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Generalizations of Numerical Radius Inequalities Related to Block Matrices

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Abstract. We establish several numerical radius inequalities related to 2×2 positive semidefinite block matrices. It is shown that if $A, B, C \in \mathbb{M}_n(\mathbb{C})$ are such that $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$, then

$w^r(B) \leq \frac{1}{2} \|A^r + C^r\|$, for $r \geq 1$.

Related numerical radius inequalities for sums and products of matrices are also given.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the space of $n \times n$ complex matrices. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called positive semidefinite if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$. We write $A \geq 0$ to mean A is positive semidefinite.

Let $w(A)$, $\|A\|$ and $r(A)$ denote the numerical radius, the usual operator norm and the spectral radius of A, respectively. Recall that

$$w(A) = \max\{|\langle Ax, x\rangle| : x \in \mathbb{C}^n, \|x\| = 1\},$$

$$\|A\| = \max\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\},$$

and

$$r(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

An alternative way to obtain the numerical radius of matrices can be found in [12] asserts that for every $A \in \mathbb{M}_n(\mathbb{C})$,

$$w(A) = \max_{\theta \in R} \left\| \mathrm{Re}(e^{i\theta} A) \right\|.$$

The power inequality is a main inequality for numerical radius, which says that for $A \in \mathbb{M}_n(\mathbb{C})$,

$$w(A^k) \leq w^k(A) \tag{1}$$

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for $k = 1, 2, ...$(see, e.g.,[6, p.118]).

It is known that the numerical radius $w(.)$ defines a norm on $\mathbb{M}_n(\mathbb{C})$, which is equivalent to the usual operator norm $\|.\|$. In fact, for any $A \in \mathbb{M}_n(\mathbb{C})$,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{2}$$

However, this norm is not unitarily invariant norm, but weakly unitarily invariant. This means that $w(UAU^*) = w(A)$ for any unitary matrix U.

A refinement of the second inequality in (2) has been given earlier in [8], that if $A \in \mathbb{M}_n(\mathbb{C})$, then

$$w(A) \leq \frac{1}{2}\||A| + |A^*|\|. \tag{3}$$

Other numerical radius inequalities improving and generalizing the inequality (2) have been given in [1],[10] and [13].

Generalizations of inequality (3) was given in [3]. It has been shown that if $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$w^r(A) \leq \frac{1}{2}\left\||A|^{2\alpha r} + |A^*|^{2(1-\alpha)r}\right\| \tag{4}$$

and

$$w^r(A+B) \leq 2^{r-2}\left\||A|^{2\alpha r} + |B|^{2\alpha r} + |A^*|^{2(1-\alpha)r} + |B^*|^{2(1-\alpha)r}\right\| \tag{5}$$

for $0 < \alpha < 1$ and $r \geq 1$.

An extension of the above inequalities has been proved in [9], it has been shown that if $A, B, C, D, X, Y \in \mathbb{M}_n(\mathbb{C})$, then

$$w(AXB + CYD) \leq \frac{1}{2}\left\|A\,|X^*|^{2(1-\alpha)}\,A^* + B^*\,|X|^{2\alpha}\,B + C\,|Y^*|^{2(1-\alpha)}\,C^* + D^*\,|Y|^{2\alpha}\,D\right\| \tag{6}$$

for $0 < \alpha < 1$. In particular,

$$w(AB \pm BA) \leq \frac{1}{2}\|A^*A + AA^* + B^*B + BB^*\|. \tag{7}$$

Several interesting inequalities for sums and products of matrices have been introduced by mathematicians. It has been shown that for $r \geq 1$, if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then

$$\|A^r + B^r\| \leq \left\|(A+B)^r\right\| \leq 2^{r-1}\|A^r + B^r\|, \tag{8}$$

and for any $A, B \in \mathbb{M}_n(\mathbb{C})$,

$$w^r(AB^*) \leq \frac{1}{2}\left\|(AA^*)^r + (BB^*)^r\right\| \tag{9}$$

and

$$\|AB \pm BA\|^r \leq 2^{r-1}\|(AA^*)^r + (BB^*)^r + (A^*A)^r + (B^*B)^r\|), \tag{10}$$

(see, e.g., [11]).

