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Bifurcation Analysis and Chaos Control of a Second-Order Exponential Difference Equation

Q. Din[a], E. M. Elabbasy[b], A. A. Elsadany[c,d,*], S. Ibrahim[d]

[a]*Department of Mathematics, University of Poonch Rawalakot, Pakistan.*
[b]*Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.*
[c]*Department of Mathematics, College of Sciences and Humanities Studies in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia.*
[d]*Department of Basic Science, Faculty of Computers and Informatics, Suez Canal University, Ismailia 41522, Egypt.*

Abstract. The aim of this article is to study the local stability of equilibria, investigation related to the parametric conditions for transcritical bifurcation, period-doubling bifurcation and Neimark-Sacker bifurcation of the following second-order difference equation

$$x_{n+1} = \alpha x_n + \beta x_{n-1} \exp(-\sigma x_{n-1})$$

where the initial conditions x_{-1}, x_0 are the arbitrary positive real numbers and α, β and σ are positive constants. Moreover, chaos control method is implemented for controlling chaotic behavior under the influence of Neimark-Sacker bifurcation and period-doubling bifurcation. Numerical simulations are provided to show effectiveness of theoretical discussion.

1. Introduction

Studying the dynamics of a difference equation means that we attempt to do the following actions: determine equilibrium points, analyze their stability, asymptotic stability, boundedness and bifurcation. The dynamic of any situation refers to how the situation changes over the course of time. In many scientific fields researchers need to study difference or differential equations that contain parameters, so it is important to study the behavior of these equations as the value of certain parameter varies. This study focuses on the concept of bifurcation. Qualitative theory of difference equations which parallels the qualitative theory of differential equations has been investigated by several authors. For more details of theory of difference equations and their applications, we refer to the books [1–8]. Investigation of the global stability character, boundedness and bifurcation of difference equations has been considered by many authors such that in [9] Metwally *et al*. studied the boundedness, the asymptotic behavior, the periodic character and the stability of solutions of the difference equation

$$x_{n+1} = \alpha + \beta x_{n-1} \exp(-x_n),$$

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*Corresponding author: A. A. Elsadany
Email address: aelsadany1@yahoo.com, a.elsadany@psau.edu.sa (A. A. Elsadany)

where the parameters α and β are positive numbers and the initial conditions are arbitrary non-negative real numbers.

Wenjie *et al*. in [10] studied the boundedness and the asymptotic behavior of the positive solutions for difference equation

$$x_{n+1} = a + bx_n \exp(-x_{n-1}),$$

where a and b are positive constants and the initial values x_{-1}, x_0 are non-negative numbers.

In [11] the global behavior of the positive solutions for difference equation

$$x_{n+1} = ax_n + bx_{n-1} \exp(-x_n),$$

is investigated, where a, b are positive constants and the initial values x_{-1}, x_0 are positive numbers.

The authors in [12] studied the system of difference equations:

$$x_{n+1} = ay_n + bx_{n-1} \exp(-y_n), \quad x_{n+1} = cx_n + dy_{n-1} \exp(-x_n),$$

where a, b, c and d are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive numbers.

Hui Feng *et al*. in [13] investigated the global stability and bounded nature of the positive solutions for difference equation

$$x_{n+1} = a + bx_{n-1} + cx_{n-1} \exp(-x_n),$$

where the parameters $a \in (0, \infty)$, $b \in (0, 1)$, $c \in (0, \infty)$ and the initial conditions are arbitrary non-negative numbers. For other papers related to the qualitative behavior of difference, we refer to [14–24]. Furthermore, for bifurcation analysis and chaos control in discrete-time models, we refer to [25–33].

The main purpose of this paper is to investigate the local stability of equilibria, bifurcation analysis and chaos control for the following second-order exponential difference equation:

$$x_{n+1} = \alpha x_n + \beta x_{n-1} \exp(-\sigma x_{n-1}), \; n = 0, 1, \cdots, \tag{1.1}$$

where the parameters α, β and σ are positive numbers and the initial conditions x_{-1} and x_0 are arbitrary positive real numbers. Moreover, introducing $y_n = x_{n-1}$, we obtain the following planar discrete-time system equivalent to (1.1):

$$\begin{aligned} x_{n+1} &= \alpha x_n + \beta y_n \exp(-\sigma y_n), \\ y_{n+1} &= x_n. \end{aligned} \tag{1.2}$$

This paper was organized as follows. In Section 2, we determine the equilibrium points and explore the parametric conditions for the the local stability of the equilibria of system (1.2). In Section 3, we investigate parametric conditions for transcritical bifurcation, period-doubling bifurcation and Neimark-Sacker bifurcation at the fixed points of a two-dimensional map associated to system (1.2). Section 4 is dedicated to chaos control for the system (1.2) under the influence of Neimark-Sacker bifurcation and period-doubling bifurcation. Finally, in Section 5, some numerical examples are presented in order to illustrate the theoretical discussions.

2. Local stability analysis of equilibria

In this section, we study local dynamical behavior for equilibria of system (1.2). First we see that steady-states of (1.2) solve the following system:

$$x = \alpha x + \beta y \exp(-\sigma y), \; y = x.$$

Solving aforementioned system yields $E_0 = (0, 0)$ and $E_1 = \left(\frac{1}{\sigma} \ln \frac{\beta}{1-\alpha}, \frac{1}{\sigma} \ln \frac{\beta}{1-\alpha}\right)$ two equilibria for system (1.2). Moreover, assume that $0 < \alpha < 1$ and $\alpha + \beta > 1$, then $\left(\frac{1}{\sigma} \ln \frac{\beta}{1-\alpha}, \frac{1}{\sigma} \ln \frac{\beta}{1-\alpha}\right)$ is unique positive equilibrium point for system (1.2). Assume that $J(x, y)$ denotes Jacobian matrix of system (1.2) evaluated at (x, y), then it follows that:

$$J(0,0) = \begin{pmatrix} \alpha & \beta \\ 1 & 0 \end{pmatrix},$$

and

$$J\left(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}, \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}\right) = \begin{pmatrix} \alpha & (1-\alpha)\left(1-\ln\left(\frac{\beta}{1-\alpha}\right)\right) \\ 1 & 0 \end{pmatrix}.$$

Moreover, characteristic polynomial for $J(0,0)$ is computed as follows:

$$P(\lambda) = \lambda^2 - \alpha\lambda - \beta, \tag{2.1}$$

and characteristic polynomial computed for $J\left(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}, \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}\right)$ is given by:

$$P(\lambda) = \lambda^2 - \alpha\lambda - 1 + \alpha + \ln\left(\frac{\beta}{1-\alpha}\right) - \alpha\ln\left(\frac{\beta}{1-\alpha}\right). \tag{2.2}$$

Keeping in view the relation between roots and coefficients for a quadratic equation, the following Lemma gives local dynamical behavior of system (1.2) at its trivial steady-state $(0,0)$.

Lemma 2.1. *The following statements hold true related to equilibrium* $(0,0)$ *of* (1.2)*:*

(*I*) $(0,0)$ *lies inside the open unit disk if and only if* $0 < \alpha < 1$ *and* $0 < \beta < 1-\alpha$*.*

(*II*) $(0,0)$ *is a saddle point for the system* (1.2) *if and only if* $0 < \alpha \leq 1$ *and* $1-\alpha < \beta < 1+\alpha$*, or* $\alpha > 1$ *and* $0 < \beta < 1+\alpha$*.*

(*III*) $(0,0)$ *is a source (repeller) for the system* (1.2) *if and only if* $\beta > 1+\alpha$*.*

(*IV*) $(0,0)$ *is a non-hyperbolic point if and only if* $1+\alpha = \beta$*, or* $\alpha+\beta = 1$*.*

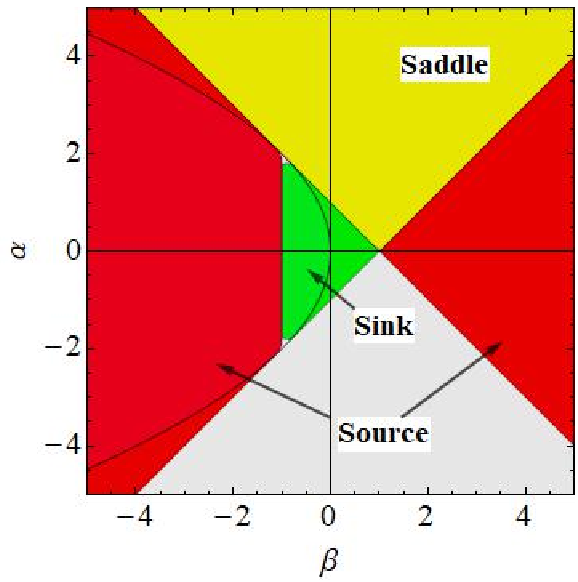

Figure 1: Sink (green), saddle (yellow) and source (red) regions for system (1.2) at $(0,0)$.

