(a) = \{g \in R : ga = 0\}$ and $\mathrm{rann}(a) = \{h \in R : ah = 0\}$ denote the sets of all left annihilators and right annihilators of a, respectively.

Definition 1.2. *Let $a, b, c, y \in R$. We say that y is a hybrid (b, c)-inverse of a if*

$$yay = y, \quad yR = bR, \quad rann(y) = rann(c).$$

If such y exists, it is unique. In this article, we use the symbol $a^{\|(b \bowtie c)}$ to denote the hybrid (b, c)-inverse of a.

Definition 1.3. *Let $a, b, c, y \in R$. We say that y is a annihilator (b, c)-inverse of a if*

$$yay = y, \quad lann(y) = lann(b), \quad lann(y) = lann(c).$$

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The topics of research on the (b, c)-inverse and the related generalized inverses attract wide interest (see [3–6, 13]).

In this paper, we mainly consider the absorption law and the reverse order law for the hybrid (b, c)-inverse in rings. The paper is organized as follows. In Section 2, the absorption law for the hybrid (b, c)-inverse are derived. It is proved that if a is hybrid (b, c)-invertible and d is hybrid (b, c)-invertible, then $a^{\|(b\bowtie c)} + d^{\|(b\bowtie c)} = a^{\|(b\bowtie c)}(a + d)d^{\|(b\bowtie c)}$. Moreover, by using Green's preorders and relations, we obtain if $a^{\|(b\bowtie c)}$ and $d^{\|(u\bowtie v)}$ exist, and conditions $b\mathcal{R}u$ and $c\mathcal{L}v$ are satisfied, then $a^{\|(b\bowtie c)} + d^{\|(u\bowtie v)} = a^{\|(b\bowtie c)}(a + d)d^{\|(u\bowtie v)}$. In Section 3, we get the reverse order law of the hybrid (b, c)-inverse. In particular, let $a^{\|(b\bowtie c)}$ and $d^{\|(b\bowtie c)}$ exist. If $aa^{\|(b\bowtie c)} = a^{\|(b\bowtie c)}a$, then ad is hybrid (b, c)-invertible and $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$. Moreover, if $a^{\|(b\bowtie c)}$ and $d^{\|(b\bowtie c)}$ exist, and conditions $ab \leq_{\mathcal{R}} ba$ and $ac \leq_{\mathcal{L}} ca$ are satisfied, then ad is hybrid (b, c)-invertible and $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$.

2. Absorption law for the hybrid (b, c)-inverse

Let $a, b \in R$ be two invertible elements. It is well known that

$$a^{-1} + b^{-1} = a^{-1}(a + b)b^{-1}.$$

The above equality is known as the absorption law of invertible elements. In general, the absorption law does not hold for generalized inverses (see [9, 10]). In this section, the absorption laws for the hybrid (b, c)-inverse are obtained. For future reference we state some known results.

Lemma 2.1. *([14, Proposition 2.1]). Let $a, b, c, y \in R$. Then the following conditions are equivalent:*
(i) a is hybrid (b, c)-invertible and y is the hybrid (b, c)-inverse of a.
(ii) $yab = b$, $cay = c$, $yR \subseteq bR$ and $rann(c) \subseteq rann(y)$.

Lemma 2.2. *([2, P.1992]). Let $a, b, c \in R$. Then a has a hybrid (b, c)-inverse if and only if $c \in cabR$ and $rann(cab) \subseteq rann(b)$.*

Lemma 2.3. *Let $a, b, c, d \in R$. If $a^{\|(b\bowtie c)}$ is the hybrid (b, c)-inverse of a and $d^{\|(b\bowtie c)}$ is the hybrid (b, c)-inverse of d, then $a^{\|(b\bowtie c)} = a^{\|(b\bowtie c)}dd^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}da^{\|(b\bowtie c)}$ and $d^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}aa^{\|(b\bowtie c)} = a^{\|(b\bowtie c)}ad^{\|(b\bowtie c)}$.*