For a positive semidefinite block matrix $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix}$, where $A, B, C \in \mathbb{M}_n(\mathbb{C})$, it is well known that

$$\left\|\begin{matrix} A & B^* \\ B & C \end{matrix}\right\| \leq \|A\| + \|C\|. \tag{11}$$

However, if the off-diagonal block B is Hermitian, then Hiroshima [5] established a stronger inequality than (11),

$$\left|\left|\left|\begin{bmatrix} A & B \\ B & C \end{bmatrix}\right|\right|\right| \leq \|A + C\|. \tag{12}$$

On the other hand, Burqan and Al-Saafin [2] gave an estimate for the numerical radius of the off-diagonal block of positive semidefinite matrix $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix}$,

$$w(B) \leq \frac{1}{2}\|A + C\|. \tag{13}$$

In this paper, we are interested in finding a generalization of inequality (13) which yields new numerical radius inequalities. More numerical radius inequalities involving sums and products of matrices will be considered.

2. Lemmas

To establish and prove our results, we need the following lemmas. The first lemma is an application of Jensen's inequality, can be found in [4]. The second lemma follows from the spectral theorem for positive matrices and Jensen's inequality (see, e.g., [7]). The third lemma is a Cauchy-Schwarz inequality involving block positive semidefinite matrices (see [14, p. 203]). The fourth lemma has been proved in [7]. The fifth lemma introduces useful estimates for the spectral radius of 2×2 block matrices, can be found in [6].

Lemma 2.1. *Let $a, b \geq 0$ and $0 \leq \alpha \leq 1$. Then*

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \leq (\alpha a^r + (1-\alpha) b^r)^{\frac{1}{r}}, \ \text{for } r \geq 1.$$

Lemma 2.2. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, and let $x \in \mathbb{C}^n$ be a unit vector. Then*

$$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle, \ \text{for } r \geq 1.$$

Lemma 2.3. *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$. Then*

$$\left|\langle Bx, y \rangle\right|^2 \leq \langle Ax, x \rangle \langle Cy, y \rangle, \ \text{for } x, y \in \mathbb{C}^n.$$

Lemma 2.4. *Let $A \in \mathbb{M}_n(\mathbb{C})$ and $0 < \alpha < 1$. Then*

$$\begin{bmatrix} |A^*|^{2\alpha} & A^* \\ A & |A|^{2(1-\alpha)} \end{bmatrix} \geq 0.$$

Lemma 2.5. *Let $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$. Then*

$$r\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq r\left(\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix}\right).$$

3. Main Results

In the beginning of this section, we introduce a generalization of inequality (13), which yields interesting new numerical radius inequalities.

Theorem 3.1. *Let* $A, B, C \in \mathbb{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$. *Then*

$$w^r(B) \leq \frac{1}{2} \|A^r + C^r\| \ \ for\ r \geq 1. \tag{14}$$

Proof. Since $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$, for every unit vector $x \in \mathbb{C}^n$, we have

$$\begin{aligned}
|\langle Bx, x\rangle| &\leq \langle Ax, x\rangle^{\frac{1}{2}} \langle Cx, x\rangle^{\frac{1}{2}} && \text{(by Lemma 2.3)} \\
&\leq \frac{1}{2}(\langle Ax, x\rangle + \langle Cx, x\rangle) \\
&\leq \left(\frac{\langle Ax, x\rangle^r + \langle Cx, x\rangle^r}{2}\right)^{\frac{1}{r}} && \text{(by Lemma 2.1)} \\
&\leq \left(\frac{\langle A^r x, x\rangle + \langle C^r x, x\rangle}{2}\right)^{\frac{1}{r}} && \text{(by Lemma 2.2)}
\end{aligned}$$

