Filomat 33:15 (2019), 4967–4979
https://doi.org/10.2298/FIL1915967B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Hermite-Hadamard Type Inequalities for Twice Differantiable Functions via Generalized Fractional Integrals

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Abstract. In this paper we first obtain two generalized identities for twice differentiable mappings involving generalized fractional integrals defined by Sarikaya and Ertuğral. Then we establish some midpoint and trapezoid type inequalities for functions whose second derivatives in absolute value are convex.

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[9], [14], [28, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics. A number of mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions such as convex mappings.

The overall structure of the study takes the form of six sections including introduction. The remainder of this work is organized as follows: we first mention some works which focus on Hermite-Hadamard inequality. In Section 2, we introduce the generalized fractional integrals defined by Sarikaya and Ertuğral along with the very first results. In section 3 we prove an identity for twice differentiable functions and using this equality we prove some trapezoid type inequalities for twice differentiable mappings. In Section 4 by giving an identity, some midpoint type inequalities for functions whose second derivatives in absolute value are convex are presented.

Barani et al. established inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality, and they used the following lemma to prove their results:

2010 *Mathematics Subject Classification*. Primary 26D07; Secondary 26D10, 26D15, 26A33
Keywords. Hermite-Hadamard inequality, midpoint inequality, fractional integral operators, convex function.
Received: 23 May 2019; Accepted: 04 September 2019
Communicated by Miodrag Spalević
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Lemma 1.1. *([4],[5]) Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f'' \in L_1[a,b]$, then we have*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \tag{2}$$

$$= \frac{(b-a)^2}{16}\int_0^1 (1-t^2)\left[f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) + f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right]dt.$$

Over the last twenty years, the numerous studies have focused on to obtain new bound for left hand side and right hand side of the inequality (1). For some examples, please refer to ([1], [3], [6], [7], [10], [15], [25], [30]-[32]).

On the other hand, Sarikaya et al. obtain the Hermite-Hadamard inequality for the Riemann-Lioville fractional integrals in [36]. Then, many authors have studied to generalize this inequality and establish Hermite-Hadamard inequality for other fractional integrals such as k-fractional integral, Hadamard fractional integrals, Katugampola fractional integrals, Conformable fractional integrals, etc. For some of them, you can check the refrences ([2], [8], [11], [12], [16]-[22], [24], [26], [27], [29], [33], [35], [37]-[42]). For more information about fractional calculus please refer to ([13], [23]).

In this paper, we obtain the new generalized trapezoid and midpoint type inequality for the generalized fractional integrals mentioned in the next section.

2. New Generalized Fractional Integral Operators

In this section, we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [34].

Let's define a function $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions :

$$\int_0^1 \frac{\varphi(t)}{t}dt < \infty.$$

We define the following generalized fractional integral operators, as follows:

$$_{a^+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t}f(t)dt, \quad x > a, \tag{3}$$

$$_{b^-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x}f(t)dt, \quad x < b. \tag{4}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integrals, k-Riemann-Liouville fractional integrals, Katugampola fractional integrals, Conformable fractional integral, Hadamard fractional integrals, etc. These important special cases of the integral operators (3) and (4) are mentioned below.

i) If we take $\varphi(t) = t$, the operator (3) and (4) reduce to the Riemann integral as follows:

$$I_{a^+}f(x) = \int_a^x f(t)dt, \quad x > a,$$

$$I_{b^-}f(x) = \int_x^b f(t)dt, \quad x < b.$$

ii) If we take $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}, \alpha > 0$, the operators (3) and (4) reduce to the Riemann-Liouville fractional integrals as follows:

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

$$I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b.$$

iii) If we take $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)} t^{\frac{\alpha}{k}}, \alpha, k > 0$, the operators (3) and (4) reduce to the k-Riemann-Liouville fractional integrals as follows:

$$I_{a^+,k}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a,$$

$$I_{b^-,k}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b$$

where

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \quad \mathcal{R}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0; k > 0$$

which are given by Mubeen and Habibullah in [26].

