Filomat 33:15 (2019), 5037–5050
https://doi.org/10.2298/FIL1915037A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Characterization of $\mathcal{S}$-Condition Pseudospectrum of Multivalued Linear Operators

Aymen Ammar[a], Aref Jeribi[a], Mayssa Zayed[a]

[a]*Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, soukra Road Km 3.5 B.P. 1171, 3000, Sfax, Tunisia*

Abstract. The main goal of this paper, is to extend the concept of condition pseudospectrum to the case of closed linear relations pencils and to reveal some of their properties. We will start by giving the definition and then we will focus on the characterization and some results concerning these pseudospectra.

1. Introduction

The linear relation made their appearance in functional analysis motivated by the need to consider adjoints of non-densely defined linear differential operators and also through the need of considering the inverses of certain operators used in the study of some Cauchy problems associated with parabolic type equation in Banach spaces. The theory of linear relation is the one of the most exciting and influential field of research in modem mathematics. Applications of this theory can be found in economic theory, non cooperative games, artificial intelligence, medicine and existence of solutions for differential inclusions, we recall as examples ([1], [2], [5], [6], [13]).

Several mathematical and physical problems lead to operators pencils, $\lambda\mathcal{S}-T$ (operator-valued functions of a complex argument). Recently the spectral theory of operator pencils attracts an attention of many mathematicians. Mainly, the completeness of the root vectors and asymptotic distributions of characteristic values are considered. In [10], A. Jeribi, N. Moalla, and S. Yengui gave a characterization of the essential spectrum of the operator pencil in order to extend many known results in the literature. After that A. Ammar, F. Abdelmouleh and A. Jeribi in [11] pursued the study of the $\mathcal{S}$-essential spectra and investigated the $\mathcal{S}$-Browder essential spectra of bounded linear operators on a Banach space X. To have further details of the properties of the operators pencils , we may refer to ([3],[9]).

The concept of condition pseudospectra of linear operators pencils are interesting objects by themselves since they carry more information than spectra and pseudospectra, especially, about transient instead of just asymptotic behaviour of dynamical systems. Also, they have better convergence and approximation properties than spectra and pseudospectra of linear operators pencils.

The definition of condition pseudospectra of linear operator pencil $\lambda\mathcal{S}-T$ in infinite dimensional Banach spaces is giving for every $0<\varepsilon<1$ by:

$$\Lambda_{\varepsilon,\mathcal{S}}(T)=\sigma_{\mathcal{S}}(T)\cup\left\{\lambda\in\mathbb{C}:\ \|\lambda\mathcal{S}-T\|\|(\lambda\mathcal{S}-T)^{-1}\|>\frac{1}{\varepsilon}\right\},$$

2010 *Mathematics Subject Classification*. 47A06
Keywords. Linear relation , $\mathcal{S}$-condition pseudospectrum
Received: 05 December 2018; Revised: 27 March 2019; Accepted: 02 April 2019
Communicated by Snežana Živković-Zlatanović
Email addresses: ammar_aymen84@yahoo.fr (Aymen Ammar), Aref.Jeribi@fss.rnu.tn (Aref Jeribi), mayssazayed@hotmail.fr (Mayssa Zayed)

with the convention that $\|\lambda\mathcal{S} - T\|\|(\lambda\mathcal{S} - T)^{-1}\| = \infty$, if $\lambda\mathcal{S} - T$ is not invertible. Note that if $\mathcal{S} = I$ (the identity operator on X), we recover the usual definition of condition pseudospectra of linear operator T.

Inspired by the notion of linear relation, T. Álvarez, A. Ammar and A. Jeribi in their works [15], thought to introduce and investigate the characterization of some $\mathcal{S}$-essential spectra of a closed linear relations in terms of certain linear relations of semi Fredholm type.

The main focus in this paper is to study some results of condition pseudospectra of closed linear relations pencils in Banach spaces and give the characterization of these condition pseudospectra. Our paper is organized as follows: In section 2, we recall some basic notation and results from the theory of linear relations that we will need to prove the main results of others sections. Section 3 is devoted to investigate some properties and useful results for the condition pseudospectra of multivalued linear operators pencils. Finally, in section 4, we will give the characterization of the condition pseudospectra of multivalued linear operators pencils.

2. Preliminary and auxiliary results

Let X and Y be a vector spaces over the some field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A linear relation T from X to Y is a mapping from a subspace

$$\mathcal{D}(T) = \{x \in X : Tx \neq \emptyset\} \subseteq X$$

called the domain of T, into the collection of nonempty subsets of Y such that

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$$

for all non zero scalars $\alpha_1,\ \alpha_2$ and $x_1,\ x_2 \in \mathcal{D}(T)$. If T maps the points of its domain to singletons, then T is said to be single valued or simply an operator, that is equivalent to $T(0) = \{0\}$. We denote by $\mathcal{L}(X, Y)$ the set of bounded operators from X to Y. The collection of linear relations is denoted by $L\mathcal{R}(X, Y)$ and we write $L\mathcal{R}(X) := L\mathcal{R}(X, X)$. A linear relation $T \in L\mathcal{R}(X, Y)$ is uniquely determined by its graph, $G(T)$, which is defined by

$$G(T) = \{(x, y) \in X \times Y : x \in \mathcal{D}(T) \textit{ and } y \in Tx\}.$$

The inverse of $T \in L\mathcal{R}(X, Y)$ is the linear relation T^{-1} defined by

$$G(T^{-1}) = \{(y, x) \in Y \times X : (x, y) \in G(T)\}.$$

Let $T, S \in L\mathcal{R}(X, Y)$, the subspaces $R(T)$, $N(T)$ and $T(0)$ stand respectively for the range and the null space of T , which are defined by

$$\begin{array}{rcl} R(T) & = & \Big\{y : (x, y) \in G(T)\Big\}, \\ N(T) & = & \Big\{x \in \mathcal{D}(T) : (x, 0) \in G(T)\Big\}, \text{ and} \\ T(0) & = & \Big\{y : (0, y) \in G(T)\Big\}. \end{array}$$

Notice that when $x \in \mathcal{D}(T)$, $y \in Tx$ if, and only if, $Tx = y + T(0)$. A linear relation T is said to be surjective, if $R(T) = Y$. Similarly, T is said to be injective, if the null spaces $N(T) = T^{-1}(0) = \{0\}$. When T is both injective and surjective, we say that T is bijective.