Similarly, for positive equilibrium point, we have the following Lemma:

Lemma 2.2. *Assume that $0 < \alpha < 1$, then for positive equilibrium of the system* (1.2) *the following statements hold true:*

(I) $\left(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}, \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}\right)$ *lies inside the unit open disk if and only if*

$$1-\alpha < \beta < e^{\frac{(2-\alpha)^2}{4(1-\alpha)}}(1-\alpha).$$

(II) $\left(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}, \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}\right)$ *is a saddle point if and only if*

$$\alpha+\beta < 1,\ \beta > e^{-\frac{2\alpha}{1-\alpha}}(1-\alpha).$$

(III) $\left(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}, \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}\right)$ *is a repeller (source) if and only if*

$$0 < \beta < e^{-\frac{2\alpha}{1-\alpha}}(1-\alpha).$$

(IV) $\left(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}, \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}\right)$ *is a non-hyperbolic if and only if*

$$\beta = 1-\alpha,$$

or

$$\beta = (1-\alpha)\exp\left(-\frac{2\alpha}{1-\alpha}\right),$$

or

$$\beta = (1-\alpha)\exp\left(\frac{2-\alpha}{1-\alpha}\right).$$

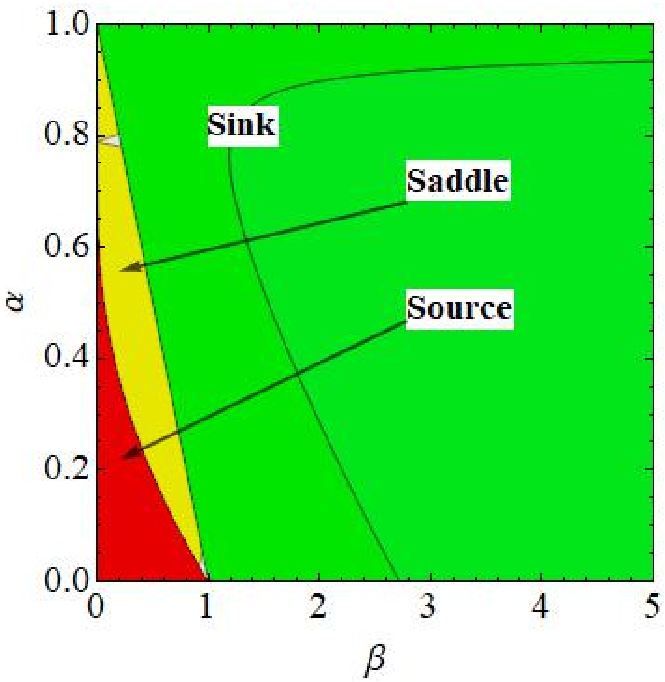

Figure 2: Sink (green), saddle (yellow) and source (red) regions for system (1.2) at positive equilibrium.

3. Bifurcation analysis

In this section, we investigate the parametric conditions for existence of transcritical bifurcation, period-doubling bifurcation and Neimark-Sacker bifurcation for the system (1.2) at its fixed points.

3.1. Transcritical bifurcation at E_0

Assume that $\beta = 1-\alpha$, then $J(0,0)$ has two eigenvalues $\nu_1 = \alpha - 1$ and $\nu_2 = 1$. In this case, we discuss a transcritical bifurcation. The first equation of the system (1.2) takes the form

$$\begin{aligned} x_{n+1} &= \alpha x_n + \beta y_n \exp(-\sigma y_n) \\ &= \alpha x_n + \beta y_n [1 - \sigma y_n + \frac{\sigma^2 y_n^2}{2} + \cdots] \\ &= \alpha x_n + \beta y_n - \beta\sigma y_n^2 + O(| y_n |^3) \end{aligned}$$

Lemma 3.1. *If $\beta = 1-\alpha$, $\alpha \neq 0$, the system* (1.2) *undergoes a transcritical bifurcation at the fixed point* $(0,0)$.

Proof. Let $\mu_n = \beta - (1-\alpha)$, the system (1.2) becomes

$$\begin{aligned} x_{n+1} &= \alpha x_n + (1-\alpha) y_n + \mu_n y_n - \sigma(1-\alpha) y_n^2 - \sigma\mu_n y_n^2, \\ \mu_{n+1} &= \mu_n, \\ y_{n+1} &= x_n, \end{aligned} \tag{3.1}$$

Let

$$T = \begin{pmatrix} \alpha-1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} \frac{1}{\alpha-2} & 0 & \frac{-1}{\alpha-2} \\ 0 & 1 & 0 \\ \frac{-1}{\alpha-2} & 0 & \frac{\alpha-1}{\alpha-2} \end{pmatrix}.$$

We use the transformation

$$\begin{pmatrix} x_n \\ \mu_n \\ y_n \end{pmatrix} = T \begin{pmatrix} u_n \\ \delta_n \\ v_n \end{pmatrix}.$$

Then, the map (3.1) can be rewritten in the following form:

$$\begin{pmatrix} u_{n+1} \\ \delta_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha-1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ \delta_n \\ v_n \end{pmatrix} + \begin{pmatrix} f_1(u_n, \delta_n, v_n) \\ 0 \\ f_2(u_n, \delta_n, v_n) \end{pmatrix} \tag{3.2}$$

where

$$\begin{aligned} f_1(u_n, \delta_n, v_n) &= \frac{1}{\alpha-2} \Big[\delta_n u_n + \delta_n v_n - \sigma(1-\alpha) u_n^2 - \sigma(1-\alpha) v_n^2 - 2\sigma(1-\alpha) u_n v_n \\ &\quad - \sigma\delta_n u_n^2 - \sigma\delta_n v_n^2 - 2\sigma\delta_n u_n v_n \Big], \\ f_2(u_n, \delta_n, v_n) &= \frac{-1}{\alpha-2} \Big[\delta_n u_n + \delta_n v_n - \sigma(1-\alpha) u_n^2 - \sigma(1-\alpha) v_n^2 - 2\sigma(1-\alpha) u_n v_n \\ &\quad - \sigma\delta_n u_n^2 - \sigma\delta_n v_n^2 - 2\sigma\delta_n u_n v_n \Big]. \end{aligned}$$

From the center manifold theory of discrete system, we can express a center manifold of the map (3.2) as follows

$$W^c(0,0) = \{(x, y, \mu) \in \mathbb{R}^3 \mid u = h(v, \delta),\ h(0,0) = Dh(0,0) = 0, | v |< \varepsilon,\ \ | \delta |< \eta\}.$$

Assume that $h(y,\mu)$ has the following form

$$u_{n+1} = h(v_n,\delta_n) = a_1 v_n^2 + a_2 y_n\delta_n + a_3\delta_n^2 + o((|v_n| + |\delta_n|)^3),$$

which must satisfy

$$h(v_n,\delta_n) = (\alpha-1)h(v_n,\delta_n) + f_1(h(v_n,\delta_n),\delta_n,v_n).$$

