



On Generalized Statistical Convergence and Boundedness of Riesz Space-Valued Sequences

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Abstract. We consider the notion of generalized density, namely, the natural density of weight g recently introduced in [4] and primarily study some sufficient and almost converse necessary conditions for the generalized statistically convergent sequence under which the subsequence is also generalized statistically convergent. Also we consider similar types of results for the case of generalized statistically bounded sequence. Some results are further obtained in a more general form by using the notion of ideals. The entire investigation is performed in the setting of Riesz spaces extending the recent results in [13].

1. Introduction

A Riesz space is an ordered vector space which is a lattice at the same time. It was first introduced by F. Riesz [32] in 1928. Riesz spaces have many applications in measure theory, operator theory and optimization. They have also some applications in economics(see [3]).

Recall that a topology on a vector space that makes the operations of addition and scalar multiplication continuous is said to be a linear topology. A vector space equipped with a linear topology is called a topological vector space. A Riesz space equipped with a linear topology that has a base at zero consisting of solid sets is called a locally solid Riesz space [3].

The notion of statistical convergence, which is an extension of the idea of usual convergence, was introduced by Fast [19] and Schoenberg [35] and its topological consequences were studied by Fridy [21] and Šalát [23]. The study of statistical convergence and its numerous extensions and, in particular, of the ideal convergence and its applications, has been one of the most active areas of research in the summability theory over the last 15 years. Subsequently, in a very recent development, the idea of statistical convergence of sequences was studied by Albayrak and Pehlivan [2] in locally solid Riesz spaces.

Naturally, it seems likely that the investigations of these generalized methods of convergence may provide

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a natural foundation for upbuilding of various tangent spaces to general metric spaces. The construction of tangent spaces in [7, 8, 15–17] is primarily based on the fundamental fact that, for a convergent sequence (x_n) in a metric space, each its subsequence (x_{n_k}) is also convergent. However, this is generally not true for the generalized methods of convergence mentioned above.

Very recently (see [25]), following the line of investigation in [27], the conditions have been studied for the density of a subsequence of a statistically bounded and also statistically convergent sequence under which the indicated subsequence is also statistically bounded or statistically convergent in the setting of metric space.

As a natural consequence, in [13] Das and Savas investigated the similar type problem that are proposed in [25] for metric valued sequences by considering the notion of natural density of weight g , which was introduced in [4].

In the present paper we continue the investigation proposed in [25] and study similar type problems for Riesz space-valued sequences by considering the notion of natural density of weight g as also the notion of f -density (introduced and studied in [2]).

2. Preliminaries

First we recall some of the basic concepts related to Riesz spaces and we refer to [32] for more details.

Definition 2.1. Let L be a real vector space and " \leq " be a partial order on this space. L is said to be an ordered vector space if it satisfies the following properties:

- (i) If $x, y \in L$ and $y \leq x$ then $y + z \leq x + z$ for each $z \in L$.
- (ii) If $x, y \in L$ and $y \leq x$ then $\lambda y \leq \lambda x$ for each $\lambda \geq 0$.

Definition 2.2. A nonempty set L is said to a lattice with respect to the partial order \leq if for each pair of elements $x, y \in L$, both the supremum and infimum of the set $\{x, y\}$ exists in L .

We shall write $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$. For $x \in L$ we further define $|x| = x \vee (-x)$.

Definition 2.3. If L is an ordered vector space as well as a lattice, then we call L a Riesz space or a Vector lattice.

A subset S of a Riesz space L is said to be solid if $y \in S$ and $|x| \leq |y|$ imply that $x \in S$.

Definition 2.4. A topological vector space (L, τ) is a real vector space L which has a topology τ , such that, the mappings $: L \times L \rightarrow L$ and $: \mathbb{R} \times L \rightarrow L$ defined by $(x, y) \rightarrow x + y$ and $(a, x) \rightarrow ax$ are continuous. In this case the topology is called linear topology.

A topological vector space L is said to be locally solid if τ has a base at zero(local base) consisting of solid sets.

Definition 2.5. A locally solid Riesz space (L, τ) is a Riesz space L equipped with a locally solid topology τ on L .

Every linear topology τ on a vector space L has a base N consisting of the neighborhoods of θ (zero) satisfying the following properties:

- (a) Each $V \in N$ is a balanced set; that is, $\lambda x \in V$ holds for all $x \in V$ and every $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$.
- (b) Each $V \in N$ is an absorbing set; that is, for every $x \in L$, there exists a $\lambda > 0$ such that $\lambda x \in V$.
- (c) For each $V \in N$ there exists some $W \in N$ with $W + W \subseteq V$

Recall that the Natural density or Asymptotic density of $A \subseteq \mathbb{N}$ is defined by [6, 34, 36]

$$d(A) = \lim_{n \rightarrow \infty} \frac{|A(n)|}{n}$$

where $A(n) = |\{k \leq n : k \in A\}| = |A \cap \{1, 2, 3, \dots, n\}|$, the number of elements of A not exceeding n . For example, a finite subset of positive integers has natural density zero.

In the sequel, by the symbol N_{sol} we will denote base at zero consisting of solid sets and satisfying the properties (a), (b) and (c) in a locally solid topology. Throughout the paper (L, τ) denotes a locally solid Riesz space and \tilde{L} denotes the set of all sequences of points from L and D be an arbitrary directed set.

3. Main Results

3.1. Results for f -density

The idea of modulus function was introduced by Nakano in 1953 [30]. He used the term concave modular, and defined it on semi-ordered linear space. Several consequences were rather studied by Ruckle[33], Maddox[28] etc.

Let $f : [0, \infty) \rightarrow [0, \infty)$. f is called modulus function if it satisfies:

- i) $f(x) = 0$ if and only if $x = 0$
- ii) $f(x + y) \leq f(x) + f(y)$ for every $x, y \in [0, \infty)$
- iii) $f(x)$ is increasing
- iv) f is continuous from the right at 0.

Some examples are:

- i) $f(x) = x, x \in [0, \infty)$
- ii) $f(x) = \frac{x}{1+x}, x \in [0, \infty)$
- iii) $f(x) = \log(1 + x), x \in [0, \infty)$
- iv) $f(x) = x^p$ with $0 < p \leq 1, x \in [0, \infty)$.

From the condition that satisfied by a modulus function, it is clear that modulus function must be continuous on \mathbb{R}^+ . Using f -density we now introduce a version of statistical convergence in locally solid Riesz space which we call it f -statistical convergence as in [2]. Throughout this section f denotes unbounded modulus function.

The notion of f -density (density function via modulus function) was introduced by Aizpuru [1] as follows:

Definition 3.1. Let f be an unbounded modulus function. The f -density of a set $A \subseteq \mathbb{N}$ is defined by

$$d_f(A) = \limsup_{n \rightarrow \infty} \frac{f(|A(n)|)}{f(n)}$$

in case the limit exists.

Definition 3.2. If $d_f(A) = 1$ then we say the set $A \subset \mathbb{N}$ is f -dense subset of \mathbb{N} .