Remark 2.1. (*i*) *T injective if, and only if, $T^{-1}T = I_{\mathcal{D}(T)}$*
(*ii*) *T single valued if, and only if, $TT^{-1} = I_{R(T)}$.*

For $T,\ S \in L\mathcal{R}(X, Y)$, then the linear relation $T + S$ is defined by

$$G(T + S) = \{(x, u + v) \in X \times Y : (x, u) \in G(T) \text{ and } (x, v) \in G(\mathcal{S})\}.$$

for $T \in L\mathcal{R}(X, Y)$ and $S \in L\mathcal{R}(Y, Z)$, then the composition or product $ST \in L\mathcal{R}(X, Z)$ is defined by

$$G(ST) = \{(x, z) \in X \times Z : (x, y) \in G(T) \text{ and } (y, z) \in G(S) \text{ for some } y \in Y\}.$$

The closure of a linear relation $T \in L\mathcal{R}(X,Y)$ is the linear relation $\overline{T}$ defined by

$$G(\overline{T}) := \overline{G(T)}.$$

A linear relation T is said to be closed if its graph is a closed subspace, continuous if $\|T\| < \infty$, bounded if it is continuous and $\mathcal{D}(T) = X$ and open if T^{-1} is continuous equivalently $\gamma(T) > 0$ where $\gamma(T)$ is the minimum modulus of T defined by

$$\gamma(T) = \sup\{\lambda \geq 0 : \lambda\, d(x, N(T)) \leq \|Tx\|,\ x \in \mathcal{D}(T)\}.$$

We denote the set of all closed and bounded linear relations from X to Y by $C\mathcal{R}(X,Y)$ and $B\mathcal{R}(X,Y)$ respectively and we write $C\mathcal{R}(X) = C\mathcal{R}(X,X)$ and $B\mathcal{R}(X) = B\mathcal{R}(X,X)$.
If M and N are subspaces of X and of the dual space $X^{'}$ respectively, then

$$M^{\perp} = \Big\{x' \in X^{'} :\ x'(x) = 0 \ \text{ for all } x \in M\Big\},$$

and

$$N^{\top} = \Big\{x \in X :\ x'(x) = 0 \ \text{ for all } x' \in N\Big\}.$$

The adjoint (or conjugate) $T^{'}$ of a linear relation $T \in L\mathcal{R}(X,Y)$ is defined by

$$G(T^{'}) = G(-T^{-1})^{\perp} \subset Y^{'} \times X^{'},$$

This means that $(y', x') \in G(T^{'})$ if, and only if, $y'(y) = x'(x)$ for all $(x,y) \in G(T)$.

Proposition 2.2. [13, Proposition I.2.8] *Let X, Y be two linear spaces and Let $T \in L\mathcal{R}(X,Y)$, then for $x \in \mathcal{D}(T)$, we have the following equivalence:*
(i) $y \in Tx$ if, and only if, $Tx = y + T(0)$. In particular,
(ii) $0 \in Tx$ if, and only if, $Tx = T(0)$. ◊

Proposition 2.3. ([13, Proposition I.4.2] *and* [14, Lemma 2]) *Let X, Y be two linear spaces and $T \in L\mathcal{R}(Y,Z)$ and $S, R \in L\mathcal{R}(X,Y)$.*

(i) If $T(0) \subset N(S)$ (or $T(0) \subset N(R)$), then $(R+S)T = RT + ST$.

(ii) If $\mathcal{D}(T)$ contains the range of both R and S (in particular, $\mathcal{D}(T)$is the whole space), then $T(R+S) = TR + TS$.

(iii) If $S(0) \subset T(0)$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. Then $T = T + S - S$. ◊

Lemma 2.4. *Let X and Y be normed spaces and $T \in L\mathcal{R}(X,Y)$ then for $x \in \mathcal{D}(T)$,*
(a) [13, Propositions II.1.4] *$\|Tx\| = d(y, T(0))$ for any $y \in Tx$.*
(b) [13, Propositions II.1.4] $\|Tx\| = d(Tx, T(0)) = d(Tx, 0)$
(c) [13, Propositions II.1.6] *$\|T\| = sup_{x \in B_X}\|Tx\|$ with $B_X := \{x \in X : \|x\| \leq 1\}$.*
(d) [13, Theorem II.2.5] $\gamma(T) = \|T^{-1}\|^{-1}$ ◊

Proposition 2.5. [13, Propositions II.1.5 and II.3.13] *Let X, Y be two normed spaces and let $S, T \in L\mathcal{R}(X,Y)$ and $R \in L\mathcal{R}(Y,Z)$*

(i) For $x \in \mathcal{D}(S+T)$ we have

$$\|S+T\| \leq \|S\| + \|T\|,$$

if additionally $S(0) \subset \overline{T(0)}$ then,

$$\|T\| - \|S\| \leq \|T - S\|.$$

(ii) If $S(0) \subset \mathcal{D}(R)$, then we have

$$\|RS\| \leq \|R\|\|S\|.$$

◊

In [15], T. Álvarez, A. Ammar and A. Jeribi are presented the $\mathcal{S}$-resolvent in this way:

Definition 2.6. *Let X a normed space and let $T \in L\mathcal{R}(X)$ and $\mathcal{S}$ a continuous linear relation such that $\mathcal{S}(0) \subset \overline{T(0)}$ and $\mathcal{D}(\mathcal{S}) \supset \mathcal{D}(T)$. We define the $\mathcal{S}$-resolvent set of T by:*

$$\rho_{\mathcal{S}}(T) := \{\lambda \in \mathbb{C} : \ \lambda\mathcal{S} - T \text{ is injective, open with dense range on } X\},$$

and the $\mathcal{S}$-spectrum of T by $\sigma_{\mathcal{S}}(T) := \mathbb{C} \setminus \rho_{\mathcal{S}}(T)$. ◊

Remark 2.7. *It is clear that if $T \in C\mathcal{R}(X)$ and X complete, we will return to the $\mathcal{S}$-spectrum definition in* [15]*, the authers defined the $\mathcal{S}$-spectrum in Banach space with closed linear relation in the following way:*

$$\begin{aligned} \rho_{\mathcal{S}}(T) &:= \{\lambda \in \mathbb{C} : \ \lambda\mathcal{S} - T \text{ is bijective on } X\}, \\ &= \{\lambda \in \mathbb{C} : \ (\lambda\mathcal{S} - T)^{-1} \text{ is a bounded linear operator on } X\}. \end{aligned}$$

◊

Proposition 2.8. [13, Proposition III.1.5 , III.1.13 and VI.1.11]) *Let X a normed linear space and let $T, S \in L\mathcal{R}(X)$*
(*i*) *If S is continuous and $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S' + T' = (S + T)'$.*
(*ii*) *$\|T'\| \leq \|T\|$ and if T is continuous, then $\|T'\| = \|T\| < \infty$.*
(*iii*) *$\sigma(T) = \sigma(T')$.* ◊

Lemma 2.9. [12, Lemma 3.5] *Let X is a normed spaces and Y is a Banach space. Let $T \in L\mathcal{R}(X, Y)$ be closed and $S \in L\mathcal{R}(X, Y)$ be continuous such that $S(0) \subset T(0)$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$, then $T + S$ is closed.* ◊

Lemma 2.10. [13, Proposition II.5.17] *Let $S \in LR(Z, X)$ be closed and let the relation T have closed range and satisfy $\alpha(T) < \infty$ and $\gamma(T) > 0$. Then, TS is closed.* ◊

Lemma 2.11. [13, Corollary III.7.7] *Let T be open and injective with dense range. Then, for any relation S such that $S(0) \subset \overline{T(0)}$, $\mathcal{D}(S) \supset \mathcal{D}(T)$ and $\|S\| < \gamma(T)$, we have $T + S$ is open injective with dense range.* ◊