Then, we can find that

$$a_1 = \frac{\sigma(1-\alpha)}{(\alpha-2)^2},\ \ a_2 = \frac{-1}{(\alpha-2)^2}\quad \text{and } a_3 = 0,$$

therefore, we have $u_n = h(v_n,\delta_n) = \frac{\sigma(1-\alpha)}{(\alpha-2)^2}v_n^2 - \frac{1}{(\alpha-2)^2}\delta_n v_n$, and

$$\begin{aligned} g_1 &= v_{n+1} = v_n + f_2(u_n,\delta_n,v_n), \\ &= v_n + f_2(\frac{\sigma(1-\alpha)}{(\alpha-2)^2}v_n^2 - \frac{1}{(\alpha-2)^2}\delta_n v_n,\delta_n,v_n), \end{aligned}$$

hence

$$\begin{aligned} g_1 &= v_{n+1} = v_n - \frac{\sigma(1-\alpha)}{(\alpha-2)^3}\delta_n v_n^2 + \frac{1}{(\alpha-2)^3}\delta_n^2 v_n - \frac{1}{\alpha-2}\delta_n v_n + \frac{\sigma^3(1-\alpha)^3}{(\alpha-2)^5}v_n^4 \\ &+\frac{\sigma(1-\alpha)}{(\alpha-2)^5}\delta_n^2 v_n^2 - \frac{2\sigma^2(1-\alpha)^2}{(\alpha-2)^5}\delta_n v_n^3 + \frac{\sigma(1-\alpha)}{\alpha-2}v_n^2 + \frac{2\sigma^2(1-\alpha)^2}{(\alpha-2)^3}v_n^3 \\ &-\frac{2\sigma(1-\alpha)}{(\alpha-2)^3}\delta_n v_n^2 + \frac{\sigma^3(1-\alpha)^2}{(\alpha-2)^5}\delta_n v_n^4 + \frac{\sigma}{(\alpha-2)^5}\delta_n^3 v_n^2 - \frac{2\sigma^2(1-\alpha)}{(\alpha-2)^5}\delta_n^2 v_n^3 \\ &+\frac{\sigma}{\alpha-2}\delta_n v_n^2 + \frac{2\sigma^2(1-\alpha)}{(\alpha-2)^3}\delta_n v_n^3 - \frac{2\sigma}{(\alpha-2)^3}\delta_n^2 v_n^2 \end{aligned}$$

Since $\frac{\partial^2 g_1}{\partial v_n^2} = \frac{2\sigma(1-\alpha)}{\alpha-2} \neq 0$, $\frac{\partial^2 g_1}{\partial v_n \partial\delta_n} = -\frac{1}{\alpha-2} \neq 0$ and $\frac{\partial^2 g_1}{\partial v_n\partial\delta_n} - \frac{\partial^2 g_1}{\partial v_n^2}\frac{\partial^2 g_1}{\partial \delta_n^2} > 0$, if $\alpha < 2$. then the map (1.2) undergoes a transcritical bifurcation at the fixed point $(0,0)$. □

3.2. Period-doubling bifurcation at E_0

Suppose that $\beta = 1+\alpha$, then $J(0,0)$ has two eigenvalues $\nu_1 = \alpha+1$ and $\nu_2 = -1$. In this case we will investigate a flip (period-doubling) bifurcation.

Lemma 3.2. *If $\beta = 1+\alpha$, $\alpha \neq 0$, the map* (1.2) *undergoes a flip bifurcation at the fixed point* $(0,0)$.

Proof. Let $\mu_n = \beta - (1+\alpha)$, the map (1.2) becomes

$$\begin{aligned} x_{n+1} &= \alpha x_n + (1+\alpha)y_n + \mu_n y_n - \sigma(1+\alpha)y_n^2 - \sigma\mu_n y_n^2, \\ \mu_{n+1} &= \mu_n, \\ y_{n+1} &= x_n, \end{aligned} \tag{3.3}$$

Let

$$T = \begin{pmatrix} \alpha+1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},\quad T^{-1} = \begin{pmatrix} \frac{1}{\alpha+2} & 0 & \frac{1}{\alpha+2} \\ 0 & 1 & 0 \\ \frac{-1}{\alpha+2} & 0 & \frac{\alpha+1}{\alpha+2} \end{pmatrix}.$$

We use the transformation

$$\begin{pmatrix} x_n \\ \mu_n \\ y_n \end{pmatrix} = T \begin{pmatrix} u_n \\ \delta_n \\ v_n \end{pmatrix}.$$

which transforms the map (3.3) into the following standard form

$$\begin{pmatrix} u_{n+1} \\ \delta_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha+1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_n \\ \delta_n \\ v_n \end{pmatrix} + \begin{pmatrix} \phi_1(u_n, \delta_n, v_n) \\ 0 \\ \phi_2(u_n, \delta_n, v_n) \end{pmatrix} \tag{3.4}$$

where

$$\begin{aligned} \phi_1(u_n, \delta_n, v_n) &= \frac{1}{\alpha+2}\Big[\delta_n u_n + \delta_n v_n - \sigma(1+\alpha)u_n^2 - \sigma(1+\alpha)v_n^2 - 2\sigma(1+\alpha)u_n v_n \\ &\quad -\sigma\delta_n u_n^2 - \sigma\delta_n v_n^2 - 2\sigma\delta_n u_n v_n\Big] \\ \phi_2(u_n, \delta_n, v_n) &= \frac{-1}{\alpha+2}\Big[\delta_n u_n + \delta_n v_n - \sigma(1+\alpha)u_n^2 - \sigma(1+\alpha)v_n^2 - 2\sigma(1+\alpha)u_n v_n \\ &\quad -\sigma\delta_n u_n^2 - \sigma\delta_n v_n^2 - 2\sigma\delta_n u_n v_n\Big]. \end{aligned}$$

From the center manifold theory of discrete dynamical systems, we can express a center manifold of the map (3.4) as follows

$$W^c(0,0) = \{(x, y, \mu) \in \mathbb{R}^3 \mid u = h(v, \delta),\ h(0,0) = Dh(0,0) = 0, \mid v \mid < \varepsilon,\ \ \mid \delta \mid < \eta\}.$$

Assume that $h(y, \mu)$ has the following form

$$u_{n+1} = h(v_n, \delta_n) = b_1 v_n^2 + b_2 y_n \delta_n + b_3 \delta_n^2 + o((\mid v_n \mid + \mid \delta_n \mid)^3),$$

which must satisfy

$$h(v_n, \delta_n) = (\alpha+1)h(v_n, \delta_n) + \phi_1(h(v_n, \delta_n), \delta_n, v_n).$$

Then, we can find that

$$b_1 = \frac{\sigma(1+\alpha)}{\alpha+2}, \ \ b_2 = \frac{-1}{\alpha+2} \quad \text{and } b_3 = 0.$$

Therefore, we have $u_n = h(v_n, \delta_n) = \frac{\sigma(1+\alpha)}{\alpha+2} v_n^2 - \frac{1}{\alpha+2}\delta_n v_n$, and

$$\begin{aligned} g_2 &= v_{n+1} = -v_n + \phi_2(u_n, \delta_n, v_n), \\ &= -v_n + \phi_2\Big(\frac{\sigma(1+\alpha)}{\alpha+2} v_n^2 - \frac{1}{\alpha+2}\delta_n v_n, \delta_n, v_n\Big). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} g_2 &= v_{n+1} = -v_n - \frac{\sigma(1+\alpha)}{(\alpha+2)^2}\delta_n v_n^2 + \frac{1}{(\alpha+2)^2}\delta_n^2 v_n - \frac{1}{\alpha+2}\delta_n v_n + \frac{\sigma^3(1+\alpha)^3}{(\alpha+2)^3} v_n^4 \\ &\quad + \frac{\sigma(1+\alpha)}{(\alpha+2)^3}\delta_n^2 v_n^2 - \frac{2\sigma^2(1+\alpha)^2}{(\alpha+2)^3}\delta_n v_n^3 + \frac{\sigma(1+\alpha)}{\alpha+2} v_n^2 + \frac{2\sigma^2(1+\alpha)^2}{(\alpha+2)^2} v_n^3 \\ &\quad - \frac{2\sigma(1+\alpha)}{(\alpha+2)^2}\delta_n v_n^2 + \frac{\sigma^3(1+\alpha)^2}{(\alpha+2)^3}\delta_n v_n^4 + \frac{\sigma}{(\alpha+2)^3}\delta_n^3 v_n^2 - \frac{2\sigma^2(1+\alpha)}{(\alpha+2)^3}\delta_n^2 v_n^3 \\ &\quad + \frac{\sigma}{\alpha+2}\delta_n v_n^2 + \frac{2\sigma^2(1+\alpha)}{(\alpha+2)^2}\delta_n v_n^3 - \frac{2\sigma}{(\alpha+2)^2}\delta_n^2 v_n^2. \end{aligned}$$

Since we have

$$\rho_1 = \left(\frac{\partial g_2}{\partial \delta_n}\frac{\partial^2 g_2}{\partial v_n^2} + 2\frac{\partial^2 g_2}{\partial v_n \partial \delta_n}\right)|_{(0.0)} = -\frac{2}{(\alpha+2)} \neq 0,$$

$$\rho_2 = \left(\frac{1}{2}\left(\frac{\partial^2 g_2}{\partial v_n^2}\right)^2 + \frac{1}{3}\frac{\partial^3 g_2}{\partial v_n^3}\right)|_{(0.0)} = \frac{2\sigma^2(1+\alpha)^2}{(\alpha+2)^2} \neq 0.$$

Then system (1.2) undergoes a subcritical flip bifurcation at $(0, 0)$. □

3.3. Transcritical bifurcation at E_1

For $\beta = (1-\alpha)$, hence $J(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}, \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha})$ two eigenvalues becomes $\nu_1 = \alpha - 1$ and $\nu_2 = 1$. The following Lemma shows that the fixed point $(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}, \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha})$ is a transcritical bifurcation point.