Thus,

$$|\langle Bx, x\rangle|^r \leq \frac{1}{2} \langle (A^r + C^r)x, x\rangle$$

and so

$$\begin{aligned}
w^r(B) &= \max\{|\langle Bx, x\rangle|^r : x \in \mathbb{C}^n, \|x\| = 1\} \\
&\leq \frac{1}{2} \max\{\langle (A^r + C^r)x, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\} \\
&= \frac{1}{2} \|A^r + C^r\|
\end{aligned}$$

as required. □

Using the fact that if $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$, then $\begin{bmatrix} X^*AX & X^*B^*Y^* \\ YBX & YCY^* \end{bmatrix} \geq 0$ for any $X, Y \in \mathbb{M}_n(\mathbb{C})$, we have the following corollary.

Corollary 3.2. *Let* $A, B, C, X, Y \in \mathbb{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$. *Then*

$$w^r(YBX) \leq \frac{1}{2} \left\|(X^*AX)^r + (YCY^*)^r\right\| \ \ for\ r \geq 1. \tag{15}$$

Our next inequality, is a refinement of inequality (11) .

Theorem 3.3. *Let* $A, B, C \in \mathbb{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$ *and let* $B = UDV^*$ *be a singular value decomposition of* B. *Then*

$$\left|\left|\left| \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \right|\right|\right| \leq \|U^*AU + V^*CV\|. \tag{16}$$

Proof. Since $\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$, it follows that

$$\begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix} \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} U^*AU & D \\ D & V^*CV \end{bmatrix} \geq 0.$$

Using unitarily invariant property and inequality (12), we get

$$\begin{aligned} \left|\left|\left| \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \right|\right|\right| &= \left|\left|\left| \begin{bmatrix} U^*AU & D \\ D & V^*CV \end{bmatrix} \right|\right|\right| \\ &\leq |||U^*AU + V^*CV|||. \end{aligned}$$

This completes the proof. □

4. Inequalities for Sums and Products of Matrices

In this section we introduce several interesting inequalities for sums and products of matrices. First inequality is a generalization of inequality (4).

Theorem 4.1. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *and* $0 < \alpha < 1$. *Then*

$$w^r(A+B) \leq \frac{1}{2} \left\| (|A^*|^{2\alpha} + |B^*|^{2\alpha})^r + (|A|^{2(1-\alpha)} + |B|^{2(1-\alpha)})^r \right\|, \textit{ for } r \geq 1. \tag{17}$$

Proof. Since the sum of positive semidefinite matrices is also positive semidefinite and by applying Lemma 2.4, we have

$$\begin{bmatrix} |A^*|^{2\alpha} + |B^*|^{2\alpha} & A^* + B^* \\ A + B & |A|^{2(1-\alpha)} + |B|^{2(1-\alpha)} \end{bmatrix} \geq 0.$$

By Theorem 3.1, we get

$$w^r(A+B) \leq \frac{1}{2} \left\| (|A^*|^{2\alpha} + |B^*|^{2\alpha})^r + (|A|^{2(1-\alpha)} + |B|^{2(1-\alpha)})^r \right\|.$$

This completes the proof. □

It is clear that inequality (17) is a refinement of inequality (5).

For $\alpha = \frac{1}{2}$ in inequality (17), we get the following power numerical radius inequality for sum matrices.

$$w^r(A+B) \leq \frac{1}{2} \|(|A^*| + |B^*|)^r + (|A| + |B|)^r\|, \ for\ r \geq 1. \tag{18}$$

In the following, we establish a numerical radius inequality for matrices that produces an estimate for the numerical radius of commutators.