Note 3.1. For density function it is clear that $d(A) = 1 - d(\mathbb{N} \setminus A)$, whenever one of the sides exists. If $A \subseteq \mathbb{N}$ and $d_f(A) = 0$ then $d_f(\mathbb{N} \setminus A) = 1 - d_f(A)$. On the other cases the relation is not true.[1]

Definition 3.3. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be a sequence in L . Then we will say that (x_n) is f -statistical convergent to x_0 and write f -st $\lim x_n = x$ if for any τ -neighborhood U of zero we have $d_f(\{n \in \mathbb{N} : x_n - x_0 \notin U\}) = 0$.

i.e.

$$\limsup_{n \rightarrow \infty} \frac{f(|\{n \in \mathbb{N} : x_n - x_0 \notin U\}|)}{f(n)} = 0.$$

Note 3.2. f -statistical convergence coincides with statistical convergence when we take the modulus function as the identity mapping.

Definition 3.4. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be a sequence in L . \tilde{x} is called bounded if for every $x \in L$ there is a τ -neighborhood U of zero such that $x_n - x \in U$ for all n .

Definition 3.5. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be a sequence in L . \tilde{x} is called f -statistically bounded if for every $x \in L$ there is a τ -neighborhood U of zero such that

$$\limsup_{n \rightarrow \infty} \frac{f(|\{k : k \leq n, x_n - x \notin U\}|)}{f(n)} = 0.$$

Note 3.3. It is clear from the definition that, a sequence $\tilde{x} = (x_n)$ in L is f -statistically bounded if and only if the sequence $(x_n - x)$ is f -statistical bounded for an arbitrary $x \in L$.

Definition 3.6. Two sequences $\tilde{x} = (x_n) \in \tilde{L}$ and $\tilde{y} = (y_n) \in \tilde{L}$ are f -statistically equivalent, if there is a f -dense set $A \subset \mathbb{N}$ such that $x_n = y_n$ for every $n \in A$.

Definition 3.7. Let $\tilde{x} = (x_n) \in \tilde{L}$. If (n_k) is an infinite strictly increasing sequence of natural number, then $\tilde{x}' = (x_{n_k})$ is called a subsequence of \tilde{x} . Let $K_{\tilde{x}'} = \{n_k : k \in \mathbb{N}\}$. \tilde{x}' is called f -dense subsequence of \tilde{x} if $K_{\tilde{x}'}$ is a f -dense subset of \mathbb{N} , i.e. $d_f(K_{\tilde{x}'}) = 1$.

We know that for any metric space if a sequence is convergent then it is bounded. We now show that similar type relation holds for f -statistical convergence.

Theorem 3.1. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be a sequence in L . The following statements hold:
(i) If \tilde{x} is bounded then \tilde{x} is f -statistically bounded.
(ii) If \tilde{x} is f -statistically convergent to $x_0 \in L$ then \tilde{x} is f -statistically bounded.

Proof. (i) Let \tilde{x} is bounded. Then for every $x \in L$, there exists a τ -neighborhood U of zero such that $x_n - x \in U$ for all n . So, $|\{k : k \leq n, x_n - x \notin U\}| = 0$. This implies $f(|\{k : k \leq n, x_n - x \notin U\}|) = 0$. Hence \tilde{x} is f -statistical bounded.

(ii) Let f -stlim $x_n = x_0$. For any arbitrary τ -neighborhood U of zero we can choose a τ -neighborhood W of zero such that $\{k : k \leq n, x_n - x_0 \notin W\} \subset \{n : x_n - x_0 \notin U\}$ i.e. $|\{k : k \leq n, x_n - x_0 \notin W\}| \leq |\{n : x_n - x_0 \notin U\}|$. As given \tilde{x} is f -statistical convergent and f is unbounded modulus function so,
$$\limsup_{n \rightarrow \infty} \frac{f(|\{k : k \leq n, x_k - x_0 \notin W\}|)}{f(n)} = 0$$
. Therefore \tilde{x} is f -statistically bounded. \square

Converses of each cases of the theorem above are not true in general. The following example gives support about this.

Example 3.1. Consider real line \mathbb{R} with usual metric and consider the sequence $\tilde{x} = (x_n)$ where $x_n = \{(-1)^n\}$. Take $f(x) = x, x \in [0, \infty)$ as a modulus function. Clearly this sequence \tilde{x} is f -statistically bounded but it is not f -statistically convergent.

Example 3.2. Consider the sequence $\tilde{x} = (x_n)$ in real line \mathbb{R} with usual metric where $x_n = 0$ if $n \neq 10^k$ and $x_n = k$ if $n = 10^k$. Also consider $f(x) = x, x \in [0, \infty)$ as a modulus function. Then \tilde{x} is f -statistically bounded which is not a bounded sequence.

Theorem 3.2. Let (L, τ) be a locally solid Riesz space and let $\tilde{x} = (x_n)$ be f -statistical convergent sequence. Also let $\tilde{x}' = (x_{n_k})$ be any subsequence of \tilde{x} . Then \tilde{x}' is f -statistically bounded.

Proof. Let $\tilde{x} = (x_n)$ be f -statistically converges to x_0 . Then obviously \tilde{x} is f -statistically bounded by above theorem. It is clear that for any τ neighborhood U of zero, $\{n_k : n_k \leq n, x_{n_k} - x_0 \notin U\} \subset \{k : k \leq n, x_k - x_0 \notin U\}$. Hence $|\{n_k : n_k \leq n, x_{n_k} - x_0 \notin U\}| \leq |\{k : k \leq n, x_k - x_0 \notin U\}|$. As \tilde{x} is f -statistically bounded so for any unbounded modulus function f , we have $0 \leq \limsup_{n \rightarrow \infty} \frac{f(|\{n_k : n_k \leq n, x_{n_k} - x_0 \notin U\}|)}{f(n)} \leq 0$. That is \tilde{x}' is f -statistically bounded. \square

We know that, every subsequence of a convergent sequence is also convergent in a metric space. But this is not generally true for statistical convergence. In the articles [25, 27] some investigation done that under what condition subsequence of a statistical convergent sequence is also convergent. Following this line of investigation some work done in [13]. We prove the next results in this direction.

Theorem 3.3. Let (L, τ) be a locally solid Riesz space, let $\tilde{x} = (x_n) \in \tilde{L}$ and let $\tilde{x}' = (x_{n_k})$ be a subsequence of \tilde{x} such that $\liminf_{n \rightarrow \infty} \frac{f(|K_{\tilde{x}'}(n)|)}{f(n)} > 0$. If \tilde{x} is f -statistically convergent to $x_0 \in L$ then \tilde{x}' is also f -statistically convergent to x_0 .

Proof. Suppose that \tilde{x} is f -statistically convergent to x_0 . Let U be a τ -neighborhood of zero. Then from definition

$$\limsup_{n \rightarrow \infty} \frac{f(\{m : m \leq n, x_n - x_0 \notin U\})}{f(n)} = 0$$

. In order to prove that \tilde{x}' is f -statistically convergent to x_0 we have to show that

$$\limsup_{n \rightarrow \infty} \frac{f(\{n_k : n_k \leq n, x_{n_k} - x_0 \notin U\})}{f(|K_{\tilde{x}'}(n)|)} = 0,$$

where $K_{\tilde{x}'}(n) = K_{\tilde{x}} \cap \{1, 2, 3, \dots, n\}$.