Lemma 3.3. *If* $(1-\alpha)(1-\ln\frac{\beta}{1-\alpha}) > 0$, $\beta = 1-\alpha$ *and* $\alpha \neq 0$, *the system* (1.2) *undergoes a transcritical bifurcation at the fixed point* $(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}, \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha})$, *and the equation* (1.2) *has only one fixed point.*

Proof. The proof is similar to the proof of Lemma 3.1. □

3.4. Period-doubling bifurcation at E_1

For $\beta = (1-\alpha)\exp(\frac{2\alpha}{\alpha-1})$, hence $J(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}, \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha})$ two eigenvalues becomes $\nu_1 = \alpha + 1$ and $\nu_2 = -1$. We will show that the fixed point $(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}, \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha})$ is a flip bifurcation point by the following Lemma:

Lemma 3.4. *If* $(1-\alpha)(1-\ln\frac{\beta}{1-\alpha}) > 0$, $\beta = (1-\alpha)\exp(\frac{2\alpha}{\alpha-1})$, *the system* (1.2) *undergoes a flip bifurcation at the fixed point* $(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}, \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha})$.

Proof. Let $\zeta_n = x_n - \frac{2\alpha}{\sigma(\alpha-1)}$, $\eta_n = y_n - \frac{2\alpha}{\sigma(\alpha-1)}$, $\mu_n = \sqrt{\beta} - \sqrt{(1-\alpha)\exp(\frac{2\alpha}{\alpha-1})}$, then the system (1.2) becomes

$$\begin{aligned}
\zeta_{n+1} &= \alpha\zeta_n + (5\alpha-1)\exp(\frac{2\alpha}{\alpha-1})\eta_n + \sigma(1-\alpha)\exp(\frac{2\alpha}{\alpha-1})\eta_n^2 \\
&+ \frac{2(5\alpha-1)}{\alpha-1}\sqrt{(1-\alpha)\exp(\frac{2\alpha}{\alpha-1})}\eta_n\mu_n + \frac{2\alpha}{\sigma(\alpha-1)}\mu_n^2 + o((|\eta_n| + |\mu_n|)^3) \\
\eta_{n+1} &= \zeta_n.
\end{aligned} \tag{3.5}$$

Let

$$T = \begin{pmatrix} \alpha+1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix},$$

and

$$T^{-1} = \begin{pmatrix} \frac{1}{\alpha+2} & 0 & \frac{1}{\alpha+2} \\ 0 & 1 & 0 \\ \frac{1}{\alpha+2} & 0 & \frac{\alpha+1}{\alpha+2} \end{pmatrix}$$

By the following transformation

$$\begin{pmatrix} \zeta_n \\ \mu_n \\ \eta_n \end{pmatrix} = T\begin{pmatrix} u_n \\ \delta_n \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha+1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} u_n \\ \delta_n \\ v_n \end{pmatrix}$$

Hence

$$\begin{pmatrix} u_{n+1} \\ \delta_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha+1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_n \\ \delta_n \\ v_n \end{pmatrix} + \begin{pmatrix} \theta_1(u_n, \delta_n, v_n) \\ 0 \\ \theta_2(u_n, \delta_n, v_n) \end{pmatrix},$$

where

$$\theta_1(u_n, \delta_n, v_n) = \theta_2(u_n, \delta_n, v_n) = \frac{\sigma(1-\alpha)}{(\alpha+2)} \exp(\frac{2\alpha}{\alpha-1}) u_n^2 + \frac{\sigma(1-\alpha)}{(\alpha+2)} \exp(\frac{2\alpha}{\alpha-1}) v_n^2$$

$$-\frac{2\sigma(1-\alpha)}{(\alpha+2)} \exp(\frac{2\alpha}{\alpha-1}) u_n v_n + \frac{2(5\alpha-1)}{(\alpha+2)(\alpha-1)} \sqrt{(1-\alpha)\exp(\frac{2\alpha}{\alpha-1})} \delta_n u_n$$

$$-\frac{2(5\alpha-1)}{(\alpha+2)(\alpha-1)} \sqrt{(1-\alpha)\exp(\frac{2\alpha}{\alpha-1})} \delta_n v_n + \frac{2\alpha}{\sigma(\alpha-1)(\alpha+2)} \delta_n^2.$$

Let

$$u_n = w(v_n, \delta_n) = d_1 v_n^2 + d_2 v_n \delta_n + d_3 \delta_n^2 + o((\mid v_n \mid + \mid \delta_n \mid)^3).$$

Which must satisfy that

$$d_1 v_n^2 + d_2 v_n \delta_n + d_3 \delta_n^2 = (\alpha+1) w(v_n, \delta_n) + \theta_1(w(v_n, \delta_n), \delta_n, v_n).$$

Then we obtain

$$d_1 = \frac{\sigma(\alpha-1)}{\alpha(\alpha+2)} \exp(\frac{2\alpha}{\alpha-1}),$$

$$d_2 = \frac{2(5\alpha-1)}{(\alpha-1)(\alpha+2)} \sqrt{(1-\alpha)\exp(\frac{2\alpha}{\alpha-1})},$$

$$d_3 = \frac{2}{\sigma(1-\alpha)(\alpha+2)}.$$

Hence $u_n = \frac{\sigma(\alpha-1)}{\alpha(\alpha+2)} \exp(\frac{2\alpha}{\alpha-1}) v_n^2 + \frac{2(5\alpha-1)}{(\alpha-1)(\alpha+2)} \sqrt{(1-\alpha)\exp(\frac{2\alpha}{\alpha-1})} v_n \delta_n + \frac{2}{\sigma(1-\alpha)(\alpha+2)} \delta_n^2,$

$$g_3 = v_{n+1} = -v_n + \frac{\sigma(1-\alpha)}{(\alpha+2)} \exp(\frac{2\alpha}{\alpha-1}) u_n^2 + \frac{\sigma(1-\alpha)}{(\alpha+2)} \exp(\frac{2\alpha}{\alpha-1}) v_n^2,$$

$$-\frac{2\sigma(1-\alpha)}{(\alpha+2)} \exp(\frac{2\alpha}{\alpha-1}) u_n v_n + \frac{2(5\alpha-1)}{(\alpha+2)(\alpha-1)} \sqrt{(1-\alpha)\exp(\frac{2\alpha}{\alpha-1})} \delta_n u_n,$$

$$-\frac{2(5\alpha-1)}{(\alpha+2)(\alpha-1)} \sqrt{(1-\alpha)\exp(\frac{2\alpha}{\alpha-1})} \delta_n v_n + \frac{2\alpha}{\sigma(\alpha-1)(\alpha+2)} \delta_n^2.$$

Then, it follows that

$$g_3 = -v_n + \frac{\sigma(1-\alpha)}{(\alpha+2)} \exp(\frac{2\alpha}{\alpha-1}) v_n^2 + \frac{2\alpha}{\sigma(\alpha-1)(\alpha+2)} \delta_n^2 + \frac{2\sigma^2(1-\alpha)^2}{\alpha(\alpha+2)^2} v_n^3$$

$$-\frac{2(5\alpha-1)}{(\alpha+2)(\alpha-1)} \sqrt{(1-\alpha)\exp(\frac{2\alpha}{\alpha-1})} \delta_n v_n + \frac{16\alpha \exp(\frac{2\alpha}{\alpha-1})}{(\alpha+2)^2(\alpha-1)} \delta_n^2 v_n$$

$$+\frac{4\sigma(5\alpha-1)}{\alpha(\alpha+2)} \exp(\frac{2\alpha}{\alpha-1}) \sqrt{(1-\alpha)\exp(\frac{2\alpha}{\alpha-1})} v_n^2 \delta_n$$

$$-\frac{4(5\alpha-1)\sqrt{(1-\alpha)\exp(\frac{2\alpha}{\alpha-1})}}{\sigma(1-\alpha)^2(\alpha+2)^2}\delta_n^3+o((\mid v_n\mid+\mid\delta_n\mid)^4).$$

Since

$$\rho_1=\left(\frac{\partial g_3}{\partial\delta_n}\frac{\partial^2 g_3}{\partial v_n^2}+2\frac{\partial^2 g_3}{\partial v_n\partial\delta_n}\right)|_{(0.0)}=-\frac{(5\alpha-1)\sqrt{(1-\alpha)\exp(\frac{2\alpha}{\alpha-1})}}{(\alpha-1)(\alpha+2)}\neq 0 \text{ for } \alpha\neq 1,\frac{1}{5}$$

$$\rho_2=\left(\frac{1}{2}\left(\frac{\partial^2 g_3}{\partial v_n^2}\right)^2+\frac{1}{3}\frac{\partial^3 g_3}{\partial v_n^3}\right)|_{(0.0)}=\frac{2\sigma^2(1-\alpha)^2(\alpha\exp(\frac{2\alpha}{\alpha-1})+1)}{\alpha(\alpha+2)^2}\neq 0.$$

Then system (1.2) undergoes a supercritical flip bifurcation at $(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha},\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha})$. □

3.5. Neimark-Sacker bifurcation at E_1

In this section, we discuss the Neimark-Sacker bifurcation analysis of Eq.(1.2). For this, first we state Neimark-Sacker bifurcation theorem, which is also known as Poincare-Andronov-Hopf bifurcation theorem for maps, see[[24], [34], [35]].