Proof. Let $x = a^{\|(b\bowtie c)}$ and $y = d^{\|(b\bowtie c)}$. Then by Lemma 2.1, we have $y \in bR$ and $xab = b$. This gives that $y = bs$ for some $s \in R$, and $xay = xa(bs) = (xab)s = bs = y$. Moreover, by Lemma 2.1, we have $c = cax = cdy$, which means that $ax - dy \in \text{rann}(c)$. Note that since $\text{rann}(c) \subseteq \text{rann}(y)$, it follows $y(ax - dy) = 0$ and $yax = ydy = y$. Here, we prove that $d^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}aa^{\|(b\bowtie c)} = a^{\|(b\bowtie c)}ad^{\|(b\bowtie c)}$. Similarly, we can also get $a^{\|(b\bowtie c)} = a^{\|(b\bowtie c)}dd^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}da^{\|(b\bowtie c)}$. □

Next, we will consider when d is hybrid (b, c)-invertible if $a^{\|(b\bowtie c)}$ exists. In fact, whether we discuss about the absorption law or the reverse order law for the hybrid (b, c)-inverse, we always assume that a and d are both hybrid (b, c)-invertible first. Moreover, this kind of problems frequently were studied in optimization theory. It is of interest to know that, in C^* algebras, if a contains some properties, wether $d = a + \varepsilon$ also contains the similar properties when $\varepsilon \to 0$. In the following, we will give existence criteria for the hybrid (b, c)-inverse of d, when a is hybrid (b, c)-invertible. By Lemma 2.1, it is easy to conclude that if a is hybrid (b, c)-invertible, then b is regular. An element $a \in R$ is called (von Neumann) regular if there exists x in R such that $a = axa$. Such an x is called an inner inverse of a and is denoted by a^-. Before we investigate the existence criteria for the hybrid (b, c)-inverse, the following lemma is necessary.

Lemma 2.4. *([8]) Let $a, e \in R$ with $e^2 = e$. Then the following statements are equivalent:*
(i) $e \in eaeR \cap Reae$.
(ii) $eae + 1 - e$ is invertible (or $ae + 1 - e$ is invertible).

Theorem 2.5. *Let $a, b, c, d \in R$. Assume that $a^{\|(b \bowtie c)}$ exists. Let b^- be any inner inverses of b and set $e = bb^-$. Then the following statements are equivalent:*

(i) d has a hybrid (b, c)-inverse.

(ii) $e \in ea^{\|(b \bowtie c)}deR \cap Rea^{\|(b \bowtie c)}de$.

(iii) $a^{\|(b \bowtie c)}de + 1 - e$ is invertible.

In this case, $d^{\|(b \bowtie c)} = (a^{\|(b \bowtie c)}de + 1 - e)^{-1}a^{\|(b \bowtie c)}$.

Proof. (i) ⇒ (ii) Suppose that $d^{\|(b \bowtie c)}$ exists. Let $x = a^{\|(b \bowtie c)}$ and $y = d^{\|(b \bowtie c)}$. It follows from Lemma 2.3 that $x = xdy$ and $y = yax$. As $a^{\|(b \bowtie c)}$ exists, we have $x \in bR$, $y \in bR$ and $b = xab$ by Lemma 2.1. Therefore $b = xab = exab = e(xdy)ab = exdeyab$ since $ey = y$. Multiplying on the right by b^- gives $e = exdeyae$ and $e \in exdeR$. Moreover, as $d^{\|(b \bowtie c)}$ exists we have $y \in bR$ and $b = ydb$. Therefore $b = ydb = eydb = e(yax)db = eyaexdb$ since $ex = x$. Multiplying on the right by b^- we obtain $e = eyaexde$ and $e \in Rexde$.

(ii) ⇒ (iii) See Lemma 2.4.

(iii) ⇒ (i) Set $x = a^{\|(b \bowtie c)}$. Firstly we note that $ex = x$ by $xR = bR$. Set $u = exde + 1 - e$. It is clear that $eu = ue$ and $eu^{-1} = u^{-1}e$. Write $y = u^{-1}x$. Next, we verify that y is the hybrid (b, c)-inverse of d.

Step 1. $ydy = y$. Indeed, using $ex = x$ and $eu^{-1} = u^{-1}e$, we can check that

$$\begin{aligned} ydy &= u^{-1}xdu^{-1}x = u^{-1}exdeu^{-1}x \\ &= u^{-1}(exde + 1 - e)eu^{-1}x \\ &= eu^{-1}x = u^{-1}x = y. \end{aligned}$$

Step 2. $bR = yR$. Indeed, from $(1 - e)b = 0$ and $x = ex$, we have

$$b = u^{-1}ub = u^{-1}(exde + 1 - e)b = u^{-1}exdeb = u^{-1}xdeb = ydeb \in yR$$

Meanwhile, $y = u^{-1}x = u^{-1}ex = eu^{-1}x \in bR$. This shows that $bR = yR$.