Then clearly, $\{n_k : n_k \leq n, x_{n_k} - x_0 \notin U\} \subseteq \{m : m \leq n, x_m - x_0 \notin U\}$. Thus we can write

$$\frac{f(\{n_k : n_k \leq n, x_{n_k} - x_0 \notin U\})}{f(|K_{\tilde{x}'}(n)|)} \leq \frac{f(\{m : m \leq n, x_m - x_0 \notin U\})}{f(|K_{\tilde{x}}(n)|)}$$

We know for any two sequences (α_n) and (β_n) of nonnegative real numbers with $0 \neq \liminf_{n \rightarrow \infty} \alpha_n < \infty$, we have $\liminf_{n \rightarrow \infty} \alpha_n \limsup_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \alpha_n \beta_n$.

Now take $\alpha_n = \frac{f(|K_{\tilde{x}'}(n)|)}{f(n)}$ and $\beta_n = \frac{f(\{m : m \leq n, x_m - x_0 \notin U\})}{f(|K_{\tilde{x}}(n)|)}$, then $\alpha_n \beta_n = \frac{f(\{m : m \leq n, x_m - x_0 \notin U\})}{f(n)}$. Therefore it follows that

$$\liminf_{n \rightarrow \infty} \frac{f(|K_{\tilde{x}'}(n)|)}{f(n)} \limsup_{n \rightarrow \infty} \frac{f(\{m : m \leq n, x_m - x_0 \notin U\})}{f(|K_{\tilde{x}}(n)|)} \leq \limsup_{n \rightarrow \infty} \frac{f(\{m : m \leq n, x_m - x_0 \notin U\})}{f(n)}.$$

Since \tilde{x} is f -statistically convergent to x_0 , the right hand side of the above inequality is zero. Also by our assumption $\liminf_{n \rightarrow \infty} \frac{f(|K_{\tilde{x}'}(n)|)}{f(n)} > 0$, hence we find

$$\limsup_{n \rightarrow \infty} \frac{f(\{m : m \leq n, x_m - x_0 \notin U\})}{f(|K_{\tilde{x}'}(n)|)} = 0.$$

Then by (1)

$$\limsup_{n \rightarrow \infty} \frac{f(\{n_k : n_k \leq n, x_{n_k} - x_0 \notin U\})}{f(|K_{\tilde{x}'}(n)|)} = 0.$$

Hence \tilde{x}' is f -statistically convergent to x_0 . \square

Theorem 3.4. Let (L, τ) be a locally solid Riesz space and let \tilde{x}, \tilde{y} be two f -statistically equivalent sequence in L . If K is a subset of \mathbb{N} such that $0 \leq \limsup_{n \rightarrow \infty} \frac{f(n)}{f(|K(n)|)} < +\infty$ and if $\tilde{x}' = (x_{n_k})$ and $\tilde{y}' = (y_{n_k})$ are subsequences of \tilde{x}, \tilde{y} respectively such that $K_{\tilde{x}'} = K_{\tilde{y}'} = K$, then \tilde{x}' and \tilde{y}' are f -statistically equivalent.

Proof. Given that \tilde{x}, \tilde{y} are f -statistically equivalent, so there exists f -dense set $M \subset \mathbb{N}$ such that $x_n = y_n$ for all $n \in M$. Equivalently, $\frac{f(\{n \in \mathbb{N} : x_n \neq y_n \text{ and } n \leq m\})}{f(m)} = 0$

We have to show that \tilde{x}' and \tilde{y}' are f -statistically equivalent, i.e. $x_{n_k} = y_{n_k}$ for all $n_k \in M$ with $d_f(M) = 1$. Equivalently we have to show that $d_f\{n_k : x_{n_k} \neq y_{n_k}\} = 0$. i.e.

$$\limsup_{m \rightarrow \infty} \frac{f(\{n_k \in K : x_{n_k} \neq y_{n_k} \text{ and } n_k \leq m\})}{f(|K(m)|)} = 0$$

Now for any $m \in \mathbb{N}$, we have

$$\{n_k \in K : x_{n_k} \neq y_{n_k} \text{ and } n_k \leq m\} \subseteq \{n \in \mathbb{N} : x_n \neq y_n \text{ and } n \leq m\}.$$

So for any unbounded modulus function f , we have

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \frac{f(\{|n_k \in K : x_{n_k} \neq y_{n_k} \text{ and } n_k \leq m\})}{f(|K(m)|)} \\ & \leq \limsup_{m \rightarrow \infty} \frac{f(\{|n \in \mathbb{N} : x_n \neq y_n \text{ and } n \leq m\})}{f(|K(m)|)} \\ & \leq \limsup_{m \rightarrow \infty} \frac{f(\{|n \in \mathbb{N} : x_n \neq y_n \text{ and } n \leq m\})}{f(m)} \limsup_{m \rightarrow \infty} \frac{f(m)}{f(|K(m)|)} \end{aligned}$$

[We know for any two bounded sequences of non-negative reals (u_n) and (v_n) , then

$$\limsup(u_n v_n) \leq (\limsup u_n)(\limsup v_n)]$$

Now using the given assumption that \tilde{x}, \tilde{y} be two f -statistically equivalent sequence in L and

$$0 \leq \limsup_{n \rightarrow \infty} \frac{f(n)}{f(|K(n)|)} < +\infty, \text{ we get}$$

$$\limsup_{m \rightarrow \infty} \frac{f(\{|n_k \in K : x_{n_k} \neq y_{n_k} \text{ and } n_k \leq m\})}{f(|K(m)|)} = 0.$$

Hence \tilde{x}' and \tilde{y} are f -statistically equivalent. \square

3.2. Results for weighted density

In [4], the notion of natural density was further extended as follows: Let $g : \mathbb{N} \rightarrow [0, \infty)$ be a function with $\lim_{n \rightarrow \infty} g(n) = \infty$. The upper density of weight g was defined in [4] by the formula $\overline{d}_g(A) = \limsup_{n \rightarrow \infty} \frac{\text{card}(A \cap [1, n])}{g(n)}$ for $A \subset \mathbb{N}$. Let $\mathcal{I}_g = \{A \subset \mathbb{N} : \overline{d}_g(A) = 0\}$. Then \mathcal{I}_g is an ideal of \mathbb{N} . Also $\mathbb{N} \in \mathcal{I}_g$ if and only if $\frac{n}{g(n)} \rightarrow 0$. Hence we assume that $\frac{n}{g(n)} \not\rightarrow 0$. Hence $\mathbb{N} \notin \mathcal{I}_g$, and it was observed in [4] that \mathcal{I}_g is a proper admissible P -ideal of \mathbb{N} . The collection of all functions g of this kind satisfying the above mentioned properties is denoted by G . As a natural consequence we can introduce the following definitions:

Definition 3.8. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be a sequence in L . \tilde{x} is called bounded if for every $x \in L$ there is a τ -neighborhood U of zero such that $x_n - x \in U$ for all n .