Lemma 2.12. [4, Lemma2.2] *Let $S \in L\mathcal{R}(X)$. Then $\|S\| = 0$ if, and only if, $R(S) \subset \overline{S(0)}$.* ◊

Lemma 2.13. [4, Lemma2.3]*Let $T \in L\mathcal{R}(X)$ and S a continuous linear relation such that $S(0) \subset \overline{T(0)}$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$. Then $\|S\| = 0$ implies $\rho_S(T) = \emptyset$ or $\mathbb{C}$.* ◊

Lemma 2.14. [4, Theorem 3.9] *Let X a normed space and let $T \in L\mathcal{R}(X)$ and $\mathcal{S}$ a continuous linear relation such that $\mathcal{S}(0) \subset \overline{T(0)}$ and $\mathcal{D}(\mathcal{S}) \supset \mathcal{D}(T)$,*

$$\sigma_{\mathcal{S}}(T) = \sigma_{\mathcal{S}'}(T').$$

◊

Definition 2.15. *Let X a complex Banach space and let $T, \mathcal{S} \in L\mathcal{R}(X)$ such that $\mathcal{S}$ is continuous, T is closed with $\mathcal{S}(0) \subset T(0)$, $\|S\| \neq 0$ and $\mathcal{D}(T) \subset \mathcal{D}(\mathcal{S})$. We define the $\mathcal{S}$-pseudospectra of linear relation for every $\varepsilon > 0$ by:*

$$\sigma_{\varepsilon,\mathcal{S}}(T) = \sigma_{\mathcal{S}}(T) \cup \left\{\lambda \in \mathbb{C} : \ \|(\lambda\mathcal{S} - T)^{-1}\| > \frac{1}{\varepsilon}\right\}.$$

The pseudoresolvent of T is denoted by $\rho_{\varepsilon,\mathcal{S}}(T)$ and is defined as,

$$\rho_{\varepsilon,\mathcal{S}}(T) = \rho_{\mathcal{S}}(T) \cap \left\{\lambda \in \mathbb{C} : \ \|(\lambda\mathcal{S} - T)^{-1}\| \leq \frac{1}{\varepsilon}\right\}.$$ ◊

3. Some properties of $\mathcal{S}$-condition pseudospectrum $\Sigma_{\varepsilon,\mathcal{S}}(T)$

In this section, we define the $\mathcal{S}$-condition pseudospectrum of linear relation in $L\mathcal{R}(X)$, where X is a complex Banach space and consider some basic properties in order to put this definition in its due place. We begin with the following definition.

Definition 3.1. *Let $T, \mathcal{S} \in L\mathcal{R}(X)$ such that $\mathcal{S}$ is continuous, T is closed with $\mathcal{S}(0) \subset T(0)$, $\|S\| \neq 0$ and $\mathcal{D}(T) \subset \mathcal{D}(\mathcal{S})$. For every $0 < \varepsilon < 1$, the $\mathcal{S}$-condition pseudospectrum of T is denoted by $\Sigma_{\varepsilon,\mathcal{S}}(T)$ and is defined as,*

$$\Sigma_{\varepsilon,\mathcal{S}}(T) = \sigma_{\mathcal{S}}(T) \cup \left\{\lambda \in \mathbb{C} :\ \|\lambda\mathcal{S} - T\|\|(\lambda\mathcal{S} - T)^{-1}\| > \frac{1}{\varepsilon}\right\}.$$

the $\mathcal{S}$-condition pseudoresolvent of T is denoted by $\rho_{\varepsilon,\mathcal{S}}(T)$ and is defined as,

$$\rho_{\varepsilon,\mathcal{S}}(T) = \rho_{\mathcal{S}}(T) \cap \left\{\lambda \in \mathbb{C} :\ \|\lambda\mathcal{S} - T\|\|(\lambda\mathcal{S} - T)^{-1}\| \leq \frac{1}{\varepsilon}\right\},$$

with the convention that $\|\lambda\mathcal{S} - T\|\|(\lambda\mathcal{S} - T)^{-1}\| = \infty$, if $\lambda\mathcal{S} - T$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma_{\mathcal{S}}(T)$. ◊

Remark 3.2. *Note that if $\mathcal{S} = I$ (the identity linear relation on X), we recover the usual definition of condition pseudospectra of closed linear reation T :*

$$\Sigma_{\varepsilon}(T) = \sigma(T) \cup \left\{\lambda \in \mathbb{C} :\ \|\lambda - T\|\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\right\},$$

and the usual definition of condition pseudoresolvent of T:

$$\rho_{\varepsilon}(T) = \rho(T) \cap \left\{\lambda \in \mathbb{C} :\ \|\lambda - T\|\|(\lambda - T)^{-1}\| \leq \frac{1}{\varepsilon}\right\}.$$ ◊

Throughout of this sequel of this section, X will denote a Banach space over the complex field $\mathbb{C}$ and $\mathcal{S}, T \in L\mathcal{R}(X)$ such that $\mathcal{S}$ is continuos, T is closed with $\mathcal{S}(0) \subset T(0)$ and $\mathcal{D}(T) \subset \mathcal{D}(\mathcal{S})$ and $\|S\| \neq 0$, except where stated otherwise.
In the next proposition we will establish the relationship between $\mathcal{S}$-condition pseudospectrum and $\mathcal{S}$-pseudospectrum of an bounded linear relation.

Proposition 3.3. *Let $T, \mathcal{S} \in B\mathcal{R}(X)$ and for every $0 < \varepsilon < 1$ such that $\varepsilon < \|\lambda\mathcal{S} - T\|$ we have*
(i) $\lambda \in \Sigma_{\varepsilon,\mathcal{S}}(T)$ if, and only if, $\lambda \in \sigma_{\varepsilon\|\lambda\mathcal{S}-T\|,\mathcal{S}}(T)$.
(ii) $\lambda \in \sigma_{\varepsilon,\mathcal{S}}(T)$ if, and only if, $\lambda \in \Sigma_{\frac{\varepsilon}{\|\lambda\mathcal{S}-T\|},\mathcal{S}}(T)$. ◊

Proof. (i) If $\lambda \in \Sigma_{\varepsilon,\mathcal{S}}(T)$, then

$$\lambda \in \sigma_{\mathcal{S}}(T) \ \text{ and } \ \|\lambda\mathcal{S} - T\|\|(\lambda\mathcal{S} - T)^{-1}\| \geq \frac{1}{\varepsilon},$$

hence,

$$\lambda \in \sigma_{\mathcal{S}}(T) \ \text{ and } \ \|(\lambda\mathcal{S} - T)^{-1}\| \geq \frac{1}{\varepsilon\|\lambda\mathcal{S} - T\|},$$

which implies that $\lambda \in \sigma_{\varepsilon\|\lambda-T\|,\mathcal{S}}(T)$. The converse is similar.
(ii) Let $\lambda \in \sigma_{\varepsilon,\mathcal{S}}(T)$, then

$$\lambda \in \sigma_{\mathcal{S}}(T) \ \text{ and } \ \|(\lambda\mathcal{S} - T)^{-1}\| \geq \frac{1}{\varepsilon},$$

thus

$$\lambda \in \sigma_{\mathcal{S}}(T) \ \text{ and } \ \|\lambda\mathcal{S} - T\|\|(\lambda\mathcal{S} - T)^{-1}\| \geq \frac{\|\lambda\mathcal{S} - T\|}{\varepsilon}.$$