Step 3. $\mathrm{rann}(c) = \mathrm{rann}(y)$. Since u is invertible element in R, we have $\mathrm{rann}(y) = \mathrm{rann}(x)$. Moreover, from Lemma 2.1, we have $\mathrm{rann}(x) = \mathrm{rann}(c)$. This leads to $\mathrm{rann}(c) = \mathrm{rann}(y)$. □

Next, the absorption law for the hybrid (b, c)-inverse is given when a and d are both hybrid (b, c)-invertible.

Theorem 2.6. *Let $a, b, c, d \in R$. If a is hybrid (b, c)-invertible and d is hybrid (b, c)-invertible, then $a^{\|(b \bowtie c)} + d^{\|(b \bowtie c)} = a^{\|(b \bowtie c)}(a + d)d^{\|(b \bowtie c)}$.*

Proof. Let $x = a^{\|(b \bowtie c)}$ and $y = d^{\|(b \bowtie c)}$. It follows from Lemma 2.3 that $xay = y$ and $xdy = x$, and consequently $x(a + d)y = xay + xdy = y + x$. □

By Theorem 2.6, we have the following corollary.

Corollary 2.7. *Let $a, b, c, d \in R$ If a is (b, c)-invertible and d is (b, c)-invertible, then $a^{\|(b,c)} + d^{\|(b,c)} = a^{\|(b,c)}(a + d)d^{\|(b,c)}$.*

Proof. If a is (b, c)-invertible and d is (b, c)-invertible, then a is hybrid (b, c)-invertible and d is hybrid (b, c)-invertible. Let $x = a^{\|(b,c)}$ and $y = d^{\|(b,c)}$. Then we have $x = a^{\|(b \bowtie c)}$ and $y = d^{\|(b \bowtie c)}$. It follows from Theorem 2.6 that $x + y = x(a + d)y$, and consequently $a^{\|(b,c)} + d^{\|(b,c)} = a^{\|(b,c)}(a + d)d^{\|(b,c)}$. □

Let $a, b, c, d \in R$. If a and d are both hybrid (b, c)-invertible, then the absorption law for the hybrid (b, c)-inverse holds by Theorem 2.6. If a is hybrid (b, c)-invertible and d is hybrid (u, v)-invertible for some $u, v \in R$, does the absorption law for $a^{\|(b \bowtie c)}$ and $d^{\|(u \bowtie v)}$ holds?

Example 2.8. *Let $\mathbb{C}^{2\times 2}$ denote the set of all 2×2 complex matrices over the complex field $\mathbb{C}$. Consider $a = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $d = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $b = c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $u = v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then it is easy to check that $a^{\|(b \bowtie c)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $d^{\|(u \bowtie v)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. It is clear that $a^{\|(b \bowtie c)} + d^{\|(b \bowtie c)} \neq a^{\|(b \bowtie c)}(a + d)d^{\|(b \bowtie c)}$.*

Following Green [7], Green's preorders and relations in a semigroup are defined. Similarly, we say the Green's preorder and relations in rings as

$$a \leq_{\mathcal{L}} b \Leftrightarrow Ra \subseteq Rb \Leftrightarrow \text{there exists } x \in R \text{ such that } a = xb.$$
$$a \leq_{\mathcal{R}} b \Leftrightarrow aR \subseteq bR \Leftrightarrow \text{there exists } x \in R \text{ such that } a = bx.$$
$$a \leq_{\mathcal{H}} b \Leftrightarrow a \leq_{\mathcal{L}} b \text{ and } a \leq_{\mathcal{R}} b.$$
$$a\mathcal{L}b \Leftrightarrow Ra = Rb \Leftrightarrow \text{there exist } x, y \in R \text{ such that } a = xb \text{ and } b = ya.$$
$$a\mathcal{R}b \Leftrightarrow aR = bR \Leftrightarrow \text{there exist } x, y \in R \text{ such that } a = bx \text{ and } b = ay.$$
$$a\mathcal{H}b \Leftrightarrow a\mathcal{L}b \text{ and } a\mathcal{R}b.$$

Before investigate the absorption law for $a^{\|(b\triangleright\triangleleft c)}$ and $d^{\|(u\triangleright\triangleleft v)}$ by using Green's preorders and relations, the following lemma is given.