Theorem 4.2. *Let* $A, B, C, D, X, Y \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$w^r(Y(AC^* + BD^*)X) \leq \frac{1}{2} \|(X^*(AA^* + BB^*)X)^r + (Y(CC^* + DD^*)Y^*)^r\|, \textit{for } r \geq 1 \tag{19}$$

Proof. We know that

$$\begin{bmatrix} AA^* + BB^* & AC^* + BD^* \\ CA^* + DB^* & CC^* + DD^* \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \geq 0,$$

for any $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$. So by Corollary 3.2, we have

$$w^r(Y(AC^* + BD^*)X) \leq \frac{1}{2} \|(X^*(AA^* + BB^*)X)^r + (Y(CC^* + DD^*)Y^*)^r\|,$$

for any $X, Y \in \mathbb{M}_n(\mathbb{C})$. □

By letting $X = Y = I$, $C^* = B$ and $D^* = \pm A$ in inequality (19), we get the following numerical radius inequality for commutators which a generalization of inequality (7).

$$w^r(AB \pm BA) \leq \frac{1}{2} \|(AA^* + BB^*)^r + (A^*A + B^*B)^r\|, \text{ for } r \geq 1 \tag{20}$$

Through inequality (8), we see that inequality (20) is a refinement of inequality (10).

The inequality (9) is produced by letting $X = Y = I$, $C = B$ and $D = B = 0$ in inequality (19).

We conclude this paper by giving numerical radius inequality involving products of matrices.

It is clear that if $AB = BA$, then $w(AB) \leq \|BA\|$. But this is not true if the hypothesis of commutativity is omitted. To see this, Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $w(AB) = \frac{1}{2} > 0 = \|BA\|$.

In the following theorem we introduce an upper bound of $w(AB)$ based on $\|BA\|$ without the hypothesis of commutativity.

Theorem 4.3. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then*

$$w(AB) \leq \frac{1}{2} (\|BA\| + \|A\|\,\|B\|)$$

Proof. For $\theta \in R$, we have

$$\begin{aligned} \left\|\mathrm{Re}(e^{i\theta}AB)\right\| &= r(\mathrm{Re}(e^{i\theta}AB)) = \frac{1}{2} r(e^{i\theta}AB + e^{-i\theta}B^*A^*) \\ &= \frac{1}{2} r\left(\begin{bmatrix} e^{i\theta}AB + e^{-i\theta}B^*A^* & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &= \frac{1}{2} r\left(\begin{bmatrix} e^{i\theta}A & B^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B & 0 \\ e^{-i\theta}A^* & 0 \end{bmatrix}\right). \end{aligned}$$

Using a commutative property of the spectral radius and Lemma 2.5, we have

$$\begin{aligned} \left\|\mathrm{Re}(e^{i\theta}AB)\right\| &= \frac{1}{2} r\left(\begin{bmatrix} B & 0 \\ e^{-i\theta}A^* & 0 \end{bmatrix}\begin{bmatrix} e^{i\theta}A & B^* \\ 0 & 0 \end{bmatrix}\right) \\ &= \frac{1}{2} r\left(\begin{bmatrix} e^{i\theta}BA & BB^* \\ A^*A & e^{-i\theta}A^*B^* \end{bmatrix}\right) \\ &\leq \frac{1}{2} r\left(\begin{bmatrix} \|BA\| & \|BB^*\| \\ \|A^*A\| & \|A^*B^*\| \end{bmatrix}\right) \\ &= \frac{1}{2}\left(\|BA\| + \sqrt{\|A^*A\|\,\|BB^*\|}\right) \\ &= \frac{1}{2} (\|BA\| + \|A\|\,\|B\|). \end{aligned}$$

Take maximum over $\theta \in R$ in two sides, we get

$$w(AB) \leq \frac{1}{2} (\|BA\| + \|A\|\,\|B\|).$$

This completes the proof. □

To show that $w(AB) \leq \frac{1}{2} (\|BA\| + \|A\|\,\|B\|)$ is sharp, consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $w(AB) = \frac{1}{2}$, $\|BA\| = 0$ and $\|A\| = \|B\| = 1$.

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