Theorem 3.5. *Let*

$$F:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2;\quad (\lambda,x)\to F(\lambda,x)$$

be a C^4 map depending on real parameter λ satisfying the following conditions:

(i) *$F(\lambda,0)=0$ for λ near some fixed λ_0 :*

(ii) *$DF(\lambda,0)$ has two non-real eigenvalues $\mu(\lambda)$ and $\bar{\mu}(\lambda)$ for λ near λ_0 with $\mid\mu(\lambda_0)\mid=1$;*

(iii) *$\frac{d}{d\lambda}\mid\mu(\lambda)\mid=d(\lambda_0)\neq 0$ at $\lambda=\lambda_0$;*

(iv) *$\mu^k(\lambda_0)\neq 1$ for $k=1,2,3,4$.*

Then there is a smooth $\alpha-$dependent change of coordinate bringing F into the form

$$F(\lambda,x)=G(\lambda,x)+O(\|x\|^5).$$

Consider a general map $F(\lambda,x)$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda)=\alpha(\lambda)+i\beta(\lambda)$ and $\bar{\mu}(\lambda)=\alpha(\lambda)-i\beta(\lambda)$ satisfying $\alpha(\lambda)^2+\beta(\lambda)^2=1$ and $\beta(\lambda)\neq 0$.

By putting the linear part of such a map into Jordan canonical form, we may assume F to have the following form near the origin

$$F(\lambda,x)=\begin{pmatrix}\alpha(\lambda) & -\beta(\lambda)\\ \beta(\lambda) & \alpha(\lambda)\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}g_1(\lambda,x_1,x_2)\\ g_2(\lambda,x_1,x_2)\end{pmatrix}$$

Moreover, for all sufficiently small positive (negative) λ F has an attracting (repelling) invariant circle if $a(\lambda_0)<0$ ($a(\lambda_0)>0$) respectively; and $a(\lambda_0)$ is given by the following formula:

$$a(\lambda_0)=Re\left[\frac{(1-2\mu(\lambda_0))\bar{\mu}^2(\lambda_0)}{1-\mu(\lambda_0)}\gamma_{11}\gamma_{20}\right]+\frac{1}{2}\mid\gamma_{11}\mid^2+\mid\gamma_{02}\mid^2-Re(\bar{\mu}(\lambda_0)\gamma_{21}),\tag{3.6}$$

where

$$\begin{aligned}
\gamma_{20} &= \frac{1}{8}\left\{(g_1)_{x_1x_1} - (g_1)_{x_2x_2} + 2(g_2)_{x_1x_2} + i\left[(g_2)_{x_1x_1} - (g_2)_{x_2x_2} - 2(g_1)_{x_1x_2}\right]\right\}\\
\gamma_{11} &= \frac{1}{4}\left\{(g_1)_{x_1x_1} + (g_1)_{x_2x_2} + i\left[(g_2)_{x_1x_1} + (g_2)_{x_2x_2}\right]\right\}\\
\gamma_{02} &= \frac{1}{8}\left\{(g_1)_{x_1x_1} - (g_1)_{x_2x_2} - 2(g_2)_{x_1x_2} + i\left[(g_2)_{x_1x_1} - (g_2)_{x_2x_2} + 2(g_1)_{x_1x_2}\right]\right\}
\end{aligned}$$

$$\gamma_{21} = \frac{1}{8}\left\{(g_1)_{x_1x_1x_1} + (g_1)_{x_1x_2x_2} + (g_2)_{x_1x_1x_2} + (g_2)_{x_2x_2x_2}\right.$$
$$\left.+i\left[(g_2)_{x_1x_1x_1} + (g_2)_{x_1x_2x_2} - (g_1)_{x_1x_1x_2} - (g_1)_{x_2x_2x_2}\right]\right\}.$$

In order to apply Theorem 3.5 we make a change of variable $y_n = x_n - \bar{x}$ in (1.1). Then, transformed equation is given by

$$y_{n+1} = \alpha(y_n + \bar{x}) + \beta(y_{n-1} + \bar{x})\exp(-\sigma(y_{n-1} + \bar{x})), \tag{3.7}$$

where

$$\bar{x} = \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}.$$

By using the substitution $u_n = y_{n-1}$, $v_n = y_n$ we write Eq.(3.7) in the equivalent form:

$$\begin{cases} u_{n+1} = v_n \\ v_{n+1} = \alpha(v_n + \bar{x}) + \beta(u_n + \bar{x})\exp(-\sigma(u_n + \bar{x})) \end{cases} \tag{3.8}$$

Let F be the function defined by:

$$F(u,v) = \begin{pmatrix} v \\ \alpha(v + \bar{x}) + \beta(u + \bar{x})\exp(-\sigma(u + \bar{x})) \end{pmatrix}.$$

Then F has the unique fixed point $(0,0)$. The Jacobian matrix of F is given by

$$J_F(u,v) = \begin{pmatrix} 0 & 1 \\ \beta(1 - \sigma(u + \bar{x}))\exp(-\sigma(u + \bar{x})) & \alpha \end{pmatrix}.$$

At $(0,0)$ $J_F(u,v)$ has the form

$$J_F(0,0) = \begin{pmatrix} 0 & 1 \\ \beta(1 - \sigma\bar{x})\exp(-\sigma\bar{x}) & \alpha \end{pmatrix}. \tag{3.9}$$

The eigenvalues of (3.9) are $\mu_{\pm}(\beta)$ where

$$\mu_{\pm}(\beta) = \frac{\alpha \pm i\sqrt{4(1-\alpha)(\sigma\bar{x} - 1) - \alpha^2}}{2},$$

Then we have that

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \beta(1 - \sigma\bar{x})\exp(-\sigma\bar{x}) & \alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.10}$$

$$+\begin{pmatrix} f_1(\beta,u,v) \\ f_2(\beta,u,v) \end{pmatrix},$$

and

$$\begin{aligned} f_1(\beta,u,v) &= 0, \\ f_2(\beta,u,v) &= \alpha(v+\bar{x}) + \beta(u+\bar{x})\exp(-\sigma(u+\bar{x})) - \beta(1-\sigma\bar{x})\exp(-\sigma\bar{x})u - \alpha v - \bar{x}. \end{aligned}$$

Let

$$\beta_0 = (1-\alpha)\exp(\frac{2-\alpha}{1-\alpha}).$$

For $\beta = \beta_0$ we obtain

$$\bar{x} = \frac{(2-\alpha)}{\sigma(1-\alpha)} \text{ and } J_F(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix}.$$