This proves that $\lambda \in \Sigma_{\frac{\varepsilon}{\|\lambda\mathcal{S}-T\|},\mathcal{S}}(T)$. The converse is similar. □

Proposition 3.4. *For all* $0 < \varepsilon < 1$, *we have*
(*i*) $\sigma_{\mathcal{S}}(T) = \bigcap\limits_{\varepsilon>0} \Sigma_{\varepsilon,\mathcal{S}}(T)$.
(*ii*) *If* $0 < \varepsilon_1 < \varepsilon_2 < 1$, *then* $\sigma_{\mathcal{S}}(T) \subset \Sigma_{\varepsilon_1,\mathcal{S}}(T) \subset \Sigma_{\varepsilon_2,\mathcal{S}}(T)$.
(*iii*) *For any* $\alpha, \beta \in \mathbb{C}$, *with* $\beta \neq 0$ *we have* $\Sigma_{\varepsilon,\mathcal{S}}(\alpha\mathcal{S} + \beta T) = \alpha + \Sigma_{\varepsilon,\mathcal{S}}(T)\beta$. ◊

Proof. (*i*) It is clear that $\sigma_{\mathcal{S}}(T) \subset \Sigma_{\varepsilon,\mathcal{S}}(T)$, then $\sigma_{\mathcal{S}}(T) \subset \bigcap\limits_{\varepsilon>0} \Sigma_{\varepsilon,\mathcal{S}}(T)$.

Conversely, if $\lambda \in \bigcap\limits_{\varepsilon>0} \Sigma_{\varepsilon,\mathcal{S}}(T)$, then for all, $\varepsilon > 0$, we have $\lambda \in \Sigma_{\varepsilon,\mathcal{S}}(T)$. We will discuss these two cases:
First case: If $\lambda \in \sigma_{\mathcal{S}}(T)$, we get the desired result.
Second case: If $\lambda \in \left\{\lambda \in \mathbb{C} : \ \|\lambda\mathcal{S} - T\|\|(\lambda\mathcal{S} - T)^{-1}\| > \dfrac{1}{\varepsilon}\right\}$, taking limits as $\varepsilon \to 0^+$, we get $\|\lambda\mathcal{S} - T\|\|(\lambda\mathcal{S} - T)^{-1}\| = \infty$.
Thus $\lambda \in \sigma_{\mathcal{S}}(T)$.
(*ii*) Let $\lambda \in \Sigma_{\varepsilon_1,\mathcal{S}}(T)$, then $\|\lambda\mathcal{S} - T\|\|(\lambda\mathcal{S} - T)^{-1}\| > \dfrac{1}{\varepsilon_1} > \dfrac{1}{\varepsilon_2}$. Hence $\lambda \in \Sigma_{\varepsilon_2,\mathcal{S}}(T)$.
(*iii*) Let $\alpha, \beta \in \mathbb{C}$, such that $\beta \neq 0$. then $\lambda \notin \Sigma_{\varepsilon,\mathcal{S}}(\alpha\mathcal{S} + \beta T)$ if, and only if,

$$\lambda \in \rho_{\mathcal{S}}(\alpha\mathcal{S} + \beta T) \ \text{ and } \ \|(\lambda - \alpha)\mathcal{S} - \beta T\|\|((\lambda - \alpha)\mathcal{S} - \beta T)^{-1}\| \leq \frac{1}{\varepsilon},$$

if, and only if, $(\lambda - \alpha)\mathcal{S} - \beta T$ is injective, open with dense range and

$$\|(\lambda - \alpha)\mathcal{S} - \beta T\|\|((\lambda - \alpha)\mathcal{S} - \beta T)^{-1}\| \leq \frac{1}{\varepsilon},$$

if, and only if, $\beta^{-1}(\lambda - \alpha)\mathcal{S} - T$ is injective, open with dense range and
$\|\beta^{-1}(\lambda - \alpha)\mathcal{S} - T\|\|(\beta^{-1}(\lambda - \alpha)\mathcal{S} - T)^{-1}\| = \|\beta^{-1}((\lambda - \alpha)\mathcal{S} - \beta T)\|\|(\beta^{-1}((\lambda - \alpha)\mathcal{S} - \beta T)^{-1}\|$
$= \|(\lambda - \alpha)\mathcal{S} - \beta T\|\|((\lambda - \alpha)\mathcal{S} - \beta T)^{-1}\| \leq \dfrac{1}{\varepsilon}$,
if, and only if, $\beta^{-1}(\lambda - \alpha) \in \rho_{\mathcal{S}}(T)$ and $\|\beta^{-1}(\lambda - \alpha)\mathcal{S} - T\|\|(\beta^{-1}(\lambda - \alpha)\mathcal{S} - T)^{-1}\| \leq \dfrac{1}{\varepsilon}$,
if, and only if, $\beta^{-1}(\lambda - \alpha) \notin \Sigma_{\varepsilon,\mathcal{S}}(T)$, if, and only if, $\lambda \notin \alpha + \Sigma_{\varepsilon,\mathcal{S}}(T)\beta$. □

Proposition 3.5. *Let* $\mathcal{S} \in B\mathcal{R}(X)$ *such that* $0 \in \rho_{\varepsilon}(\mathcal{S})$, $\mathcal{S}(0) = T(0)$ *and for all* $0 < \varepsilon < 1$, *we have*

$$\Sigma_{\frac{\varepsilon}{\|\mathcal{S}\|\|\mathcal{S}^{-1}\|}}(\mathcal{S}^{-1}T) \subset \Sigma_{\varepsilon,\mathcal{S}}(T) \subset \Sigma_{\varepsilon\|\mathcal{S}\|\|\mathcal{S}^{-1}\|}(\mathcal{S}^{-1}T).$$

◊

Proof. First of all, it is necessary to show that

$$\lambda - \mathcal{S}^{-1}T = \mathcal{S}^{-1}(\lambda\mathcal{S} - T), \tag{1}$$

and

$$\lambda\mathcal{S} - T = \mathcal{S}(\lambda - \mathcal{S}^{-1}T). \tag{2}$$

Since $\mathcal{S}$ is a bounded linear relation, then $I_{\mathcal{D}(\mathcal{S})} = \mathcal{S}^{-1}\mathcal{S} = I_X$. Using Proposition 2.3, we have

$$\begin{aligned} \mathcal{S}^{-1}(\lambda\mathcal{S} - T) &= \lambda\mathcal{S}^{-1}\mathcal{S} - \lambda\mathcal{S}^{-1}T, \\ &= \lambda - S^{-1}T. \end{aligned}$$