Lemma 2.9. *Let $a, b, c, u, v \in R$. If $b\mathcal{R}u$ and $c\mathcal{L}v$, then a is hybrid (b, c)-invertible if and only if a is hybrid (u, v)-invertible. In this case, we have $a^{\|(b\triangleright\triangleleft c)} = a^{\|(u\triangleright\triangleleft v)}$.*

Proof. We present a proof of the necessity. As $b\mathcal{R}u$, then we have $u = b\gamma$ and $b = u\delta$ for some $\gamma, \delta \in R$. Moreover, by $c\mathcal{L}v$, it gives that $v = \alpha c$ and $c = \beta v$ for some $\alpha, \beta \in R$. Since a is hybrid (b, c)-invertible, by Lemma 2.2, there is $w \in R$ such that $c = cabw$. It follows that $v = \alpha c = \alpha(cabw) = (\alpha c)abw = vabw = vau\delta w$, and consequently $vR = vauR$. For any $x \in \text{rann}(vau)$, by $c = \beta v$, then $vaux = 0$ and $caux = (\beta v)aux = \beta vaux = 0$. Note that $u = b\gamma$, then $caux = cab\gamma x = 0$. Again, from Lemma 2.2, it follows $\gamma x \in \text{rann}(cab) = \text{rann}(b)$. This implies that $b\gamma x = 0$ and $ux = 0$, which gives $\text{rann}(vau) \subseteq \text{rann}(u)$. So, by Lemma 2.2, one can see that a is hybrid (u, v)-invertible. Moreover, from Lemma 2.1, it is not difficult to directly check that $a^{\|(b\triangleright\triangleleft c)} = a^{\|(u\triangleright\triangleleft v)}$. □

Theorem 2.10. *Let $a, b, c, d, u, v \in R$ with $b\mathcal{R}u$ and $c\mathcal{L}v$. If $a^{\|(b\triangleright\triangleleft c)}$ and $d^{\|(u\triangleright\triangleleft v)}$ exist, then $a^{\|(b\triangleright\triangleleft c)} + d^{\|(u\triangleright\triangleleft v)} = a^{\|(b\triangleright\triangleleft c)}(a + d)d^{\|(u\triangleright\triangleleft v)}$.*

Proof. Since $b\mathcal{R}u$ and $c\mathcal{L}v$, by Lemma 2.9 we have $a^{\|(b\triangleright\triangleleft c)} = a^{\|(u\triangleright\triangleleft v)}$. Therefore, by Theorem 2.6, one can see that

$$\begin{aligned} a^{\|(b\triangleright\triangleleft c)} + d^{\|(u\triangleright\triangleleft v)} &= a^{\|(u\triangleright\triangleleft v)} + d^{\|(u\triangleright\triangleleft v)} \\ &= a^{\|(u\triangleright\triangleleft v)}(a + d)d^{\|(u\triangleright\triangleleft v)} \\ &= a^{\|(b\triangleright\triangleleft c)}(a + d)d^{\|(u\triangleright\triangleleft v)}. \end{aligned}$$

□

An involutory ring R means that R is a unital ring with involution, i.e., a ring with unity 1, and a mapping $a \mapsto a^*$ from R to R such that $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a + b)^* = a^* + b^*$, for all $a, b \in R$. Let R be an involutory ring and $a \in R$. By [2, P.1910] and [12, Theorem 3.10], we have that a is Moore-Penrose invertible if and only if a is (a^*, a^*)-invertible if and only if a is hybrid (a^*, a^*)-invertible. Let R be an associative ring and $a \in R$. a is Drazin invertible if and only if there exists $k \in \mathbb{N}$ such that a is (a^k, a^k)-invertible if and only if there exists $k \in \mathbb{N}$ such that a is hybrid (a^k, a^k)-invertible, where the positive integer k is the Drazin index of a, denoted by $\text{ind}(a)$. a is group invertible if and only if a is (a, a)-invertible if and only if a is hybrid (a, a)-invertible. As applications of Theorem 2.10, we have the following corollary. We use the symbols $a^\dagger$, $a^\sharp$ and a^D to denote the Moore-Penrose inverse, the group inverse and the Drazin inverse of a.