iv) If we take $\varphi(t) = t(x-t)^{\alpha-1}$, the operator (3) reduces to the conformable fractional operators as follows:

$$I_a^{\alpha} f(x) = \int_a^x t^{\alpha-1} f(t)dt = \int_a^x f(t)d_{\alpha}t, \quad x > a, \ \alpha \in (0,1)$$

which is given by Khalil et.al in [22].

v) If we take

$$\varphi(t) = \frac{1}{\Gamma(\alpha)} \frac{[(\log x - \log(x-t)]^{\alpha-1}}{x-t}$$

and

$$\varphi(t) = \frac{1}{\Gamma(\alpha)} t \frac{[(\log(t-x) - \log x]^{\alpha-1}}{t-x},$$

in the operators (3) and (4), respectively, the operator (3) and (4) reduce to the right-sided and left-sided Hadamard fractional integrals as follows [23]:

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\log x - \log t)^{\alpha-1} \frac{f(t)}{t} dt, \quad 0 < a < x < b,$$

$$I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\log t - \log x)^{\alpha-1} \frac{f(t)}{t} dt, \quad 0 < a < x < b.$$

vii) If we take $\varphi(t) = \frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$ in the operators (3) and (4), respectively, the operator (3) and (4) reduce to the right-sided and left-sided fractional integral operators with exponential kernel for $\alpha \in (0,1)$ as follows:

$$\mathcal{I}^{\alpha}_{a^+} f(x) = \frac{1}{\alpha}\int_a^x \exp\left(-\frac{1-\alpha}{\alpha}(x-t)\right) f(t)dt, \quad a < x,$$

$$\mathcal{I}^{\alpha}_{b^-} f(x) = \frac{1}{\alpha}\int_x^b \exp\left(-\frac{1-\alpha}{\alpha}(t-x)\right) f(t)dt, \quad x < b$$

which are defined by Kirane and Torebek in [24].

Sarıkaya and Ertuğral also establish the following Hermite-Hadamard inequality for the generalized fractional integral operators:

Theorem 2.1. *[34] Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ with $a < b$, then the following inequalities for fractional integral operators hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Psi(1)}\left[{}_{a+}I_{\varphi} f(b) + {}_{b-}I_{\varphi} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{5}$$

where the mapping $\Psi : [0,1] \to \mathbb{R}$ is defined by

$$\Psi(x) = \int_0^x \frac{\varphi((b-a)t)}{t} dt.$$

3. Trapezoid Type Inequalities for Generalized Fractional Integral Operators

In this section, we obtain some trapezoid type inequalities for functions whose second derivatives in absolute value are convex.

Lemma 3.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be an absolutely continuous mapping on I° such that $f'' \in L([a,b])$, where $a, b \in I^{\circ}$ with $a < b$. Then the following equality for generalized fractional integrals holds:*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{1}{2\nabla(1)}\left[{}_{b-}I_{\varphi} f\left(\frac{a+b}{2}\right) + {}_{a+}I_{\varphi} f\left(\frac{a+b}{2}\right)\right] \\
&= \frac{(b-a)^2}{8\nabla(1)} \int_0^1 \Delta(t)\left[f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) + f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right] dt
\end{aligned} \tag{6}$$

where

$$\Delta(t) = \int_t^1 \nabla(s)ds, \quad \nabla(s) = \int_0^s \frac{\varphi\left(\frac{b-a}{2}u\right)}{u} du. \tag{7}$$

Proof. First, we consider

$$I = \int_0^1 \Delta(t) f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt + \int_0^1 \Delta(t) f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)dt = I_1 + I_2. \tag{8}$$