Hence, (1) holds. Let $x \in X$, then $x \in \mathcal{D}(\mathcal{S}^{-1}) = R(\mathcal{S})$ hence $\exists a \in X$, $(a, x) \in S$ such that $(x, a) \in \mathcal{S}^{-1}$, so we have $(x, x) \in \mathcal{S}\mathcal{S}^{-1}$. This implies that $I \subset \mathcal{S}\mathcal{S}^{-1}$, therefore $\lambda\mathcal{S} - T \subset \mathcal{S}\mathcal{S}^{-1}(\lambda\mathcal{S} - T) = \mathcal{S}(\lambda - \mathcal{S}^{-1})$. Which results that

$$\lambda\mathcal{S} - T \subset \mathcal{S}(\lambda - \mathcal{S}^{-1}T).$$

Furthemore, we have

$$\begin{aligned}
\mathcal{S}(\lambda-\mathcal{S}^{-1}T)(0) &= \mathcal{S}(\lambda(0)-\mathcal{S}^{-1}T(0))\\
&= \mathcal{S}(0-\mathcal{S}^{-1}T(0))\\
&= \mathcal{S}\mathcal{S}^{-1}T(0)\\
&= \mathcal{S}\mathcal{S}^{-1}\mathcal{S}(0)\\
&= \mathcal{S}(0)\\
&= T(0)\\
&= (\lambda\mathcal{S}-T)(0),
\end{aligned}$$

and moreover $\mathcal{D}(\lambda\mathcal{S}-T)=\mathcal{D}(\lambda\mathcal{S})\cap\mathcal{D}(T)=\mathcal{D}(T)$ and

$$\mathcal{D}(\mathcal{S}(\lambda-\mathcal{S}^{-1}T))=\Big\{x\in\mathcal{D}(\lambda-\mathcal{S}^{-1}T):(\lambda-\mathcal{S}^{-1}T)x\cap\mathcal{D}(\mathcal{S})=\lambda-\mathcal{S}^{-1}T)x\cap X\neq\emptyset\Big\}.$$

On the other hand,

$$\begin{aligned}
\mathcal{D}(\lambda-\mathcal{S}^{-1}T) &= \mathcal{D}(\lambda)\cap\mathcal{D}(\mathcal{S}^{-1}T)\\
&= \mathcal{D}(\mathcal{S}^{-1}T)\\
&= \Big\{x\in\mathcal{D}:Tx\cap\mathcal{D}(\mathcal{S}^{-1})=Tx\cap R(\mathcal{S})\neq\emptyset\Big\}\\
&= \mathcal{D}(T)\\
&= \mathcal{D}(\lambda\mathcal{S}-T).
\end{aligned}$$

Then, from these properties we infer that

$$\lambda\mathcal{S}-T=\mathcal{S}(\lambda-\mathcal{S}^{-1}T).$$

So, that (2) holds. Further $C=\mathcal{S}^{-1}T$ is closed. Indeed, $\mathcal{S}$ is continuos so that by [13, II.5.1], we have $\mathcal{S}$ is closed then $\mathcal{S}^{-1}$ is closed. Moreover, $\alpha(\mathcal{S}^{-1})=0<\infty$ and $\gamma(\mathcal{S}^{-1})=\dfrac{1}{\|\mathcal{S}\|}>0$. Hence by Lemma 2.10, we have $\mathcal{S}^{-1}T$ is closed.

For the first inclusion, we suppose that $\lambda\in\Sigma_{\frac{\varepsilon}{\|\mathcal{S}\|\|\mathcal{S}^{-1}\|}}(\mathcal{S}^{-1}T)$, so we have

$$\begin{aligned}
\frac{\|\mathcal{S}\|\|\mathcal{S}^{-1}\|}{\varepsilon} &< \|\lambda-\mathcal{S}^{-1}T\|\|(\lambda-\mathcal{S}^{-1}T)^{-1}\|\\
&= \|\mathcal{S}^{-1}(\lambda\mathcal{S}-T)\|\|(\mathcal{S}^{-1}(\lambda\mathcal{S}-T))^{-1}\|\\
&= \|\mathcal{S}^{-1}(\lambda\mathcal{S}-T)\|\|(\lambda\mathcal{S}-T)^{-1}\mathcal{S}\|.
\end{aligned}$$

Since we have $\mathcal{D}(\mathcal{S}^{-1})=R(\mathcal{S})=X$, and

$$\begin{aligned}
\mathcal{D}(\lambda\mathcal{S}-T)^{-1} &= R(\lambda\mathcal{S}-T)\\
&= \mathcal{S}R(\lambda-\mathcal{S}^{-1}T)\\
&= R(\mathcal{S})\\
&= X.
\end{aligned}$$

Then , $(\lambda\mathcal{S}-T)(0)\subset\mathcal{D}(S^{-1})$ and $\mathcal{S}(0)\subset\mathcal{D}(\lambda\mathcal{S}-T)^{-1}$. Therefore from Proposition 2.5, we have

$$\|\mathcal{S}^{-1}(\lambda\mathcal{S}-T)\|\|(\lambda\mathcal{S}-T)^{-1}S\|\leq\|\mathcal{S}^{-1}\|\|\mathcal{S}\|\|(\lambda\mathcal{S}-T)\|\|(\lambda\mathcal{S}-T)^{-1}\|.$$

Hence,

$$\frac{1}{\varepsilon}<\|(\lambda\mathcal{S}-T)\|\|(\lambda\mathcal{S}-T)^{-1}\|.$$

Consequently, we obtain $\lambda \in \Sigma_{\varepsilon,\mathcal{S}}(T)$. For the second inclusion, we assume that $\lambda \in \Sigma_{\varepsilon,\mathcal{S}}(T)$, so we have

$$\begin{aligned}\frac{1}{\varepsilon} &< \|(\lambda\mathcal{S}-T)\|\|(\lambda\mathcal{S}-T)^{-1}\| \\ &= \|\mathcal{S}(\lambda-\mathcal{S}^{-1}T)\|\|(\mathcal{S}(\lambda-\mathcal{S}^{-1}T))^{-1}\| \\ &= \|\mathcal{S}(\lambda-\mathcal{S}^{-1}T)\|\|(\lambda-\mathcal{S}^{-1}T)^{-1}\mathcal{S}^{-1}\|.\end{aligned}$$

Since we have $\mathcal{D}(\mathcal{S}) = X$ and

$$\begin{aligned}\mathcal{D}((\lambda-\mathcal{S}^{-1}T)^{-1}) &= R(\lambda-\mathcal{S}^{-1}T) \\ &= \mathcal{S}^{-1}R(\lambda\mathcal{S}-T) \\ &= R(\mathcal{S}^{-1}) \\ &= \mathcal{D}(\mathcal{S}) = X,\end{aligned}$$

therefore, we have $\mathcal{S}^{-1}(0) \subset \mathcal{D}((\lambda-\mathcal{S}^{-1}T)^{-1})$ and $(\lambda-\mathcal{S}^{-1}T)(0) \subset \mathcal{D}(\mathcal{S})$. Then from Proposition 2.5, we have

$$\|\mathcal{S}(\lambda-\mathcal{S}^{-1}T)\|\|(\lambda-\mathcal{S}^{-1}T)^{-1}\mathcal{S}^{-1}\| \leq \|\mathcal{S}\|\|\mathcal{S}^{-1}\|\|\lambda-\mathcal{S}^{-1}T\|\|(\lambda-\mathcal{S}^{-1}T)^{-1}\|.$$