Definition 3.9. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be a sequence in L . \tilde{x} is called d_g -statistically bounded if for every $x \in L$ there is a τ -neighborhood U of zero such that

$$\limsup_{n \rightarrow \infty} \frac{|\{k : k \leq n, x_n - x \notin U\}|}{g(n)} = 0.$$

Definition 3.10. Let (L, τ) be a locally solid Riesz space and $(x_n) \in \tilde{L}$. Then (x_n) is said to be d_g -statistically convergent to $x_0 \in L$ if for any τ -neighborhood U of zero we have $d_g(A_U) = 0$, where $A_U = \{n \in \mathbb{N} : x_n - x_0 \notin U\}$.

In what follows, we present several more basic definition required throughout the paper.

Definition 3.11. [11] A set $K \subset \mathbb{N}$ is called d_g -dense subset of \mathbb{N} if $d_g(K^c) = 0$.

Definition 3.12. Let (L, τ) be a locally solid Riesz space. A sequence $(x_n) \in \tilde{L}$ is said to be $\mathcal{I}(\tau)$ -convergent to an element $x_0 \in L$ if for each τ -neighborhood U of zero $\{k \in \mathbb{N} : x_k - x_0 \notin U\} \in \mathcal{I}$.

Definition 3.13. [13] A set $K \subset \mathbb{N}$ is called \mathcal{I} -dense subset of \mathbb{N} if $K \in \mathcal{F}(\mathcal{I})$.

Definition 3.14. (cf. [13]) If $(n(k))$ is an infinite strictly increasing sequence of natural numbers and $\tilde{x} = (x_n) \in \tilde{L}$, then we write $\tilde{x}' = (x_{n(k)})$ and $K_{\tilde{x}'} = \{n(k) : k \in \mathbb{N}\}$. A subsequence \tilde{x}' is called an \mathcal{I} -dense subsequence of \tilde{x} , if $K_{\tilde{x}'}$ is an \mathcal{I} -dense subset of \mathbb{N} .

Definition 3.15. (cf. [13]) Two sequences $\tilde{x} = (x_n) \in \tilde{L}$ and $\tilde{y} = (y_n) \in \tilde{L}$ are \mathcal{I} -equivalent, $\tilde{x} \asymp \tilde{y}$, if there is an \mathcal{I} -dense set $M \subset \mathbb{N}$ such that $x_n = y_n$ for every $n \in M$.

The following definitions are special case of the last two definitions:

Definition 3.16. If $(n(k))$ is an infinite strictly increasing sequence of natural numbers and $\tilde{x} = (x_n) \in \tilde{L}$, then we write $\tilde{x}' = (x_{n(k)})$ and $K_{\tilde{x}'} = \{n(k) : k \in \mathbb{N}\}$. A subsequence \tilde{x}' is called an d_g -dense subsequence of \tilde{x} , if $K_{\tilde{x}'}$ is an d_g -dense subset of \mathbb{N} .

Definition 3.17. Two sequences $\tilde{x} = (x_n) \in \tilde{L}$ and $\tilde{y} = (y_n) \in \tilde{L}$ are d_g -statistically equivalent, $\tilde{x} \asymp \tilde{y}$ (d_g -statistically), if there is an d_g -dense set $M \subset \mathbb{N}$ such that $x_n = y_n$ for every $n \in M$.

The first result shows that there is a one-to-one correspondence between topologies on L and the subsets of \tilde{L} consisting of all \mathcal{I} -convergent net for certain special types of ideals. The result is in line of Theorem 3.1 in [13].

Theorem 3.5. Let (L, τ_1) and (L, τ_2) be two locally solid Riesz spaces. Let \mathcal{I} be a DP-ideal, which is not maximal. Then the following statement are equivalent: (i) The set of all $\tau_1 - \mathcal{I}$ -convergent nets coincides with the set of all $\tau_2 - \mathcal{I}$ -convergent nets. (ii) The set of all nets convergent in (L, τ_1) coincides with the set of all nets convergent in (L, τ_2) . (iii) The topologies τ_1 and τ_2 are homeomorphic on L .

Proof. (ii) \Leftrightarrow (iii): The result is well known.

(ii) \rightarrow (i): Let $\tilde{x} = (x_n)$ be $\tau_1 - \mathcal{I}$ -convergent. Since \mathcal{I} is a DP-ideal, we conclude that \tilde{x} is $\tau_1 - \mathcal{I}^*$ convergent i.e. there is a set $M \in \mathcal{F}(\mathcal{I})$ such that $(\tilde{x})_M$ is τ_1 -convergent [26]. By (ii) $(\tilde{x})_M$ is τ_2 -convergent, and hence \tilde{x} is $\tau_2 - \mathcal{I}^*$ convergent, which evidently implies that \tilde{x} is $\tau_2 - \mathcal{I}$ convergent.

(i) \rightarrow (iii): Assume that (i) holds. If possible suppose that the topologies are distinct. Then there exists $x_0 \in L$ and a τ_1 -neighborhood U_0 of zero such that $\{x \in L : x - x_0 \in U_0\} \not\supseteq \{x \in L : x - x_0 \in U_2\}$ for all τ_2 -neighborhood U_2 of zero or the opposite inclusion. Without loss of any generality we can assume that the first one holds. For any $n \in D$ we can choose $x_n \in L$ and a neighborhood U_n of zero such that $x_n - x_0 \in U_n$ and $x_n - x_0 \notin U_0$ for each $n \in D$. We choose a set $K \subset D$ such that $K \notin \mathcal{I}$ as well as $K^c \notin \mathcal{I}$ (because \mathcal{I} is not maximal). Further we define a net $\tilde{y} = (y_n) \in \tilde{L}$ by

$$y_n = \begin{cases} x_n & \text{if } n \in K \\ x_0 & \text{if } n \notin K. \end{cases}$$

Clearly $\{n \in \mathbb{N} : y_n - x_0 \in U_0\} = K \notin \mathcal{I}$. We now observe that the net $\tilde{y} = (y_n)$ converges to x_0 in (L, τ_2) and therefore is $\tau_2 - \mathcal{I}$ -convergent. By virtue of (i), $\tilde{y} = (y_n)$ is also $\tau_1 - \mathcal{I}$ -convergent. Note that \tilde{y} must be $\tau_1 - \mathcal{I}$ -convergent to x_0 because, otherwise if \tilde{y} is $\tau_1 - \mathcal{I}$ -convergent to $y_0 \neq x_0$, then taking a τ_1 -neighborhood U' of zero with $x_0 - y_0 \notin U'$, we obtain that $\{n : y_n - y_0 \notin U'\} \supset K^c$. Since $K^c \notin \mathcal{I}$, we get $\{n : y_n - y_0 \notin U'\} \notin \mathcal{I}$. Which contradicts the fact that $\tilde{y} = (y_n)$ is $\tau_1 - \mathcal{I}$ -convergent to y_0 . However if \tilde{y} is $\tau_1 - \mathcal{I}$ -convergent to x_0 then we must have $\{n : y_n - x_0 \in U_0\} = K \in \mathcal{I}$. Which contradicts the fact $K \notin \mathcal{I}$. Thus (i) \rightarrow (iii) holds. \square

If the given sequence is d_g -statistical convergent, it is natural to ask how we can check that its subsequence is d_g -statistical convergent to the same limit. Also it is natural to ask when the converse assertion is true. We prove the next results in this direction, as also in the case of d_g -statistical bounded sequences.