Corollary 2.11. *Let $a, b \in R$. Then*

(*i*) *If $a^\dagger$ and $b^\dagger$ exist with $a\mathcal{H}b$, then $a^\dagger + b^\dagger = a^\dagger(a + b)b^\dagger$.*

(*ii*) *If $a^\sharp$ and $b^\sharp$ exist with $a\mathcal{H}b$, then $a^\sharp + b^\sharp = a^\sharp(a + b)b^\sharp$.*

(*iii*) *If a^D and b^D exist with $a^n\mathcal{H}b^m$, where $ind(a) = n$ and $ind(b) = m$, then $a^D + b^D = a^D(a + b)b^D$.*

3. Reverse order law for the hybrid (b, c)-inverse

Let $a, b \in R$ be two invertible elements. It is well known that

$$(ab)^{-1} = b^{-1}a^{-1}.$$

The above equality is known as the reverse order law of invertible elements. In general, the reverse order law does not hold for generalized inverses (see [1, 11]). In this section, the reverse order laws for the hybrid (b, c)-inverse are obtained.

Theorem 3.1. *Let $a, b, c, d \in R$ such that $a^{\|(b\bowtie c)}$ and $d^{\|(b\bowtie c)}$ exist. If $aa^{\|(b\bowtie c)} = a^{\|(b\bowtie c)}a$, then ad is hybrid (b, c)-invertible and $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$.*

Proof. Let $x = a^{\|(b\bowtie c)}$, $y = d^{\|(b\bowtie c)}$ and $z = yx$. We verify that z is the hybrid (b, c)-inverse of ad.

Step 1. $zadz = z$. Indeed, by Lemma 2.3, we know that $xdy = x$ and $yax = y$, which give that $z(ad)z = yxadyx = y(xa)dyx = yaxdyx = ya(xdy)x = yaxx = (yax)x = yx = z$.

Step 2. $zR = bR$. Indeed, as $yR = bR$, then we have $zR = yxR \subseteq bR = yR = yaxR = yxaR \subseteq yxR = zR$, which gives $zR = bR$.

Step 3. $\mathrm{rann}(z) = \mathrm{rann}(c)$. It is easy to get $\mathrm{rann}(c) = \mathrm{rann}(x) \subseteq \mathrm{rann}(yx) = \mathrm{rann}(z)$.

Next, we claim that $\mathrm{rann}(z) \subseteq \mathrm{rann}(c)$. Given any $t \in \mathrm{rann}(z)$, then $yxt = 0$, i.e., $xt \in \mathrm{rann}(y) = \mathrm{rann}(c)$. Moreover, since $ax = xa$ and $x = xax$, it gives that $x = ax^2$. It follows from $xR = bR$ that $xt = ax^2t \in abR$. Hence, one can see that $xt \in \mathrm{rann}(c) \cap abR$. By [14, Theorem 2.4], we know that $\mathrm{rann}(c) \cap abR = \{0\}$, which gives $xt = 0$. Therefore, it implies $t \in \mathrm{rann}(x) = \mathrm{rann}(c)$, and consequently $\mathrm{rann}(z) \subseteq \mathrm{rann}(x) = \mathrm{rann}(c)$. □

Remark 3.2. *By [12, Proposition 3.3], we know that if a is (b, c)-invertible, then b and c are both regular. Moreover, from Theorem 3.1, if $a^{\|(b\bowtie c)}$ and $d^{\|(u\bowtie v)}$ exist with $aa^{\|(b\bowtie c)} = a^{\|(b\bowtie c)}a$, then $z = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$ is regular and $\mathrm{rann}(z) = \mathrm{rann}(c)$.*

Lemma 3.3. *[12, Lemma 3.2] Let $a \in R$ be regular. Then $\mathrm{lann}(\mathrm{rann}(a)) = Ra$.*

In view of Remark 3.2 and Lemma 3.3, we obtain the following result.