Hence,

$$\frac{1}{\|\mathcal{S}\|\|\mathcal{S}^{-1}\|\varepsilon} < \|\lambda-\mathcal{S}^{-1}T\|\|(\lambda-\mathcal{S}^{-1}T)^{-1}\|.$$

Now, it remains to shows that $0 < \|\mathcal{S}\|\|\mathcal{S}^{-1}\|\varepsilon < 1$. Since $0 \in \rho_\varepsilon(\mathcal{S})$, then we have $0 < \|\mathcal{S}\|\|\mathcal{S}^{-1}\| < \frac{1}{\varepsilon}$. Thus, $0 < \|\mathcal{S}\|\|\mathcal{S}^{-1}\|\varepsilon < 1$. Finally, we conclude that $\lambda \in \Sigma_{\varepsilon\|\mathcal{S}\|\|\mathcal{S}^{-1}\|}(\mathcal{S}^{-1}T)$. □

Theorem 3.6. *Let $T, \mathcal{S}$ be continuous linear relation and $0 < \varepsilon < 1$.*

$$\Sigma_{\varepsilon,\mathcal{S}}(T) = \Sigma_{\varepsilon,\mathcal{S}'}(T').$$

◊

Proof. At the first, it is clear from the proof of Lemma 2.14 that $(\lambda\mathcal{S}-T)' = \lambda\mathcal{S}'-T'$. Now let $\lambda \in \rho_{\varepsilon,\mathcal{S}'}(T')$. Then,

$$\lambda \in \rho_{\mathcal{S}'}(T') \text{ and } \|\lambda\mathcal{S}'-T'\|\|(\lambda\mathcal{S}'-T')^{-1}\| \leq \frac{1}{\varepsilon}.$$

So from Lemma 2.14, we obtain $\lambda \in \rho(T, S)$ and therefore $(\lambda\mathcal{S}-T)^{-1}$ is continuous. We also have $(\lambda\mathcal{S}-T)$ is continuous, hence from Proposition 2.8, it follows that

$$\begin{aligned}\|\lambda\mathcal{S}-T\|\|(\lambda\mathcal{S}-T)^{-1}\| &= \|(\lambda\mathcal{S}-T)'\|\|((\lambda\mathcal{S}-T)')^{-1}\| \leq \frac{1}{\varepsilon} \\ &= \|\lambda\mathcal{S}'-T'\|\|(\lambda\mathcal{S}'-T')^{-1}\| \leq \frac{1}{\varepsilon},\end{aligned}$$

furthermore,

$$\lambda \in \rho_{\varepsilon,\mathcal{S}}(T).$$

However, the opposite inclusion follows by symmetry. □

Proposition 3.7. *Let $T \in B\mathcal{R}(X)$ and S is a bounded and closed operator such that $0 \in \rho_{\varepsilon\|S\|\|S^{-1}\|}(T)$, with $k = \|S\|\|S^{-1}\|\|T\|\|T^{-1}\|$ and $0 < \varepsilon < 1$. We have:*
(i) If $\lambda \in \Sigma_{\varepsilon,S}(T) \setminus \{0\}$, then $\frac{1}{\lambda} \in \Sigma_{\varepsilon k,S^{-1}}(T^{-1}) \setminus \{0\}$.
(ii) If $\lambda \in \Sigma_{\varepsilon,S^{-1}}(T^{-1}) \setminus \{0\}$, then $\frac{1}{\lambda} \in \Sigma_{\varepsilon k,S}(T) \setminus \{0\}$. ◊

Proof. (*i*) Let $\lambda \in \Sigma_{\varepsilon,S}(T) \setminus \{0\}$. Then,

$$\frac{1}{\varepsilon} < \|\lambda S - T\|\|(\lambda S - T)^{-1}\|$$

By using Remark 2.1 and Proposition 2.3, it follows that

$$\begin{aligned}
\|\lambda S - T\|\|(\lambda S - T)^{-1}\| &= \|S(\lambda - S^{-1}T)\|\|(S(\lambda - S^{-1}T))^{-1}\| \\
&= \| - \lambda S(\frac{S^{-1}}{\lambda} - T^{-1})T\|\|(-\lambda(S(\frac{S^{-1}}{\lambda} - T^{-1})T)^{-1}\| \\
&= \| - \lambda S(\frac{S^{-1}}{\lambda} - T^{-1})T\|\|(-\lambda(T^{-1}(\frac{S^{-1}}{\lambda} - T^{-1})^{-1}S^{-1})\| \\
&\leq \|S\|\|S^{-1}\|\|T\|\|T^{-1}\|\|(\frac{S^{-1}}{\lambda} - T^{-1})\|\|(\frac{S^{-1}}{\lambda} - T^{-1})^{-1}\|.
\end{aligned}$$

Which implies that

$$\frac{1}{\varepsilon\|S\|\|S^{-1}\|\|T\|\|T^{-1}\|} < \|(\frac{S^{-1}}{\lambda} - T^{-1})\|\|(\frac{S^{-1}}{\lambda} - T^{-1})^{-1}\|.$$

Now, it remains to shows that $0 < \|S\|\|S^{-1}\|\|T\|\|T^{-1}\|\varepsilon < 1$. Indeed, $0 \in \rho_{\varepsilon\|S\|\|S^{-1}\|}(T)$, then we have

$$0 < \|T\|\|T^{-1}\| < \frac{1}{\varepsilon\|S\|\|S^{-1}\|}.$$

Hence, we obtain

$$0 < \|S\|\|S^{-1}\|\|T\|\|T^{-1}\| < \frac{1}{\varepsilon}.$$

Thus, $0 < \|S\|\|S^{-1}\|\|T\|\|T^{-1}\|\varepsilon < 1$. Consequently,

$$\frac{1}{\lambda} \in \Sigma_{\varepsilon k,S^{-1}}(T^{-1}) \setminus \{0\}.$$

(*ii*) Let $\lambda \in \Sigma_{\varepsilon,S^{-1}}(T^{-1}) \setminus \{0\}$. Then,

$$\frac{1}{\varepsilon} < \|\lambda S^{-1} - T^{-1}\|\|(\lambda S^{-1} - T^{-1})^{-1}\|.$$

On the other hand, we have

$$\begin{aligned}
\|\lambda S^{-1} - T^{-1}\|\|(\lambda S^{-1} - T^{-1})^{-1}\| &= \| - \lambda T^{-1}(\frac{S}{\lambda} - T)S^{-1}\|\| - \lambda^{-1}(T^{-1}(\frac{S}{\lambda} - T)S^{-1})^{-1}\| \\
&= \| - \lambda T^{-1}(\frac{S}{\lambda} - T)S^{-1}\|\| - \lambda^{-1}S(\frac{S}{\lambda} - T)^{-1}T\| \\
&\leq \|S\|\|S^{-1}\|\|T\|\|T^{-1}\|\|(\frac{S}{\lambda} - T)\|\|(\frac{S}{\lambda} - T)^{-1}\|.
\end{aligned}$$