Calculating I_1 and I_2 by integration by parts twice, we have

$$\begin{aligned}
I_1 &= \int_0^1 \Delta(t) f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)dt \\
&= \Delta(t)\frac{2}{b-a}f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\Big|_0^1 - \frac{2}{b-a}\int_0^1 \nabla(t) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt \\
&= \Delta(0)\frac{2}{b-a}f'\left(\frac{a+b}{2}\right) \\
&\quad -\frac{2}{b-a}\left[-\nabla(t)\frac{2}{b-a}f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\Big|_0^1 + \frac{2}{b-a}\int_0^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt\right] \\
&= \Delta(0)\frac{2}{b-a}f'\left(\frac{a+b}{2}\right) + \nabla(1)\frac{4}{(b-a)^2}f(a) - \frac{4}{(b-a)^2}\int_0^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt \\
&= \Delta(0)\frac{2}{b-a}f'\left(\frac{a+b}{2}\right) + \nabla(1)\frac{4}{(b-a)^2}f(a) - \frac{4}{(b-a)^2}\int_a^{\frac{a+b}{2}} \frac{\varphi\left(\frac{a+b}{2}-x\right)}{\frac{2}{b-a}\left(\frac{a+b}{2}-x\right)} f(x)\frac{2}{b-a}dx \\
&= \Delta(0)\frac{2}{b-a}f'\left(\frac{a+b}{2}\right) + \frac{4}{(b-a)^2}\left[\nabla(1)f(a) - {}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]
\end{aligned}$$

and similarly,

$$\begin{aligned}
I_2 &= \int_0^1 \Delta(t) f''\left(\frac{1-t}{2}b + \frac{1+t}{2}a\right)dt \\
&= -\Delta(0)\frac{2}{b-a}f'\left(\frac{a+b}{2}\right) + \frac{4}{(b-a)^2}\left[\nabla(1)f(b) - {}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right].
\end{aligned}$$

Substituting I_1 and I_2, then multiplying the result by $\frac{(b-a)^2}{8\nabla(1)}$, we get the desired result. □

Remark 3.2. *If we choose $\varphi(t) = t$ in Lemma 3.1, then the identity (6) reduces to the inequality (2)..*

Corollary 3.3. *If we choose $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Lemma 3.1, then we have the following identity*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I_{b-}^{\alpha}f\left(\frac{a+b}{2}\right) + I_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right] \\
= &\frac{(b-a)^2}{8(\alpha+1)}\int_0^1 \left(1-t^{\alpha+1}\right)\left[f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) + f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right]dt.
\end{aligned}$$

Theorem 3.4. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L([a,b])$,where $a, b \in I^\circ$ with $a < b$. If the function $\left|f''\right|$ is convex on $[a,b]$, then we have the following inequality for generalized fractional integral* *operators*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\nabla(1)}\left[{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)+_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{(b-a)^2}{4\nabla(1)}\left[\int_0^1 \Delta(t)dt\right]\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right]$$

where $\Delta(t)$ *is defined as in (7).*

Proof. Taking modulus in Lemma 3.1 and using the convexity of $\left|f''\right|$, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\nabla(1)}\left[{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)+_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$= \frac{(b-a)^2}{8\nabla(1)}\left|\int_0^1 \Delta(t)f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt+\int_0^1 \Delta(t)f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right|$$

$$\leq \frac{(b-a)^2}{8\nabla(1)}\left[\int_0^1 |\Delta(t)|\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1 |\Delta(t)|\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt\right]$$

$$\leq \frac{(b-a)^2}{8\nabla(1)}\left[\int_0^1 |\Delta(t)|\left(\frac{1+t}{2}\left|f''(a)\right|+\frac{1-t}{2}\left|f''(b)\right|\right)dt+\int_0^1 |\Delta(t)|\left(\frac{1-t}{2}\left|f''(a)\right|+\frac{1+t}{2}\left|f''(b)\right|\right)dt\right]$$

$$= \frac{(b-a)^2}{8\nabla(1)}\left(\left|f''(a)\right|+\left|f''(b)\right|\right)\left(\int_0^1 |\Delta(t)|\,dt\right).$$

Hence, the proof is completed. □

Remark 3.5. *If we choose* $\varphi(t)=t$ *in Theorem 3.4, then we have the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq \frac{(b-a)^2}{12}\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right]$$

which is given by Sarıkaya and Aktan [33].

Corollary 3.6. *If we choose* $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ *in Theorem 3.4, then we have the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I_{b-}f\left(\frac{a+b}{2}\right)+I_{a+}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{(b-a)^2}{4(\alpha+2)}\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right].$$

Theorem 3.7. *Let* $f: I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ *be twice differentiable function on* I° *such that* $f''\in L([a,b])$*,where* $a,b\in I^{\circ}$ *with* $a<b$. *If the function* $\left|f''\right|^{q}, q>1$ *is convex on* $[a,b]$*, then we have the following inequality for generalized fractional* *integral operators*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\nabla(1)}\left[{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]\right| \tag{9}$$

$$\leq \frac{(b-a)^2}{8\nabla(1)}\left(\int_0^1|\Delta(t)|^p\,dt\right)^{\frac{1}{p}}\left\{\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}$$

$$\leq \frac{(b-a)^2}{2^{\frac{2}{q}}\nabla(1)}\left(\int_0^1|\Delta(t)|^p\,dt\right)^{\frac{1}{p}}\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right]$$

where $\frac{1}{q}+\frac{1}{p}=1$.