Theorem 3.6. Let (L, τ) be a locally solid Riesz space, let $\tilde{x} = (x_n) \in \tilde{L}$ and let $\tilde{x}' = (x_{n(k)})$ be a subsequence of \tilde{x} such that $\liminf_{n \rightarrow \infty} \frac{g(|K_{\tilde{x}'}(n)|)}{g(n)} > 0$. If \tilde{x} is d_g -statistically convergent to $x_0 \in L$, then \tilde{x}' is also d_g -statistically convergent to x_0 .

Proof. Suppose that \tilde{x} is d_g -statistically convergent to x_0 . Let U be a τ -neighborhood of zero. Then clearly $\{n(k) : n(k) \leq n, x_{n(k)} - x_0 \notin U\} \subseteq \{m : m \leq n, x_m - x_0 \notin U\}$. Thus we can write

$$\frac{1}{g(|K_{\tilde{x}'}(n)|)} |\{n(k) : n(k) \leq n, x_{n(k)} - x_0 \notin U\}| \leq \frac{|\{m : m \leq n, x_m - x_0 \notin U\}|}{g(|K_{\tilde{x}'}(n)|)}$$

In order to prove that \tilde{x}' is d_g -statistically convergent to x_0 we have to show that

$$\limsup_{n \rightarrow \infty} \frac{|\{n(k) : n(k) \leq n, x_{n(k)} - x_0 \notin U\}|}{g(|K_{\tilde{x}'}(n)|)} = 0.$$

Now we know for any two sequences (α_n) and (β_n) of nonnegative real numbers with $0 \neq \liminf_{n \rightarrow \infty} \alpha_n < \infty$, we have $\liminf_{n \rightarrow \infty} \alpha_n \limsup_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \alpha_n \beta_n$. Take $\alpha_n = \frac{g(|K_{\tilde{x}'}(n)|)}{g(n)}$ and $\beta_n = \frac{|\{m : m \leq n, x_m - x_0 \notin U\}|}{g(|K_{\tilde{x}'}(n)|)}$ then $\alpha_n \beta_n = \frac{|\{m : m \leq n, x_m - x_0 \notin U\}|}{g(n)}$. Therefore it follows that

$$\liminf_{n \rightarrow \infty} \frac{g(|K_{\tilde{x}'}(n)|)}{g(n)} \limsup_{n \rightarrow \infty} \frac{|\{m : m \leq n, x_m - x_0 \notin U\}|}{g(|K_{\tilde{x}'}(n)|)} \leq \limsup_{n \rightarrow \infty} \frac{|\{m : m \leq n, x_m - x_0 \notin U\}|}{g(n)}.$$

Since \tilde{x} is d_g -statistically convergent to x_0 , the right hand side of the above inequality is zero. Also by our assumption $\liminf_{n \rightarrow \infty} \frac{g(|K_{\tilde{x}'}(n)|)}{g(n)} > 0$, hence we find

$$\limsup_{n \rightarrow \infty} \frac{|\{m : m \leq n, x_m - x_0 \notin U\}|}{g(|K_{\tilde{x}'}(n)|)} = 0.$$

Hence \tilde{x}' is d_g -statistically convergent to x_0 . \square

The relation between bounded sequences and convergent sequences in an arbitrary metric space is known. How will it be for d_g -statistical boundedness and d_g -statistical convergence? The next results will answer this question and give some relations between d_g -statistical boundedness and d_g -statistical convergence of Riesz space-valued sequences.

Theorem 3.7. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be a sequence in L . Then the following statements hold:

- (i) If \tilde{x} is bounded then \tilde{x} is d_g -statistically bounded.
- (ii) If \tilde{x} is d_g -statistically convergent to $x_0 \in L$ then \tilde{x} is d_g -statistically bounded.

Proof. (i) The proof is trivial.

(ii) For any arbitrary τ -neighborhood U of zero we can choose a τ -neighborhood U_0 of zero such that $\{k : k \leq n, x_k - x_0 \notin U_0\} \subset \{k : k \leq n, x_k - x_0 \in U\}$ i.e. $|\{k : k \leq n, x_k - x_0 \notin U_0\}| \leq |\{k : k \leq n, x_k - x_0 \in U\}|$. For this inequality we have $\limsup_{n \rightarrow \infty} \frac{|\{k : k \leq n, x_k - x_0 \notin U_0\}|}{g(n)} = 0$. Therefore \tilde{x} is d_g -statistically bounded. \square

Note 3.4. The converse of (i) and (ii) does not hold generally as can be seen from [24].

Theorem 3.8. Let (L, τ) be a locally solid Riesz space and let $\tilde{x} = (x_n) \in \tilde{L}$. Let $\tilde{x}' = (x_{n_k})$ be a subsequence of \tilde{x} which is dense in (x_n) . If \tilde{x} is d_g -statistically bounded then \tilde{x}' is also d_g -statistically bounded.

Proof. Suppose \tilde{x} is d_g -statistically bounded. It is clear that there exists a τ -neighborhood U_0 of zero and $x_0 \in L$ such that $\{n_k : n_k \leq n, x_{n_k} - x_0 \notin U_0\} \subset \{k : k \leq n, x_k - x_0 \notin U_0\}$. Then since $|\{n_k : n_k \leq n, x_{n_k} - x_0 \notin U_0\}| \leq |\{k : k \leq n, x_k - x_0 \notin U_0\}|$ we have $0 \leq \limsup_{n \rightarrow \infty} \frac{|\{n_k : n_k \leq n, x_{n_k} - x_0 \notin U_0\}|}{g(n)} \leq 0$. That is \tilde{x}' is d_g -statistically bounded. \square

Theorem 3.9. Let (L, τ) be a locally solid Riesz space and let $\tilde{x} = (x_n) \in \tilde{L}$. Then the following statements are equivalent:

(a) \tilde{x} is d_g -statistically convergent;

(b) Every subsequence \tilde{x}' of \tilde{x} with $\liminf_{n \rightarrow \infty} \frac{g(|K_{\tilde{x}'}(n)|)}{g(n)} > 0$ is d_g -statistically convergent;

(c) Every d_g -statistically dense subsequence \tilde{x}' of \tilde{x} is d_g -statistically convergent provided that $g \in G$ is such that $0 < \liminf_{n \rightarrow \infty} \frac{n}{g(n)} < \infty$.

Proof. (a) \Rightarrow (b) follows from the Theorem 3.2. Since it is obvious that \tilde{x} is a d_g -dense subsequence of itself, we conclude that (c) \Rightarrow (a).

(b) \Rightarrow (c) The proof is similar to the proof of Theorem 3.3 [13]. \square

The next results are given in the more general version in terms of ideals.