Thus,

$$\frac{1}{\varepsilon\|S\|\|S^{-1}\|\|T\|\|T^{-1}\|} \leq \|(\frac{S}{\lambda} - T)\|\|(\frac{S}{\lambda} - T)^{-1}\|.$$

Moreover, we have

$$0 < \|S\|\|S^{-1}\|\|T\|\|T^{-1}\|\varepsilon < 1.$$

Consequently,

$$\frac{1}{\lambda} \in \Sigma_{\varepsilon k,S}(T) \setminus \{0\}.$$

□

4. Characterization of $\mathcal{S}$-condition pseudospectrum

Throughout of this section, X will denote a Banach space over the complex field $\mathbb{C}$ and $\mathcal{S}, T \in L\mathcal{R}(X)$ such that $\mathcal{S}$ is continuos, T is closed with $\mathcal{S}(0) \subset T(0)$, $\mathcal{D}(\mathcal{S}) \supset \mathcal{D}(T)$ and $\|S\| \neq 0$, except where stated otherwise. The purpose of this section is to give a characterization of the $\mathcal{S}$-condition pseudospectrum of linear relation. Our first result is the following.

Lemma 4.1. *Let $T, \mathcal{S} \in \mathcal{BR}(X)$ and $0 < \varepsilon < 1$. Then, $\lambda \in \Sigma_{\varepsilon,\mathcal{S}}(T) \setminus \sigma_{\mathcal{S}}(T)$ if, and only if, there exists $x \in X$, such that*

$$\|(\lambda\mathcal{S} - T)x\| < \varepsilon\|\lambda\mathcal{S} - T\|\|x\|. \qquad \diamond$$

Proof. Assume that $\lambda \in \Sigma_{\varepsilon,\mathcal{S}}(T) \setminus \sigma_{\mathcal{S}}(T)$, then

$$\|\lambda\mathcal{S} - T\|\|(\lambda\mathcal{S} - T)^{-1}\| > \frac{1}{\varepsilon},$$

and thus we have

$$\|(\lambda\mathcal{S} - T)^{-1}\| > \frac{1}{\varepsilon\|\lambda\mathcal{S} - T\|}.$$

Moreover,

$$\sup_{y \in X\setminus\{0\}} \frac{\|(\lambda\mathcal{S} - T)^{-1}y\|}{\|y\|} > \frac{1}{\varepsilon\|\lambda\mathcal{S} - T\|}.$$

Hence, there exists a nonzero $y \in X$, such that

$$\|(\lambda\mathcal{S} - T)^{-1}y\| > \frac{\|y\|}{\varepsilon\|\lambda\mathcal{S} - T\|}. \tag{3}$$

Put $x = (\lambda\mathcal{S} - T)^{-1}y$, so

$$\begin{aligned}(\lambda\mathcal{S} - T)x &= (\lambda\mathcal{S} - T)(\lambda\mathcal{S} - T)^{-1}y \\ &= y + (\lambda\mathcal{S} - T)(0).\end{aligned}$$

Knowing that $(\lambda\mathcal{S} - T)(0) = \lambda\mathcal{S}(0) - T(0) = T(0)$ (as $\mathcal{S}(0) \subset T(0)$), this allows us to deduce that

$$\begin{aligned}\|(\lambda\mathcal{S} - T)x\| &= d(y, (\lambda\mathcal{S} - T)(0)) \\ &= d(y, T(0)) \\ &\leq d(y, 0) \ \ (\textit{since} \ \ 0 \in T(0)) \\ &\leq \|y\|.\end{aligned}$$

Therefore, from Eq (3), we infer that

$$\|x\| > \frac{\|y\|}{\varepsilon\|\lambda\mathcal{S} - T\|} \geq \frac{\|(\lambda\mathcal{S} - T)x\|}{\varepsilon\|\lambda\mathcal{S} - T\|}.$$

Finally, we have as a result,

$$\|(\lambda\mathcal{S} - T)x\| < \varepsilon\|\lambda\mathcal{S} - T\|\|x\|.$$

For the revese inclusion, we assume there exists $x \in X$ such that

$$\|(\lambda\mathcal{S} - T)x\| < \varepsilon\|\lambda\mathcal{S} - T\|\|x\|.$$

Since $\lambda \in \rho_{\mathcal{S}}(T)$, then $\lambda\mathcal{S} - T$ is injective and open. Moreover, we have

$$\gamma(\lambda\mathcal{S} - T)\|x\| \leq \|(\lambda\mathcal{S} - T)x\| < \varepsilon\|\lambda\mathcal{S} - T\|\|x\|,$$

therefore,

$$0 < \gamma(\lambda\mathcal{S} - T) < \varepsilon\|\lambda\mathcal{S} - T\|.$$

We already have

$$\gamma(\lambda\mathcal{S} - T) = \|(\lambda\mathcal{S} - T)^{-1}\|^{-1},$$

which shows that

$$\|(\lambda\mathcal{S} - T)^{-1}\| > \frac{1}{\varepsilon\|\lambda\mathcal{S} - T\|},$$

and, consequently

$$\lambda \in \sigma_{\varepsilon\|\lambda\mathcal{S}-T\|}(T),$$

equivalently, in virtue of Proposition3.3, we obtain $\lambda \in \Sigma_{\varepsilon,\mathcal{S}}(T)$. $\square$

Theorem 4.2. *Let V is a closed and bounded operator such that $\mathcal{S}V = V\mathcal{S}$ and $0 \in \rho(V)$. Let $R = VTV^{-1}$ Then, for all $0 < \varepsilon < 1$, $k = \|V\|\|V^{-1}\|$ and $0 < k^2\varepsilon < 1$ we have*

$$\Sigma_{\frac{\varepsilon}{k^2},\mathcal{S}}(T) \subseteq \Sigma_{\varepsilon,\mathcal{S}}(R) \subseteq \Sigma_{k^2\varepsilon,\mathcal{S}}(T). \qquad \diamond$$

Proof. First of all, we have from [5, Theorem 3.1] R is closed. On the other hand, we have

$$\begin{aligned} \lambda\mathcal{S} - R &= \lambda\mathcal{S} - VTV^{-1} \\ (\lambda\mathcal{S} - R)V &= (\lambda\mathcal{S} - VTV^{-1})V. \end{aligned}$$

Since V is injective and by virtue of Proposition 2.3(i), we obtain that

$$\begin{aligned} (\lambda\mathcal{S} - R)V &= \lambda\mathcal{S}V - TV^{-1}V \\ &= \lambda\mathcal{S}V - VT. \end{aligned}$$

Moreover, since V is injective and by using Proposition 2.3(ii), we deduce that

$$\begin{aligned} V^{-1}(\lambda\mathcal{S} - R)V &= V^{-1}(\lambda\mathcal{S}V - VT) \\ &= \lambda V^{-1}V\mathcal{S} - V^{-1}VT \\ &= \lambda\mathcal{S} - T. \end{aligned}$$