Proof. Using the convexity of $\left|f''\right|^q$ on $[a,b]$, Lemma 3.1 and Hölder's inequality we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\nabla(1)}\left[{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{(b-a)^2}{8\nabla(1)}\left[\int_0^1|\Delta(t)|\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1|\Delta(t)|\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt\right]$$

$$\leq \frac{(b-a)^2}{8\nabla(1)}\left\{\left(\int_0^1|\Delta(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1|\Delta(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq \frac{(b-a)^2}{8\nabla(1)}\left(\int_0^1|\Delta(t)|^p\,dt\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_0^1\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(b-a)^2}{8\nabla(1)}\left(\int_0^1|\Delta(t)|^p\,dt\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_0^1\left[\frac{1+t}{2}\left|f''(a)\right|^q+\frac{1-t}{2}\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}+\left(\int_0^1\left[\frac{1-t}{2}\left|f''(a)\right|^q+\frac{1+t}{2}\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(b-a)^2}{8\nabla(1)}\left(\int_0^1|\Delta(t)|^p\,dt\right)^{\frac{1}{p}}\left\{\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}$$

which completes the proof of first inequality in (9).

For the proof of second inequality, let $a_1 = 3\left|f'(a)\right|^q$, $b_1 = \left|f'(b)\right|^q$, $a_2 = \left|f'(a)\right|^q$ and $b_2 = 3\left|f'(b)\right|^q$. Using the facts that

$$\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s,\ 0 \leq s < 1 \tag{10}$$

and $3^{\frac{1}{q}}+1 \leq 4$, the desired result can be obtained straightforwardly. □

Corollary 3.8. *If we choose $\varphi(t) = t$ in Theorem 3.7, then we have the following inequality*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{2}{(b-a)}\int_a^b f(t)dt\right| \\
\leq\ & \frac{(b-a)^2}{4}\left(\frac{2p}{2p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{3\left|f''(a)\right|^q + \left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{\left|f''(a)\right|^q + 3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\} \\
\leq\ & \frac{(b-a)^2}{2^{1+\frac{2}{q}}}\left(\frac{2p}{2p+1}\right)^{\frac{1}{p}}\left[\frac{\left|f''(a)\right| + \left|f''(b)\right|}{2}\right].
\end{aligned}$$

Proof. The proof is obvious from the using the fact that

$$(A-B)^p \leq A^p - B^p \tag{11}$$

for $A > B \geq 0$ and $p \geq 0$, thus

$$\int_0^1 (1-t^2)^p dt \leq \int_0^1 (1-t^{2p})dt = \frac{2p}{2p+1}.$$

□

Corollary 3.9. *If we choose $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 3.7, then we have the following inequality*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I_{b-}^{\alpha} f(\frac{a+b}{2}) + I_{a+}^{\alpha} f\left(\frac{a+b}{2}\right)\right]\right| \\
\leq\ & \frac{(b-a)^2}{8(\alpha+1)^{\frac{1}{p}}}\left(\frac{p(\alpha+1)}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left\{\left(\frac{3\left|f''(a)\right|^q + \left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{\left|f''(a)\right|^q + 3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\} \\
\leq\ & \frac{(b-a)^2}{2^{\frac{2}{q}}(\alpha+1)}\left(\frac{p(\alpha+1)}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left[\frac{\left|f''(a)\right| + \left|f''(b)\right|}{2}\right].
\end{aligned}$$

Proof. The proof is obvious from the inequality (11). □

4. Midpoint Type Inequalities for Generalized Fractional Integral Operators

In this section, we obtain some midpoint type inequalities for twice differantiable mappings.