Lemma 3.1. Let (L, τ) be a locally solid Riesz space with $|L| > 2$, let $\tilde{x} = (x_n) \in \tilde{L}$ and let $\tilde{x}' = (x_{n(k)})$ be an infinite subsequence of \tilde{x} such that $K_{\tilde{x}'} \in \tilde{I}$. Then there exists a sequence $\tilde{y} \in \tilde{L}$ and a subsequence \tilde{y}' of \tilde{y} such that $K_{\tilde{x}'} = K_{\tilde{y}'}$, where \tilde{y}' is not \tilde{I} -convergent provided that \tilde{I} is not a maximal ideal.

Proof. Choose a and b be two disjoint elements from L and a subset $M \subset \mathbb{N}$ such that $M \notin \tilde{I}$ and in addition $M \notin \mathcal{F}(\tilde{I})$. Now let us define a sequence $\tilde{y} = (y_n) \in \tilde{L}$ as:

$$y_n = \begin{cases} x_n & \text{if } n \in \mathbb{N} \setminus K_{\tilde{x}'} \\ a & \text{if } n = n(k) \in K_{\tilde{x}'}, \text{ where } k \in M \\ b & \text{if } n = n(k) \in K_{\tilde{x}'}, \text{ where } k \notin M \end{cases}$$

Since $K_{\tilde{x}'} \in \tilde{I}$, we get $\mathbb{N} \setminus K_{\tilde{x}'} \in \mathcal{F}(\tilde{I})$, which shows that $\tilde{x} \approx \tilde{y}$ (ideally). Obviously taking $\tilde{y}' = (y_{n(k)})$ we have $K_{\tilde{x}'} = K_{\tilde{y}'}$. Hence for any $c \in L$ choose a τ -neighborhood U of zero with $(a - c) \vee (b - c) \in U$, we observe that $\{k : y_{n(k)} - c \notin U\} \supset M$ or M^c and thus cannot belong to \tilde{I} . This shows that \tilde{y}' is not \tilde{I} -convergent. \square

Lemma 3.2. Let (L, τ) be a locally solid Riesz space, let $a \in L$ and $\tilde{x} = (x_n), \tilde{y} = (y_n) \in \tilde{L}$. If \tilde{x} is \tilde{I} -convergent to a and $\tilde{x} \approx \tilde{y}$ (ideally), then \tilde{y} is also \tilde{I} -convergent to a .

Proof. Since $\tilde{x} \approx \tilde{y}$ (ideally), there is $M \in \mathcal{F}(\tilde{I})$ such that $x_n = y_n$ for all $n \in M$. Hence clearly for any τ -neighborhood U of zero $\{n : y_n - a \notin U\} \subset M^c \cup \{n : x_n - a \notin U\}$. Since \tilde{x} is \tilde{I} -convergent to a , $\{n : x_n - a \notin U\} \in \tilde{I}$. Which implies that $\{n : y_n - a \notin U\} \in \tilde{I}$ and hence \tilde{y} is also \tilde{I} -convergent to a . \square

Using these Lemma 3.1 and Lemma 3.2 we can formulate the following theorem:

Theorem 3.10. Let (L, τ) be a locally solid Riesz space with $|L| > 2$, let $a \in L$ and \tilde{I} be not maximal. Also let $\tilde{x} = (x_n)$ be \tilde{I} -convergent to a . Then for every infinite subsequence \tilde{x}' of \tilde{x} with $K_{\tilde{x}'} \in \tilde{I}$ there exists a sequence \tilde{y} in L and a subsequence \tilde{y}' of \tilde{y} such that:

(i) $\tilde{x} \approx \tilde{y}$ (ideally) and $K_{\tilde{x}'} = K_{\tilde{y}'}$

(ii) \tilde{y} is \tilde{I} -convergent to a .

(iii) \tilde{y}' is not \tilde{I} -convergent.

Lemma 3.3. Let (L, τ) be a locally solid Riesz space and $\tilde{x}, \tilde{y} \in \tilde{L}$ with $\tilde{x} \approx \tilde{y}$ (d_g -statistically). If K is a subset of \mathbb{N} such that $\liminf_{n \rightarrow \infty} \frac{g(|K(n)|)}{g(n)} > 0$ and if $\tilde{x}' = (x_{n(k)})$ and $\tilde{y}' = (y_{n(k)})$ are subsequences of \tilde{x}, \tilde{y} respectively such that $K_{\tilde{x}'} = K_{\tilde{y}'} = K$ then the relation $\tilde{x}' \approx \tilde{y}'$ (d_g -statistically) is true.

Proof. The proof is similar to the usual case with some trivial modification and so we omit it. \square

Theorem 3.11. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be d_g -statistically convergent to a . Suppose that $\tilde{x}' = (x_{n(k)})$ is a subsequence of \tilde{x} for which there are $\tilde{y} = (y_n)$ and \tilde{y}' such that (i) $\tilde{x} \asymp \tilde{y}$ (d_g -statistically) and $K_{\tilde{x}'} = K_{\tilde{y}'}$. (ii) \tilde{y}' is not d_g -statistically convergent. Then $\liminf_{n \rightarrow \infty} \frac{|K_{\tilde{x}}(n)|}{g(n)} = 0$ provided that $g : \mathbb{N} \rightarrow [0, \infty)$ satisfying the inequalities $0 < \liminf_{n \rightarrow \infty} \frac{n}{g(n)}$ and $\limsup_{n \rightarrow \infty} \frac{n}{g(n)} < \infty$.

Proof. If possible suppose that $\liminf_{n \rightarrow \infty} \frac{|K_{\tilde{x}}(n)|}{g(n)} > 0$. Then

$$\liminf_{n \rightarrow \infty} \frac{g(|K_{\tilde{x}'}(n)|)}{g(n)} \geq \liminf_{n \rightarrow \infty} \frac{g(|K_{\tilde{x}}(n)|)}{|K_{\tilde{x}'}(n)|} \liminf_{n \rightarrow \infty} \frac{|K_{\tilde{x}}(n)|}{g(n)} > 0$$

Let $\tilde{y} \in \tilde{L}$ and \tilde{y}' be a subsequence of \tilde{y} such that (i) and (ii) hold. Then we have $K_{\tilde{x}'} = K_{\tilde{y}'}$ and $\tilde{x} \asymp \tilde{y}$ (d_g -statistically). Thus it follows from Lemma 3.2 that $\tilde{x}' \asymp \tilde{y}'$ (d_g -statistically). Now applying Theorem 3.4 we conclude that \tilde{x}' is d_g -statistically convergent to a . Since $\tilde{x}' \asymp \tilde{y}'$ (d_g -statistically), by Lemma 3.2, \tilde{y}' is also d_g -statistically convergent to a . Which contradicts (ii). Hence the theorem. \square

Now we will give some relations between d_g -statistical boundedness with usual boundedness.

Lemma 3.4. Let (L, τ) be a locally solid Riesz space and let $\tilde{x} = (x_n) \in \tilde{L}$. Then the sequence \tilde{x} is d_g -statistically bounded if and only if the sequence $(x_n - x)$ is d_g -statistical bounded for an arbitrary $x \in L$.