The eigenvalues of $J_F(0,0)$ are $\mu(\beta_0)$ and $\bar{\mu}(\beta_0)$ where

$$\mu(\beta_0) = \frac{\alpha + i\sqrt{4-\alpha^2}}{2}.$$

The eigenvectors corresponding to $\mu(\beta_0)$ and $\bar{\mu}(\beta_0)$ are $v(\beta_0)$ and $\bar{v}(\beta_0)$ where

$$v(\beta_0) = (\frac{\alpha - i\sqrt{4-\alpha^2}}{2}, 1).$$

we have

$$\mu(\beta_0) = \frac{\alpha + i\sqrt{4-\alpha^2}}{2},$$

One can prove that

$$\begin{aligned} |\mu(\beta_0)| &= 1, \\ \mu^2(\beta_0) &= \frac{\alpha^2}{2} - 1 + \frac{i\alpha\sqrt{4-\alpha^2}}{2}, \\ \mu^3(\beta_0) &= \frac{\alpha(\alpha^2-2)}{4} + \frac{\alpha(\alpha^2-4)}{4} + i\left[\frac{\alpha^2\sqrt{4-\alpha^2}}{4} + (\alpha^2-2)\frac{\sqrt{4-\alpha^2}}{2}\right], \\ \mu^4(\beta_0) &= \frac{(\alpha^2-2)^2}{4} - \frac{\alpha^2(4-\alpha^2)}{4} + i\alpha(\alpha^2-2)\frac{\sqrt{4-\alpha^2}}{2}. \end{aligned}$$

From which follows that $\mu^k(\beta_0) \neq 1$ for $k = 1,2,3,4$. Substituting $\beta = \beta_0$ and $\bar{x}$ into (3.10) we get

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix},$$

and

$$h_1(u,v) = f_1(\beta_0,u,v) = 0$$

$$h_2(u,v) = f_2(\beta_0,u,v) = \alpha(v + \frac{(2-\alpha)}{\sigma(1-\alpha)}) + \beta(u + \frac{(2-\alpha)}{\sigma(1-\alpha)})\exp(-\sigma(u + \frac{(2-\alpha)}{\sigma(1-\alpha)}))$$

$$+u - \alpha v - \frac{(2-\alpha)}{\sigma(1-\alpha)}$$

Hence, for $\beta = \beta_0$ system (3.10) is equilvalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} h_1(u_n, v_n) \\ h_2(u_n, v_n) \end{pmatrix}. \tag{3.11}$$

Let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix},$$

where

$$P = \begin{pmatrix} \frac{\alpha}{2} & \frac{-\sqrt{4-\alpha^2}}{2} \\ 1 & 0 \end{pmatrix},$$

$$P^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{2}{\sqrt{4-\alpha^2}} & -\frac{\alpha}{\sqrt{4-\alpha^2}} \end{pmatrix}.$$

Then system (3.10) is equivalent to its normal form

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2} & -\frac{\sqrt{4-\alpha^2}}{2} \\ \frac{\sqrt{4-\alpha^2}}{2} & \frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + G \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix},$$

where

$$H \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} h_1(u, v) \\ h_2(u, v) \end{pmatrix}.$$

Let

$$G \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g_1(u, v) \\ g_2(u, v) \end{pmatrix} = P^{-1} H \left(P \begin{pmatrix} u \\ v \end{pmatrix} \right).$$

$$P \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2} & \frac{\sqrt{4-\alpha^2}}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\alpha}{2} u + \frac{\sqrt{4-\alpha^2}}{2} v \\ u \end{pmatrix},$$

$$H \left(P \begin{pmatrix} u \\ v \end{pmatrix} \right) = \begin{pmatrix} h_1(\frac{\alpha}{2} u + \frac{\sqrt{4-\alpha^2}}{2} v, u) \\ h_2(\frac{\alpha}{2} u + \frac{\sqrt{4-\alpha^2}}{2} v, u) \end{pmatrix} = \begin{pmatrix} 0 \\ h_2(\frac{\alpha}{2} u + \frac{\sqrt{4-\alpha^2}}{2} v, u) \end{pmatrix},$$

$$P^{-1} H \left(P \begin{pmatrix} u \\ v \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ \frac{2}{\sqrt{4-\alpha^2}} & -\frac{\alpha}{\sqrt{4-\alpha^2}} \end{pmatrix} \begin{pmatrix} 0 \\ h_2(\frac{\alpha}{2} u + \frac{\sqrt{4-\alpha^2}}{2} v, u) \end{pmatrix}.$$

Then

$$g_1(u, v) = h_2(\frac{\alpha}{2} u + \frac{\sqrt{4-\alpha^2}}{2} v, u),$$

$$= \alpha(u + \frac{(2-\alpha)}{\sigma(1-\alpha)}) + \beta(\frac{\alpha}{2} u + \frac{\sqrt{4-\alpha^2}}{2} v + \frac{(2-\alpha)}{\sigma(1-\alpha)}) \exp(-\sigma(\frac{\alpha}{2} u + \frac{\sqrt{4-\alpha^2}}{2} v + \frac{(2-\alpha)}{\sigma(1-\alpha)}))$$

$$+ \frac{\alpha}{2} u + \frac{\sqrt{4-\alpha^2}}{2} v - \alpha u - \frac{(2-\alpha)}{\sigma(1-\alpha)}$$

$$g_2(u,v) = -\frac{\alpha}{\sqrt{4-\alpha^2}} g_1(u,v).$$

Other calculation gives

$$\begin{aligned}
\frac{\partial^2 g_1(0,0)}{\partial u^2} &= \frac{\sigma\alpha^3}{4},\\
\frac{\partial^2 g_1(0,0)}{\partial v^2} &= \frac{\sigma\alpha\sqrt{4-\alpha^2}}{4},\\
\frac{\partial^2 g_1(0,0)}{\partial uv} &= \frac{\sigma\alpha^2\sqrt{4-\alpha^2}}{4},\\
\frac{\partial^2 g_1(0,0)}{\partial u^3} &= \frac{\sigma^2\alpha^3(1-2\alpha)}{8}\\
\frac{\partial^2 g_1(0,0)}{\partial u\partial v^2} &= \frac{\sigma\alpha^2(1-2\alpha)(4-\alpha^2)}{8}\\
\frac{\partial^2 g_1(0,0)}{\partial u^2\partial v} &= \frac{\sigma^2\alpha^2(1-2\alpha)\sqrt{4-\alpha^2}}{8}\\
\frac{\partial^2 g_1(0,0)}{\partial v^3} &= \frac{\sigma^2(1-2\alpha)(4-\alpha^2)^{\frac{3}{2}}}{8}
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{20}(0,0) &= -\frac{\sigma\alpha}{8}\left[1+i\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\\
\gamma_{11}(0,0) &= \frac{\sigma\alpha}{4}\left[1-i\frac{\alpha}{\sqrt{4-\alpha^2}}\right]\\
\gamma_{02}(0,0) &= \frac{\sigma\alpha}{8}\left[(\alpha^2-1)+i\frac{\alpha[3-\alpha^2]}{\sqrt{4-\alpha^2}}\right]\\
\gamma_{21}(0,0) &= -\frac{\sigma^2}{64}\left[\alpha^4+i\frac{[2\alpha^4-3\alpha^5+28\alpha^3-16\alpha^2+32\alpha+16]}{\sqrt{4-\alpha^2}}\right].
\end{aligned}$$

Now we have that

$$Re\left(\bar{\mu(\beta_0)}\gamma_{21}(\beta_0)\right) = \frac{\sigma^2\left(\alpha^4-\alpha^5+14\alpha^3-8\alpha^2+16\alpha+8\right)}{64}, \tag{3.12}$$

$$\left[\frac{(1-2\mu(\beta_0))\bar{\mu(\beta_0)}^2}{1-\mu(\beta_0)}\right] = \frac{\left(2\alpha^3+\alpha^2-14\alpha+8\right)-i(2\alpha^3-3\alpha^2-6\alpha+6)\sqrt{4-\alpha^2}}{4(2-\alpha)},$$

$$Re\left[\frac{(1-2\mu(\beta_0))\bar{\mu(\beta_0)}^2}{1-\mu(\beta_0)}\gamma_{11}\gamma_{20}\right] = \frac{\sigma^2\alpha^2\left(2\alpha^3+\alpha^2-14\alpha+8\right)}{32(2-\beta)}, \tag{3.13}$$

$$\gamma_{11}(\beta_0)\bar{\gamma_{11}}(\beta_0) = \frac{\sigma^2\alpha^2}{4(4-\alpha^2)}, \tag{3.14}$$

$$\gamma_{02}(\beta_0)\bar{\gamma_{02}}(\beta_0) = \frac{\sigma^2\alpha^2(4+3\alpha^4)}{64(4-\alpha^2)} \tag{3.15}$$

Then by using (3.12), (3.13), (3.14) and (3.15) we get that

$$\begin{aligned} a(\beta_0) &= \frac{\sigma^2\alpha^2\left(2\alpha^3+\alpha^2-14\alpha+8\right)}{32(2-\alpha)} + \frac{\sigma^2\alpha^2(12+3\alpha^4)}{64(4-\alpha^2)} \\ &\quad - \frac{\sigma^2\left(\alpha^4-\alpha^5+14\alpha^3-8\alpha^2+16\alpha+8\right)}{64} \\ &< 0 \end{aligned}$$

One can see that

$$\begin{aligned} &| \quad \mu(\beta) |^2 = \mu(\beta)\bar{\mu(\beta)} = (1-\alpha)(\sigma\bar{x}-1) \\ &= \quad (1-\alpha)(\ln\frac{\beta}{1-\alpha}-1), \end{aligned}$$

from which we obtain

$$\begin{aligned} \frac{d}{d\beta} \quad | \quad \mu(\beta) |_{\beta=\beta_0} &= \frac{(1-\alpha)^2}{2\beta\sqrt{(1-\alpha)(\ln\frac{\beta}{1-\alpha}-1)}} |_{\beta=\beta_0} \\ &= \frac{1-\alpha}{2\exp(\frac{2-\alpha}{1-\alpha})} > 0 \end{aligned}$$

From the above analysis, we have the following Lemma:

Lemma 3.6. *If* $4 > \alpha^2$, *then system* (1.2) *undergoes a Neimark-Sacker bifurcation at its positive equilibrium* E_1 *when* $\beta = \beta_0$.