Lemma 4.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be an absolutely continuous mapping on I° such that $f'' \in L([a.b])$,where $a, b \in I^{\circ}$ with $a < b$. Then the following equality for generalized fractional integrals holds:*

$$\frac{1}{2\Phi(0)}\left[{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right) +_{a+} I_{\varphi}f\left(\frac{a+b}{2}\right)\right] - f\left(\frac{a+b}{2}\right)$$

$$= \frac{(b-a)^2}{8\Phi(0)} \int_0^1 \Lambda(t)\left[f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) + f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right]dt \tag{12}$$

where

$$\Lambda(t) = \int_t^1 \Phi(s)ds, \quad \Phi(s) = \int_s^1 \frac{\varphi\left(\frac{b-a}{2}u\right)}{u}du. \tag{13}$$

Proof. Firstly, we take

$$I = \int_0^1 \Lambda(t)f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt + \int_0^1 \Lambda(t)f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)dt = I_1 + I_2. \tag{14}$$

Calculating I_1 and I_2 by integrating by parts twice, we have

$$\begin{aligned}
I_1 &= \int_0^1 \Lambda(t)f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt \\
&= -\Lambda(t)\frac{2}{b-a}f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\Big|_0^1 - \frac{2}{b-a}\int_0^1 \Phi(t)f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt \\
&= \frac{2}{b-a}\Lambda(0)f'\left(\frac{a+b}{2}\right) \\
&\quad -\frac{2}{b-a}\left[-\Phi(t)\frac{2}{b-a}f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\Big|_0^1 - \int_0^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t}f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt\right] \\
&= \frac{2}{b-a}\Lambda(0)f'\left(\frac{a+b}{2}\right) + \frac{4}{(b-a)^2}\left[-\Phi(0)f\left(\frac{a+b}{2}\right) +_{a+} I_{\varphi}f\left(\frac{a+b}{2}\right)\right]
\end{aligned}$$

and similarly

$$\begin{aligned}
I_2 &= \int_0^1 \Lambda(t)f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)dt \\
&= -\frac{2}{b-a}\Lambda(0)f'\left(\frac{a+b}{2}\right) + \frac{4}{(b-a)^2}\left[-\Phi(0)f\left(\frac{a+b}{2}\right) +_{b-} I_{\varphi}f\left(\frac{a+b}{2}\right)\right].
\end{aligned}$$

Substituting I_1 and I_2 in (14) and multiplying the result by $\frac{(b-a)^2}{8\Phi(0)}$, we get the desired result. □

Remark 4.2. *If we choose $\varphi(t) = t$ in Lemma 4.1, then we have the following equality*

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{16}\int_0^1 (1-t)^2\left[f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) + f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right]dt$$

which is given by Noor and Awan in [27].

Corollary 4.3. *If we choose $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Lemma 4.1, then we have the following equality*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)+I_{a+}^{\alpha} f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)$$

$$= \frac{(b-a)^2}{8(\alpha+1)}\int_0^1\left[(\alpha+1)(1-t)-1+t^{\alpha+1}\right]\left[f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt.$$

Theorem 4.4. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L([a,b])$,where $a, b \in I^\circ$ with $a < b$. If the function $\left|f''\right|$ is convex on $[a,b]$, then we have the following inequality for generalized fractional integral operators*

$$\left|\frac{1}{2\Phi(0)}\left[{}_{b-}I_\varphi f\left(\frac{a+b}{2}\right)+{}_{a+}I_\varphi f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^2}{4\Phi(0)}\left(\int_0^1|\Lambda(t)|\,dt\right)\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right].$$

Proof. In Lemma 4.1, by using the convexity of $\left|f''\right|$, we have

$$\left|\frac{1}{2\Phi(0)}\left[{}_{b-}I_\varphi f\left(\frac{a+b}{2}\right)+{}_{a+}I_\varphi f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^2}{8\Phi(0)}\left[\int_0^1|\Lambda(t)|\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1|\Lambda(t)|\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt\right]$$

$$\leq \frac{(b-a)^2}{8\Phi(0)}\left\{\int_0^1|\Lambda(t)|\left[\frac{1+t}{2}\left|f''(a)\right|+\frac{1-t}{2}\left|f''(b)\right|\right]dt+\int_0^1|\Lambda(t)|\left[\frac{1-t}{2}\left|f''(a)\right|+\frac{1+t}{2}\left|f''(b)\right|\right]dt\right\}$$