Corollary 3.4. *Let $a, b, c, d \in R$ such that $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist. If $a^{\|(b,c)}a = aa^{\|(b,c)}$ then ad is (b, c)-invertible and $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$.*

Proof. From Theorem 3.1 and Remark 3.2, one can see $z = d^{\|(b,c)}a^{\|(b,c)}$ is regular and $\mathrm{rann}(z) = \mathrm{rann}(c)$. As $a^{\|(b,c)}$ exists, it follows from [12, Proposition 3.3] that c is regular. Then, we obtain $Rz = \mathrm{lann}(\mathrm{rann}(z)) = \mathrm{lann}(\mathrm{rann}(c)) = Rc$. On account of [2, Proposition 6.1], we conclude that ad is (b, c)-invertible and $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$. □

Lemma 3.5. *Let $a, b, c \in R$ with $ab \leq_{\mathcal{R}} ba$ and $ac \leq_{\mathcal{L}} ca$. If $a^{\|(b\bowtie c)}$ exists, then $aa^{\|(b\bowtie c)} = a^{\|(b\bowtie c)}a$.*

Proof. Let $x = a^{\|(b\bowtie c)}$. Since $ab \leq_{\mathcal{R}} ba$ and $ac \leq_{\mathcal{L}} ca$, there is $ab = ba\mu$ and $ca = \nu ac$ for some $\mu, \nu \in R$. Hence, it follows from $c = cax$ that $ca = \nu ac = \nu a(cax) = (\nu ac)ax = caax$. Note that $a^{\|(b\bowtie c)}$ exists, it gives $\mathrm{rann}(x) = \mathrm{rann}(c)$, and consequently $a - aax \in \mathrm{rann}(c) = \mathrm{rann}(x)$, which implies $xa = xaax$. Moreover, by $bR = xR$, we have $x = bs$ for some $s \in R$. On account of $b = xab$, we conclude that $ax = a(bs) = (ab)s = (ba\mu)s = (xab)a\mu s = xa(ba\mu)s = xaa(bs) = xaax$. Thus, $ax = xa$, as required. □

In view of Theorem 3.1 and Lemma 3.5, we obtain the following result.

Theorem 3.6. *Let $a, b, c, d \in R$ with $ab \leq_{\mathcal{R}} ba$ and $ac \leq_{\mathcal{L}} ca$. If $a^{\|(b\bowtie c)}$ and $d^{\|(b\bowtie c)}$ exist, then ad is hybrid (b, c)-invertible and $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$.*

Corollary 3.7. *Let $a, b, c, d \in R$ such that $ab = ba$ and $ac = ca$. If $a^{\|(b\bowtie c)}$ and $d^{\|(b\bowtie c)}$ exist, then ad is hybrid (b, c)-invertible and $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$.*

In view of Lemma 3.3 and Corollary 3.7, we obtain the following result.

Corollary 3.8. *Let $a, b, c, d \in R$ such that $ab = ba$ and $ac = ca$. If $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist, then ad is (b, c)-invertible and $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$.*

Theorem 3.9. *Let $a, b, c, d \in R$ with $db \leq_{\mathcal{R}} bd$ and $ca \leq_{\mathcal{L}} ac$. If $a^{\|(b\bowtie c)}$ and $d^{\|(b\bowtie c)}$ exist, then ad is hybrid (b, c)-invertible and $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$.*

Proof. Let $x = a^{\|(b\bowtie c)}$, $y = d^{\|(b\bowtie c)}$ and $z = yx$. Since $db \leq_{\mathcal{R}} bd$ and $ca \leq_{\mathcal{L}} ac$, there is $db = bd\mu$ and $ca = \nu ac$ for some $\mu, \nu \in R$. It follows from $b = xab = ydb$ that $z(ad)b = yxa(db) = y(xab)d\mu = y(bd\mu) = ydb = b$. Moreover, by $c = cdy = cax$, we have $c(ad)z = (ca)dyx = \nu a(cdy)x = (\nu ac)x = cax = c$. Since $yxR \subseteq yR = bR$ and $bR = z(ad)bR \subseteq zR$, we have $zR = bR$. Note that $\mathrm{rann}(c) = \mathrm{rann}(x) \subseteq \mathrm{rann}(yx) = \mathrm{rann}(z)$ and $c = c(ad)z$. Then $\mathrm{rann}(c) = \mathrm{rann}(z)$. On account of [14, Proposition 2.1] we conclude that ad is hybrid (b, c)-invertible and $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$. □

Corollary 3.10. *Let $a, b, c, d \in R$ such that $bd = db$ and $ac = ca$. If $a^{\|(b\bowtie c)}$ and $d^{\|(b\bowtie c)}$ exist, then ad is hybrid (b, c)-invertible and $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$.*

In view of Lemma 3.3 and Corollary 3.10, we obtain the following result.