Proof. It follows from the definition. \square

Theorem 3.12. Let (L, τ) be a locally solid Riesz space and let $\tilde{x} = (x_n) \in \tilde{L}$ be a d_g -statistically bounded sequence. Then the sequence has at least one bounded subsequence.

Proof. From the Lemma 3.4 the sequence $(x_n - x)$ is d_g -statistical bounded for an arbitrary $x \in L$. Thus there exists a τ -neighborhood U_0 of zero such that $d_g(A) = 1$ and $d_g(B) = 0$. Where, $A = \{k : x_k - x \in U_0\}$ and $B = \{k : x_k - x \notin U_0\}$. Let $k_1 \in \mathbb{N}$ be the minimal element of A and $x_{k_1} - x \in U_0$. Since $d_g(A) = 1$, it can be chosen $k_2 \geq k_1$ such that the minimal element of the set $\{k : k > k_1, k \in A\}$ satisfying $x_{k_2} - x \in U_0$. In the n -th step we can chose $k_n \geq k_{n-1}$ which is the minimal element of the set $\{k : k > k_{n-1}, k \in A\}$ such that $x_k - x \in U_0$. So we obtain a non-decreasing sequence (k_n) such that $\tilde{x}' = (x_{n_k})$ is the subsequence of \tilde{x} satisfying $x_{k_n} - x \in U_0$ for all $k_n \in \mathbb{N}$. This shows that the subsequence \tilde{x}' is bounded. \square

Theorem 3.13. Let (L, τ) be a locally solid Riesz space. Also let $\tilde{x} = (x_n), \tilde{y} = (y_n) \in \tilde{L}$ and \tilde{x} is d_g -statistically bounded. If $\tilde{x} \asymp \tilde{y}$ (d_g -statistically), then \tilde{y} is also d_g -statistical bounded.

Proof. The proof is parallel to the proof in [24]. \square

Note 3.5. Let (L, τ) be a locally solid Riesz space and $\tilde{x} \in \tilde{L}$. If every subsequence \tilde{x}' of \tilde{x} with

$$\liminf_{n \rightarrow \infty} \frac{g(|K_{\tilde{x}'}(n)|)}{g(n)} > 0 \text{ is } d_g\text{-statistically bounded then } \tilde{x} \text{ must be } d_g\text{-statistical bounded.}$$

3.3. Some further investigation for convergence of functions:

In this section we investigate certain aspects of ideal convergence of Riesz space valued functions, in a very general context. We consider the situation when an \mathcal{I} -convergent function f from S to a Riesz space L will have a F -subfunction which is K -convergent to the same limit. Further we also introduce the notion of \mathcal{I} -cluster points of Riesz space valued functions and make some observations.

Let S be an arbitrary infinite set and (L, τ) be a Locally solid Riesz space. Let \mathcal{I}, K be ideals of S . $F(\mathcal{I})$ denotes filter associated with ideal \mathcal{I} . We recall that a topological space (L, τ) is called finitely generated space (Alexandroff space) if any intersection of open subsets of L , is open set. L is finitely generated if and only if each point of L has a smallest neighborhood.

The notion of \mathcal{I}^K -convergence was first introduced by Maćaj [27].

Definition 3.18. Any function $f : S \rightarrow L$ is said to be \mathcal{I} -convergent to $x \in L$ if for every τ neighborhood U of zero, the set $\{s \in S : f(s) - x \notin U\} \in \mathcal{I}$.

In this case we say $\mathcal{I}_\tau - \lim f = x$

Definition 3.19. Any function $f : S \rightarrow L$ is said to be \mathcal{I}_τ^* -convergent to $x \in L$ if there exists set $M \in F(\mathcal{I})$ such that g is defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ x & \text{if } s \notin M. \end{cases}$$

is convergent to x .

In this case we say $\mathcal{I}_\tau^* - \lim f = x$

Definition 3.20. Any function $f : S \rightarrow L$ is said to be \mathcal{I}_τ^K -convergent to $x \in L$ if there exists set $M \in F(\mathcal{I})$ such that g is defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ x & \text{if } s \notin M. \end{cases}$$

is K -convergent to x .

In this case we say $\mathcal{I}_\tau^K - \lim f = x$

Note 3.6. If $S = \mathbb{N}$, then we obtain usual \mathcal{I} -convergence, \mathcal{I}^* -convergence, \mathcal{I}^K -convergence

From the above definitions the theorem easily follows.

Theorem 3.14. Let \mathcal{I} and K be two ideals of S , and $f : S \rightarrow L$ be a function such that $K_\tau - \lim f = x$ then $\mathcal{I}_\tau^K - \lim f = x$

Definition 3.21. $f : S \rightarrow X$ is an arbitrary function. Let $F \subseteq 2^S$. A nonempty function g is defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in A \\ x & \text{if } s \notin A. \end{cases}$$

where $A \in F$ and $x \in L$ is called a F -subfunction of f with respect to x .

Theorem 3.15. Let \mathcal{I} and K be two ideals on a set S such that $K \subset \mathcal{I}$, and $F \subset 2^S$. Let (L, τ) be Riesz space where the topology τ on L is first countable and not finitely generated. Then the following two conditions are equivalent :

(i) for any \mathcal{I} -convergent function $f : S \rightarrow L$ has a F -subfunction which is K -convergent to the same limit.

(ii) for any sequence of sets $(A_n)_{n \in \mathbb{N}}$ from \mathcal{I} there exist $A \in F$ such that $A \cap A_n \in K$ for all $n \in \mathbb{N}$

Proof. (i) \Rightarrow (ii): As L is not finitely generated, so we get $x \in L$, such that there is a sequence of distinct points x_n in X convergent to x . Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of sets from \mathcal{I} . We have to prove that there exist $A \in F$ such that $A \cap A_n \in K$ for all $n \in \mathbb{N}$.

Let $U_n = \bigcup_{k=1}^n A_k$, then clearly $A_n \subset U_n$ and each U_n belongs to \mathcal{I} .

Now we define $f : S \rightarrow L$ by

$$f(s) = \begin{cases} x_n & \text{when } s \in U_{n+1} \setminus U_n \\ x & \text{when } s \notin \bigcup_{n \in \mathbb{N}} A_n. \end{cases}$$

Take any τ neighborhood U of zero, then there exists $m \in \mathbb{N}$ such that $x_n - x \in U$ for all $n \geq m$. Hence $\{s \in S : f(s) - x \notin U\} \subset U_{m+1} \in \mathcal{I}$. Hence f is \mathcal{I} -convergent to x .

Now from the given condition f has a F -subfunction g which is K -convergent to x . i.e., there is $A \in F$ such that the function g defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in A \\ x & \text{if } s \notin A. \end{cases}$$

is K -convergent to x . Which implies $A \cap U_n \in K$ for all n .
Consequently $A \cap A_n \subset A \cap U_n \in K$.

(ii) \Rightarrow (i): Let $f : S \rightarrow L$ be a function such that $I_\tau\text{-}\lim f = x$, where $x \in L$. As (L, τ) is first countable space, let U_n be the monotonically decreasing τ -neighborhoods of zero. Now define, $V_n = \{s \in S : f(s) - x \notin U_n\}$, clearly V_n is sequence of sets from \mathcal{I} . So from the given condition there exists $A \in F$ such that $A \cap V_n \in K$ for all $n \in \mathbb{N}$.