4. Chaos control

In this section, a feedback control methodology (that is, OGY method) is implemented to system (1.2) for controlling chaos under the influence of Hopf and period-doubling bifurcation at positive steady-state of system (1.2). Ott *et al.* [36] introduced a feedback control method for controlling chaos in discrete-time systems and this method is known as OGY method.

For the application of OGY method, we write system (1.2) in the following form:

$$\begin{aligned} x_{n+1} &= \alpha x_n + \beta y_n \exp(-\sigma y_n) = f(x_n, y_n, \beta), \\ y_{n+1} &= x_n = g(x_n, y_n, \beta), \end{aligned} \tag{4.1}$$

where β denotes parameter for chaos control. Suppose that β lies in a small interval, that is, $\beta \in \left(\hat{\beta}-\eta, \hat{\beta}+\eta\right)$ such that $\eta > 0$ and $\hat{\beta}$ represents nominal value for β which belongs to some chaotic region. Suppose that $(x^*, y^*) = \left(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}, \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}\right)$ denotes an unstable fixed point for (1.2) in chaotic region which is produced under the influence of Hopf bifurcation or flip bifurcation. In this case, (4.1) is approximated in the neighborhood of $(x^*, y^*) = \left(\frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}, \frac{1}{\sigma}\ln\frac{\beta}{1-\alpha}\right)$ as follows:

$$\begin{bmatrix} x_{n+1}-x^* \\ y_{n+1}-y^* \end{bmatrix} \approx J(x^*, y^*, \hat{\beta}) \begin{bmatrix} x_n-x^* \\ y_n-y^* \end{bmatrix} + B[\beta-\hat{\beta}], \tag{4.2}$$

where

$$\begin{aligned} J(x^*, y^*, \hat{\beta}) &= \begin{bmatrix} \frac{\partial f(x^*,y^*,\hat{\beta})}{\partial x} & \frac{\partial f(x^*,y^*,\hat{\beta})}{\partial y} \\ \frac{\partial g(x^*,y^*,\hat{\beta})}{\partial x} & \frac{\partial g(x^*,y^*,\hat{\beta})}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} \alpha & (1-\alpha)\left(1-\ln\left(\frac{\hat{\beta}}{1-\alpha}\right)\right) \\ 1 & 0 \end{bmatrix} \end{aligned}$$

and

$$B = \begin{bmatrix} \frac{\partial f(x^*,y^*,\hat{\beta})}{\partial \beta} \\ \frac{\partial g(x^*,y^*,\hat{\beta})}{\partial \beta} \end{bmatrix} = \begin{bmatrix} -\frac{(1-\alpha)\ln\left(\frac{1-\alpha}{\hat{\beta}}\right)}{\hat{\beta}\sigma} \\ 0 \end{bmatrix}.$$

Furthermore, the system (4.1) is controllable if the following matrix has rank 2

$$C = [B : JB] = \begin{bmatrix} -\frac{(1-\alpha)\ln\left(\frac{1-\alpha}{\hat{\beta}}\right)}{\hat{\beta}\sigma} & -\frac{\alpha(1-\alpha)\ln\left(\frac{1-\alpha}{\hat{\beta}}\right)}{\hat{\beta}\sigma} \\ 0 & -\frac{(1-\alpha)\ln\left(\frac{1-\alpha}{\hat{\beta}}\right)}{\hat{\beta}\sigma} \end{bmatrix}. \tag{4.3}$$

According to assumption for existence of positive equilibrium, we have $(1-\alpha)\ln\left(\frac{\hat{\beta}}{1-\alpha}\right) > 0$ and this implies that rank of C is 2. Furthermore, we set $[\beta - \hat{\beta}] = -K\begin{bmatrix} x_n - x^* \\ y_n - y^* \end{bmatrix}$, where $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, then system (4.2) is written as follows

$$\begin{bmatrix} x_{n+1} - x^* \\ y_{n+1} - y^* \end{bmatrix} \approx [J - BK]\begin{bmatrix} x_n - x^* \\ y_n - y^* \end{bmatrix}. \tag{4.4}$$

In this case, the corresponding control system of (1.2) is given as follows

$$\begin{aligned} x_{n+1} &= \alpha x_n + \left(\hat{\beta} - k_1(x_n - x^*) - k_2(y_n - y^*)\right) y_n \exp(-\sigma y_n), \\ y_{n+1} &= x_n. \end{aligned} \tag{4.5}$$

Moreover, the positive equilibrium point (x^*, y^*) of (4.5) is locally stable if and only if absolute values of both eigenvalues of $J - BK$ are less than one. Moreover, the matrix $J - BK$ is given as follows:

$$J - BK = \begin{bmatrix} \alpha - \frac{k_1(1-\alpha)\ln\left(\frac{\hat{\beta}}{1-\alpha}\right)}{\sigma\hat{\beta}} & -\frac{(1-\alpha)\left(\sigma\hat{\beta} + (k_2+\sigma\hat{\beta})\ln\left(\frac{\hat{\beta}}{1-\alpha}\right)\right)}{\sigma\hat{\beta}} \\ 1 & 0 \end{bmatrix}.$$

The characteristic equation for the matrix $J - BK$ is given as follows

$$\mathbb{P}(\lambda) = \lambda^2 - \left(\alpha - k_1(1-\alpha)\ln\left(\frac{\hat{\beta}}{1-\alpha}\right)\right)\lambda - 1 + \alpha + \frac{1}{\hat{\beta}\sigma}(1-\alpha)(k_2 + \hat{\beta}\sigma)\ln\left(\frac{\hat{\beta}}{1-\alpha}\right) = 0. \tag{4.6}$$

Assume that λ_1 and λ_2 represent the roots of (4.6), then it follows that

$$\lambda_1 + \lambda_2 = \alpha - k_1(1-\alpha)\ln\left(\frac{\hat{\beta}}{1-\alpha}\right), \tag{4.7}$$

and

$$\lambda_1\lambda_2 = -1 + \alpha + \frac{1}{\hat{\beta}\sigma}(1-\alpha)(k_2 + \hat{\beta}\sigma)\ln\left(\frac{\hat{\beta}}{1-\alpha}\right). \tag{4.8}$$

Moreover, we take $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$. Then, the lines of marginal stability for (4.5) are computed as follows:

$$L_1 : k_1 + k_2 + \hat{\beta}\sigma = 0, \tag{4.9}$$

$$L_2 : 2\alpha\hat{\beta}\sigma + (1-\alpha)(k_2 - k_1 + \hat{\beta}\sigma)\ln\left(\frac{\hat{\beta}}{1-\alpha}\right) = 0, \tag{4.10}$$

and

$$L_3 : \alpha + \frac{1}{\hat{\beta}\sigma}(1-\alpha)(k_2 + \hat{\beta}\sigma)\ln\left(\frac{\hat{\beta}}{1-\alpha}\right) = 2. \tag{4.11}$$

Then, stability region for (4.5) is triangular region bounded by L_1, L_2 and L_3 in k_1k_2-plane.

5. Numerical simulations

In order to illustrate theoretical discussion, first we take $\alpha = 0.0005$, $\sigma = 5.8$ and $\beta \in [2, 20]$. Then, system (1.2) undergoes flip bifurcation and corresponding diagrams are depicted in Fig. 3 and Fig. 4.