Corollary 3.11. *Let $a, b, c, d \in R$ such that $bd = db$ and $ac = ca$. If $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist, then ad is (b, c)-invertible and $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$.*

Since $a^{\|(b\bowtie c)}$ is an outer inverse of a when it exists, both $aa^{\|(b\bowtie c)}$ and $a^{\|(b\bowtie c)}a$ are idempotents. These will be referred to as the hybrid (b, c)-idempotents associated with a. We are interested in finding characterizations of those elements in the ring with equal hybrid (b, c)-idempotents. In fact, it is also closely related to the reverse order law. We use the symbol $R^\sharp$ to denote the set of all group invertible elements.

Theorem 3.12. *Let $a, b, c, d \in R$ such that $a^{\|(b\bowtie c)}$ and $d^{\|(b\bowtie c)}$ exist. Then the following statements are equivalent:*
(i) $aa^{\|(b\bowtie c)} = dd^{\|(b\bowtie c)}$.
(ii) $aa^{\|(b\bowtie c)}dd^{\|(b\bowtie c)} = dd^{\|(b\bowtie c)}aa^{\|(b\bowtie c)}$.
(iii) $ad^{\|(b\bowtie c)}da^{\|(b\bowtie c)} = da^{\|(b\bowtie c)}ad^{\|(b\bowtie c)}$.
(iv) $ad^{\|(b\bowtie c)} \in R^\sharp$ *and* $(ad^{\|(b\bowtie c)})^\sharp = da^{\|(b\bowtie c)}$.
(v) $da^{\|(b\bowtie c)} \in R^\sharp$ *and* $(da^{\|(b\bowtie c)})^\sharp = ad^{\|(b\bowtie c)}$.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$. Let $x = a^{\|(b\bowtie c)}$ and $y = d^{\|(b\bowtie c)}$. From Lemma 2.3 we obtain

$$\begin{aligned} &x = xdy = ydx;\\ &y = yax = xay. \end{aligned} \tag{1}$$

Hence,

$$\begin{aligned} ax = dy \quad &\Leftrightarrow \quad axdy = dyax\\ &\Leftrightarrow \quad aydx = dxay. \end{aligned}$$

$(iii) \Leftrightarrow (iv)$. Set $g = da^{\|(b\bowtie c)}$. We will prove that x is the group inverse of $ad^{\|(b\bowtie c)}$. Using (iii) and Lemma 2.3, we get

$$\begin{aligned} gad^{\|(b\bowtie c)} &= dxay = aydx = ad^{\|(b\bowtie c)}g;\\ ad^{\|(b\bowtie c)}gad^{\|(b\bowtie c)} &= a(ydx)ay = a(xay) = ay = ad^{\|(b\bowtie c)};\\ gad^{\|(b\bowtie c)}g &= gaydx = ga(ydx) = gax = dxax = dx = g. \end{aligned}$$

This implies that $ad^{\|(b\bowtie c)} \in R^\sharp$ and $(ad^{\|(b\bowtie c)})^\sharp = da^{\|(b\bowtie c)}$.

Conversely, if the latter holds, then $gad^{\|(b\bowtie c)} = ad^{\|(b\bowtie c)}g$ i.e., $da^{\|(b\bowtie c)}ad^{\|(b\bowtie c)} = ad^{\|(b\bowtie c)}da^{\|(b\bowtie c)}$.

$(iii) \Leftrightarrow (v)$. The proof is similar to the previous equivalence. □

Next, we consider conditions under which the reverse order law for the hybrid (b,c)-inverse of the product ad, $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$ holds.

Theorem 3.13. *Let $a,b,c,d \in R$ such that $a^{\|(b\bowtie c)}$ and $d^{\|(b\bowtie c)}$ exist. Then the following statements are equivalent:*
(i) ad has a hybrid (b,c)-inverse of the form $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$.
(ii) $d^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}add^{\|(b\bowtie c)}a^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}add^{\|(b\bowtie c)}$.
(iii) $a^{\|(b\bowtie c)} = a^{\|(b\bowtie c)}add^{\|(b\bowtie c)}a^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}ada^{\|(b\bowtie c)}$.