$$\leq \frac{(b-a)^2}{8\Phi(0)}\left\{\left|f''(a)\right|\int_0^1|\Lambda(t)|\left(\frac{1+t}{2}\right)dt+\left|f''(b)\right|\int_0^1|\Lambda(t)|\left(\frac{1-t}{2}\right)dt\right.$$

$$\left.+\left|f''(a)\right|\int_0^1|\Lambda(t)|\left(\frac{1-t}{2}\right)+\left|f''(b)\right|\int_0^1|\Lambda(t)|\left(\frac{1+t}{2}\right)dt\right\}$$

$$\leq \frac{(b-a)^2}{8\Phi(0)}\left(\int_0^1|\Lambda(t)|\,dt\right)\left(\left|f''(a)\right|+\left|f''(b)\right|\right).$$

This completes the proof. □

Remark 4.5. *If we choose $\varphi(t) = t$ in Theorem 4.4, then we have the following inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)^2}{24}\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right]$$

which was proved by Sarikaya et al. in [32].

Corollary 4.6. *If we choose $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 4.4, then we have the following inequality*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+I_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|\\ \leq\ &\frac{(b-a)^2}{4}\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right].\end{aligned}$$

Theorem 4.7. *Let $f:I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be twice differentiable function on I° such that $f''\in L([a,b])$,where $a,b\in I^{\circ}$ with $a<b$. If the function $\left|f''\right|^{q}$, $q>1$ is convex on $[a,b]$,then we have the following inequality for generalized fractional integral operators*

$$\begin{aligned}&\left|\frac{1}{2\Phi(0)}\left[{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \qquad (15)\\ \leq\ &\frac{(b-a)^2}{8\Phi(0)}\left(\int_0^1|\Lambda(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]\\ \leq\ &\frac{(b-a)^2}{2^{\frac{2}{q}}\Phi(0)}\left(\int_0^1|\Lambda(t)|^p\,dt\right)^{\frac{1}{p}}\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right]\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using the convexity of $\left|f''\right|^{q}$ on $[a,b]$ and Hölder's inequality in Lemma 4.1, we obtain

$$\begin{aligned}&\left|\frac{1}{2\Phi(0)}\left[{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|\\ \leq\ &\frac{(b-a)^2}{8\Phi(0)}\left[\int_0^1|\Lambda(t)|\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1|\Lambda(t)|\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt\right]\\ \leq\ &\frac{(b-a)^2}{8\Phi(0)}\left\{\left(\int_0^1|\Lambda(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1|\Lambda(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\ \leq\ &\frac{(b-a)^2}{8\Phi(0)}\left(\int_0^1|\Lambda(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]\end{aligned}$$

$$\leq \frac{(b-a)^2}{8\Phi(0)}\left(\int_0^1 |\Lambda(t)|^p\,dt\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_0^1\left(\frac{1+t}{2}\left|f''(a)\right|^q+\frac{1-t}{2}\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}}+\left(\int_0^1\left(\frac{1-t}{2}\left|f''(a)\right|^q+\frac{1+t}{2}\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(b-a)^2}{8\Phi(0)}\left(\int_0^1 |\Lambda(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right].$$

This completes the proof of first inequality in (15).

The proof of second inequality in (15) is obvious from the inequality (10). □

Corollary 4.8. *If we choose $\varphi(t) = t$ in Theorem 4.7, then we have the following inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^2}{16\,(2p+1)^{\frac{1}{p}}}\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(b-a)^2}{8}\left(\frac{4}{2p+1}\right)^{\frac{1}{p}}\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right].$$

Remark 4.9. *In all Theorems in this paper, if we choose $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)}t^{\frac{\alpha}{k}}$, $k > 0$, $\varphi(t) = t(x-t)^{\alpha-1}$ and $\varphi(t) = \frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, $\alpha \in (0,1)$, then we obtain trapezoid and midpoint type inequalities involving k-Riemann-Liouville fractional, conformable fractional integrals and fractional integral operators with exponential kernel, respectively.*

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