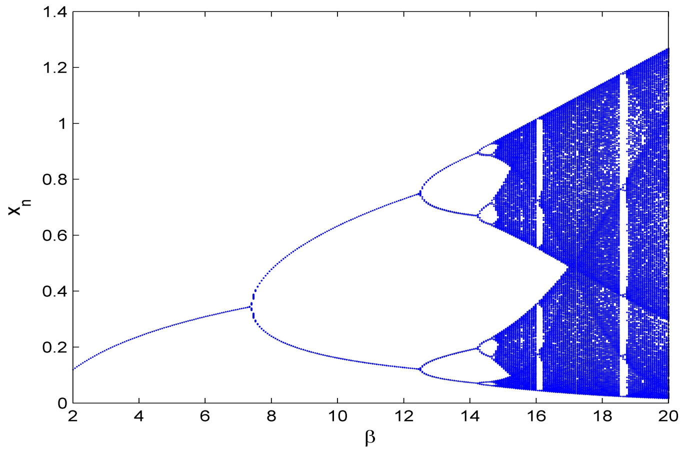

Figure 3: Bifurcation diagram for x_n.

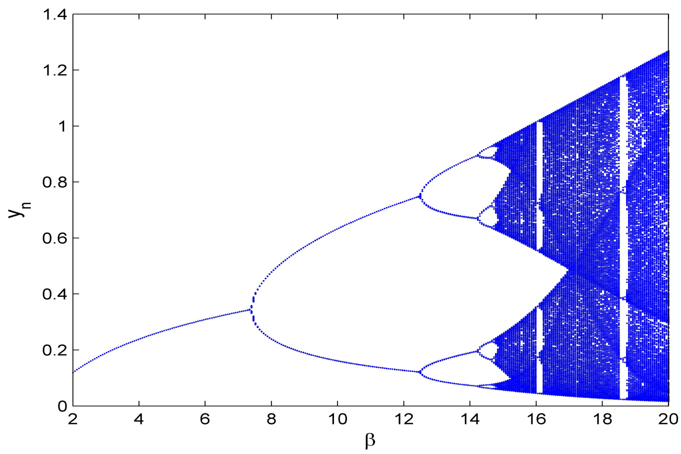

Figure 4: Bifurcation diagram for y_n.

Next, we choose $\alpha = 0.3$, $\sigma = 0.1$ and $\beta \in [5, 35]$ with initial conditions $x_0 = y_0 = 20$, then system (1.2) undergoes Neimark-Sacker bifurcation at $\beta = 7.93987$. Moreover, at $(\alpha, \sigma, \beta) = (0.3, 0.1, 7.93987)$ the system

(1.2) has unique positive fixed point (24.2857, 24.2857), and characteristic equation for Jacobian matrix of system (1.2) is computed as follows:

$$\lambda^2 - 0.3\lambda + 1 = 0.$$

The roots of aforementioned characteristic equation are $\lambda_1 = 0.15 - 0.988686i$ and $\lambda_2 = 0.15 + 0.988686i$ with absolute values equal to one. Thus for these parametric values, system (1.2) undergoes Neimark-Sacker bifurcation, and bifurcation diagrams are depicted in Fig. 5 and Fig. 6.

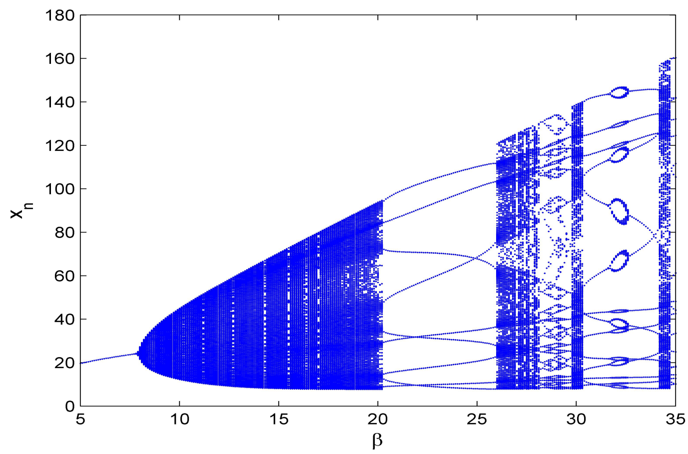

Figure 5: Bifurcation diagram for x_n.

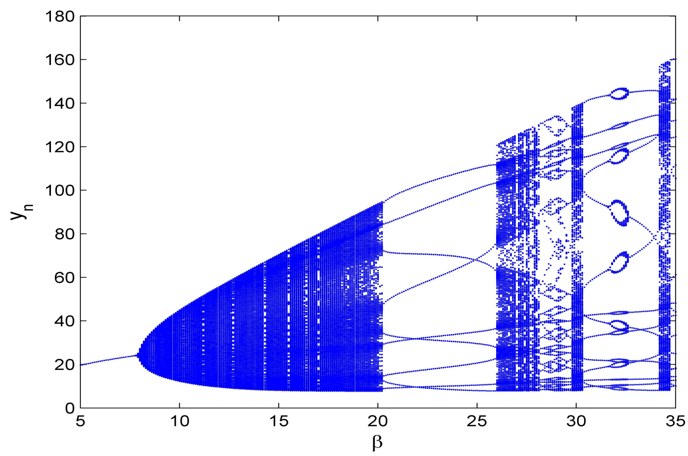

Figure 6: Bifurcation diagram for y_n.

In order to apply OGY control method to system (1.2), we take $(\alpha, \sigma, \beta) = (0.3, 0.1, 20)$. In this case system (1.2) has a unique positive unstable equilibrium point (33.5241, 33.5241) located in chaotic region. Moreover, implementation of OGY control strategy yields the following controlled system:

$$\begin{aligned} x_{n+1} &= 0.3x_n + (20 - k_1(x_n - 33.524) - k_2(y_n - 33.524))\, y_n \exp(-0.1y_n), \\ y_{n+1} &= x_n. \end{aligned} \tag{5.1}$$

The Jacobian matrix of (5.1) is computed as follows:

$$\begin{bmatrix} 0.3 - 1.17334k_1 & -1.64669 - 1.17334k_2 \\ 1 & 0 \end{bmatrix}.$$

The characteristic equation for aforementioned Jacobian matrix is given as follows:

$$\lambda^2 - (0.3 - 1.1733425261224528k_1)\lambda + 1.646685052244906 + 1.1733425261224528k_2 = 0.$$

Moreover, the lines of marginal stability are calculated as follows:

$$L_1 : 2.34669 + 1.17334k_1 + 1.17334k_2 = 0,$$

$$L_2 : 2.94669 - 1.17334k_1 + 1.17334k_2 = 0,$$

and

$$L_3 : 0.646685 + 1.17334k_2 = 0.$$

The stability region bounded by L_1, L_2 and L_3 is depicted in Fig. 7.

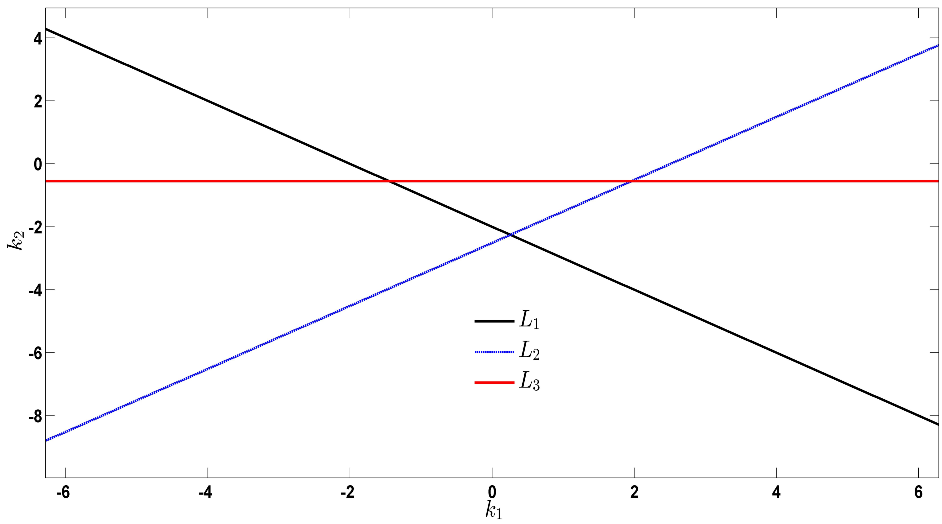

Figure 7: Stability region for controlled system (5.1).

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