Proof. $(i) \Leftrightarrow (ii)$. Suppose that ad has a hybrid (b,c)-inverse, and $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$. Then Lemma 2.3 is true for $(ad)^{\|(b\bowtie c)}$ in place of $a^{\|(b\bowtie c)}$. It follows that

$$d^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}ad(ad)^{\|(b\bowtie c)} = (ad)^{\|(b\bowtie c)}add^{\|(b\bowtie c)}.$$

Substituting $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$ yields

$$d^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}add^{\|(b\bowtie c)}a^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}add^{\|(b\bowtie c)}.$$

Conversely, if the latter identities hold, we claim $z = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$ is the hybrid (b,c)-inverse of ad. Write $x = a^{\|(b\bowtie c)}$ and $y = d^{\|(b\bowtie c)}$. Indeed, it is clear that $z = yx \in yR = bR$. Moreover, it is also easy to find $\mathrm{rann}(c) = \mathrm{rann}(x) \subseteq \mathrm{rann}(yx) = \mathrm{rann}(z)$. On account of $ydb = b$ and $y = yxady$ in the condition (ii), we conclude that

$$zadb = yxadb = yxad(ydb) = (yxady)db = ydb = b.$$

Similarly, in view of $y = yadyx$ in the condition (ii) and $cdy = c$, one can see that

$$cadz = cadyx = (cdy)adyx = cd(yadyx) = cdy = c.$$

Then ad has a hybrid (b,c)-inverse of the form $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$ by [14, Proposition 2.1].

$(ii) \Rightarrow (iii)$. By Lemma 2.3 we have $x = xdy = ydx$. From the condition (ii), one can see that

$$x = xdy = xd(yadyx) = (xdy)adyx = xadyx.$$

That is, $a^{\|(b\bowtie c)} = a^{\|(b\bowtie c)}add^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$. Moreover, again from the condition (ii), it follows

$$x = ydx = (yxady)dx = yxad(ydx) = yxadx.$$

That is, $a^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}ada^{\|(b\bowtie c)}$.

$(iii) \Rightarrow (ii)$. The proof is similar to $(ii) \Rightarrow (iii)$. □

We close this section with the characterization of $a^{\|(b\bowtie c)}a = dd^{\|(b\bowtie c)}$ in rings.

Theorem 3.14. *Let $a,b,c,d \in R$ such that $a^{\|(b\bowtie c)}$ and $d^{\|(b\bowtie c)}$ exist. Then the following statements are equivalent:*
(i) $a^{\|(b\bowtie c)}a = dd^{\|(b\bowtie c)}$.
(ii) $a^{\|(b\bowtie c)}dd^{\|(b\bowtie c)}a = dd^{\|(b\bowtie c)}aa^{\|(b\bowtie c)}$.
(iii) $d^{\|(b\bowtie c)}da^{\|(b\bowtie c)}a = da^{\|(b\bowtie c)}ad^{\|(b\bowtie c)}$.
(iv) $a^{\|(b\bowtie c)} = dd^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$ and $d^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}a$.
(v) $a^{\|(b\bowtie c)}ad^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}a$ and $a^{\|(b\bowtie c)}dd^{\|(b\bowtie c)} = dd^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$.
If any of the previous statements is valid, then $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$.

Proof. Let $x = a^{\|(b\bowtie c)}$ and $y = d^{\|(b\bowtie c)}$. From Lemma 2.3 we obtain (3.1), that is,

$$x = xdy = ydx;$$
$$y = yax = xay.$$

$(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$. By (1), it is clear that

$$xa = xdya = ydxa;$$
$$dy = dyax = dxay.$$

Hence, it follows that

$$\begin{aligned} xa = dy \quad &\Leftrightarrow \quad xdya = dyax \\ &\Leftrightarrow \quad ydxa = dxay. \end{aligned}$$

$(i) \Leftrightarrow (iv)$. The necessary condition is immediate. Next, we assume that $x = dyx$ and $y = yxa$. Then we have $xa = dyxa$ and $dy = dyxa$, consequently $xa = dy$, as desired.

$(v) \Leftrightarrow (i)$. The proof is similar to the above.

Finally, we will prove that $dy = xa$ implies that ad has a hybrid (b, c)-inverse given by $(ad)^{\|(b\bowtie c)} = d^{\|(b\bowtie c)}a^{\|(b\bowtie c)}$. From $y = ydy$ and $dy = xa$, it gives $y = yxa$, and consequently $y = ydy = (yxa)dy$. Moreover, note that $y = yax$ and $dy = xa$, it follows that $y = yax = (yax)ax = ya(dy)x$. By Theorem 3.13 (ii) our assertion is proved. □

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