We define g by,

$$g(s) = \begin{cases} f(s) & \text{if } s \in A \\ x & \text{if } s \notin A. \end{cases}$$

Clearly g is F -subfunction of f . We have to show that $K_\tau\text{-}\lim g = x$.

Consider any τ -neighborhood U of zero, as U_n is the monotonically decreasing τ -neighborhoods of zero. So we get a $n_0 \in \mathbb{N}$ such that $U_{n_0} \subset U$. Then $\{s \in S : g(s) - x \notin U\} \subset \{s \in S : g(s) - x \notin U_{n_0}\} \subset A \cap V_{n_0} \in K$. Hence g is K -convergent to x . \square

Let S be an arbitrary infinite set and (L, τ) be a locally solid Riesz space. Also let \mathcal{I} be an ideal of S .

Definition 3.22. Let $f : S \rightarrow L$ be a function. $x \in L$ is said to be limit point of f , if for every τ neighborhood U of x , the set $\{s \in S : f(s) \in U\}$ is infinite.

By $L(f)$ we denote the set of all limit points of f .

Definition 3.23. $x \in L$ is said to be \mathcal{I} -cluster point of a function $f : S \rightarrow L$ if for every τ neighborhood U of x , $\{s \in S : f(s) \in U\} \notin \mathcal{I}$.

By $C_{\mathcal{I}}(f)$ we denote the set of all \mathcal{I} -cluster points of f with respect to the ideal \mathcal{I} .

Definition 3.24. A function $f : S \rightarrow L$ is said to be \mathcal{I} -maximal if for any set $A \subset L$, either $\{s \in S : f(s) \notin A\} \in \mathcal{I}$ or $\{s \in S : f(s) \notin L \setminus A\} \in \mathcal{I}$.

Theorem 3.16. Let the function $f : S \rightarrow L$ be \mathcal{I} -maximal. If $x_0 \in C_{\mathcal{I}}(f)$ then f is \mathcal{I} -convergent to x_0 .

Proof. Let $x_0 \in C_{\mathcal{I}}(f)$. Let U be any τ -neighborhood of x_0 . By our assumption f is \mathcal{I} -maximal. So either $\{s \in S : f(s) \notin U\} \in \mathcal{I}$ or $\{s \in S : f(s) \notin L \setminus U\} \in \mathcal{I}$. If $\{s \in S : f(s) \notin L \setminus U\} \in \mathcal{I}$ then this implies $\{s \in S : f(s) \in U\} \in \mathcal{I}$. Which contradicts the fact that $x_0 \in C_{\mathcal{I}}(f)$. So $\{s \in S : f(s) \notin U\} \in \mathcal{I}$. Hence f is \mathcal{I} -convergent to x_0 . \square

Theorem 3.17. If $f : S \rightarrow L$ and $g : S \rightarrow L$ be two functions such that $\{s \in S : f(s) \neq g(s)\} \in \mathcal{I}$, then $C_{\mathcal{I}}(f) = C_{\mathcal{I}}(g)$.

Proof. Let $x_0 \in C_{\mathcal{I}}(f)$ then for any τ -neighborhood U of x_0 , $\{s \in S : f(s) \in U\} \notin \mathcal{I}$. If possible let $x_0 \notin C_{\mathcal{I}}(g)$ then there exist τ -neighborhood W of x_0 such that $\{s \in S : g(s) \in W\} \in \mathcal{I}$. Now,

$$\{s \in S : f(s) \in W\} \subset \{s \in S : g(s) \in W\} \cup \{s \in S : f(s) \neq g(s)\} \in \mathcal{I}.$$

So we get a τ -neighborhood W of x_0 , that $\{s \in S : f(s) \in W\} \in \mathcal{I}$. This contradicts the assumption that $x_0 \in C_{\mathcal{I}}(f)$. Hence $x_0 \in C_{\mathcal{I}}(g)$. This implies $C_{\mathcal{I}}(f) \subseteq C_{\mathcal{I}}(g)$. Similarly $C_{\mathcal{I}}(g) \subseteq C_{\mathcal{I}}(f)$. Hence $C_{\mathcal{I}}(f) = C_{\mathcal{I}}(g)$. \square

Theorem 3.18. Let $f : S \rightarrow L$ be a function. Let $x_0 \in L$. Then the following two conditions are equivalent

(i) $x_0 \in \overline{C_{\mathcal{I}}(f)}$

(ii) $x_0 \in \overline{f(M)}$, for every $M \in F(\mathcal{I})$.

Here $f(M) = \{f(m) : m \in M\}$, $F(\mathcal{I})$ is filter associated with the ideal \mathcal{I} and $\overline{f(M)}$ denotes the closure of $f(M)$.

Proof. (i) \Rightarrow (ii) : Let $x_0 \in C_I(f)$. Let U be any τ -neighborhood of x_0 . So $\{s \in S : f(s) \in U\} \notin \mathcal{I}$. This implies for any $M \in F(\mathcal{I})$, $M \not\subseteq \{s \in S : f(s) \in L \setminus U\}$. Hence there exists $m \in M$ such that $f(m) \in U$. Which implies $U \cap f(M) \neq \phi$, this is true for any arbitrary τ -neighborhood U of x_0 . Hence $x_0 \in \overline{f(M)}$, for every $M \in F(\mathcal{I})$.

(ii) \Rightarrow (i) : Let $x_0 \in \overline{f(M)}$, for every $M \in F(\mathcal{I})$. If possible let $x_0 \notin C_I(f)$. Then we get a τ -neighborhood V of x_0 such that $\{s \in S : f(s) \in V\} \in \mathcal{I}$. Hence for $M = \{s \in S : f(s) \in L \setminus V\} \in F(\mathcal{I})$ and $x_0 \in \overline{f(M)}$. So $V \cap f(M) \neq \phi$. Then there exists $y_0 \in V$ such that $y_0 = f(m_0)$ for some $m_0 \in M \subseteq F(\mathcal{I})$, which implies $y_0 = f(m_0) \in L \setminus V$. We get a contradiction. Hence $x_0 \in C_I(f)$. \square

Theorem 3.19. Let $f : S \rightarrow L$ be a function. Then for any compact subset G of L if $\{s \in S : f(s) \in G\} \notin \mathcal{I}$ then $G \cap C_I(f) \neq \phi$.

Proof. If possible let $G \cap C_I(f) = \phi$. Then for every $x \in G$, $x \notin C_I(f)$. So we get a τ -neighborhood U_x of x such that $\{s \in S : f(s) \in U_x\} \in \mathcal{I}$. Clearly $\{U_x : x \in G\}$ is an τ -open cover of compact set G hence $\{s \in S : f(s) \in G\} \subset \bigcup_{i=1}^n \{s \in S : f(s) \in U_{x_i}\} \in \mathcal{I}$. Which contradicts the given assumption. Hence $G \cap C_I(f) \neq \phi$. \square

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