Because V is a single valued, this, further, implies that

$$(\lambda\mathcal{S} - R) = V(\lambda\mathcal{S} - T)V^{-1}.$$

Then, it is obvious that

$$(\lambda\mathcal{S} - T)^{-1} = V^{-1}(\lambda\mathcal{S} - R)^{-1}V \ \text{ and } \ (\lambda\mathcal{S} - R)^{-1} = V(\lambda\mathcal{S} - T)^{-1}V^{-1}.$$

Next, we show that $\sigma_{\mathcal{S}}(T) = \sigma_{\mathcal{S}}(R)$. To see this, let $\lambda \in \rho_{\mathcal{S}}(T)$ then the closed relation $\lambda\mathcal{S} - T$ is injective, surjective and open. By [13, Proposition VI.5.2] it follows that $V(\lambda\mathcal{S} - T)V^{-1} = \lambda\mathcal{S} - R$ is also closed, bounded below (injective and open), surjective. Hence $\lambda \in \rho_{\mathcal{S}}(R)$. Conversely, if $\lambda \in \rho_{\mathcal{S}}(R)$, then the closed relation $\lambda\mathcal{S} - R$ is injective, surjective and open. Taking into account the [13, Proposition VI.5.2], it follows that $V^{-1}(\lambda\mathcal{S} - R)V = \lambda\mathcal{S} - T$ is also closed, bounded below (injective and open), surjective. Hence $\lambda \in \rho_{\mathcal{S}}(T)$. Now, we start with the first inclusion. So, we can write

$$\begin{aligned} \|\lambda\mathcal{S} - R\|\|(\lambda\mathcal{S} - R)^{-1}\| &= \|V(\lambda\mathcal{S} - T)V^{-1}\|\|V(\lambda\mathcal{S} - T)^{-1}V^{-1}\| \\ &\leq \|V(\lambda\mathcal{S} - T)\|\|V^{-1}\|\|V(\lambda\mathcal{S} - T)^{-1}\|\|V^{-1}\| \\ &\leq \|V\|\|(\lambda\mathcal{S} - T)\|\|V^{-1}\|\|V\|\|(\lambda\mathcal{S} - T)^{-1}\|\|V^{-1}\| \\ &\leq \left(\|V\|\|V^{-1}\|\right)^2\|(\lambda\mathcal{S} - T)\|\|(\lambda\mathcal{S} - T)^{-1}\| \\ &\leq k^2\|(\lambda\mathcal{S} - T)\|\|(\lambda\mathcal{S} - T)^{-1}\|. \end{aligned}$$

In the similar way,

$$\begin{array}{rcl} \|\lambda\mathcal{S}-T\|\|(\lambda\mathcal{S}-T)^{-1}\| & = & \|V^{-1}(\lambda\mathcal{S}-R)V\|\|V^{-1}(\lambda\mathcal{S}-R)^{-1}V\| \\ & \leq & \left(\|V\|\|V^{-1}\|\right)^{2}\|(\lambda\mathcal{S}-R)\|\|(\lambda\mathcal{S}-R)^{-1}\| \\ & \leq & k^{2}\|(\lambda\mathcal{S}-R)\|\|(\lambda\mathcal{S}-R)^{-1}\|. \end{array}$$

For $\lambda \in \Sigma_{\varepsilon/k^2,\mathcal{S}}(T)$, we have

$$\lambda \in \sigma_{\mathcal{S}}(T) \ \text{ and } \ \|\lambda\mathcal{S}-T\|\|(\lambda\mathcal{S}-T)^{-1}\| > \frac{k^2}{\varepsilon},$$

then,

$$\lambda \in \sigma_{\mathcal{S}}(R) \ \text{ and } \ \|\lambda\mathcal{S}-R\|\|(\lambda\mathcal{S}-R)^{-1}\| \geq \frac{1}{k^2}\|\lambda\mathcal{S}-T\|\|(\lambda\mathcal{S}-T)^{-1}\| > \frac{1}{\varepsilon},$$

hence,

$$\lambda \in \Sigma_{\varepsilon,\mathcal{S}}(R).$$

Therefore,

$$\Sigma_{\varepsilon/k^2,\mathcal{S}}(T) \subseteq \Sigma_{\varepsilon,\mathcal{S}}(R).$$

For the second inclusion, let $\lambda \in \Sigma_{\varepsilon,\mathcal{S}}(R)$, then

$$\lambda \in \sigma_{\mathcal{S}}(R) \ \text{ and } \ \|\lambda\mathcal{S}-R\|\|(\lambda\mathcal{S}-R)^{-1}\| > \frac{1}{\varepsilon},$$

this induces that,

$$\lambda \in \sigma_{\mathcal{S}}(T) \ \text{ and } \ \|\lambda\mathcal{S}-T\|\|(\lambda\mathcal{S}-T)^{-1}\| \geq \frac{1}{k^2}\|\lambda\mathcal{S}-R\|\|(\lambda\mathcal{S}-R)^{-1}\| > \frac{1}{k^2\varepsilon},$$

hence,

$$\lambda \in \Sigma_{k^2\varepsilon,\mathcal{S}}(T).$$

Consequently,

$$\Sigma_{\varepsilon,\mathcal{S}}(R) \subseteq \Sigma_{k^2\varepsilon,\mathcal{S}}(T).$$

□

In the sequel of this section, we suppose that X is a Banach space and A is a linear relation satisfying the following property $(\mathcal{P})$:

$$(\mathcal{P}) : \forall\, A \in L\mathcal{R}(X) \text{ with } 0 \in \rho(A),\ \exists\, B \in L\mathcal{R}(X) \text{ with } 0 \notin \rho(B) \ \text{ such that } \|A-B\| = \frac{1}{\|A^{-1}\|}.$$

The following example shows the above property.

Example 4.3. *Let X be a Banach space and we consider*

$$A = \begin{pmatrix} I & 0 \\ 0 & \frac{I}{2} \end{pmatrix} \ \text{ and } \ B = \begin{pmatrix} \frac{I}{2} & 0 \\ 0 & 0 \end{pmatrix}.$$

So we have,

$$A^{-1} = \begin{pmatrix} I & 0 \\ 0 & 2I \end{pmatrix} \ \text{ and } \ A-B = \begin{pmatrix} \frac{I}{2} & 0 \\ 0 & \frac{I}{2} \end{pmatrix}.$$

This implies that,

$$\|A^{-1}\| = \max\left\{\|I\|, \|2I\|\right\} = 2 \ \text{ and } \ \|A-B\| = \max\left\{\|\frac{I}{2}\|, \|\frac{I}{2}\|\right\} = \frac{1}{2}.$$

Hence we have,

$$\|A-B\| = \frac{1}{2} = \frac{1}{\|A^{-1}\|}.$$

Theorem 4.4. *Let X is a Banach space and $0 < \varepsilon < 1$. Suppose that there exists a non invertible linear relation R such that $\mathcal{D}(T) \supset \mathcal{D}(R)$, $T(0) \subset R(0)$ then, $\lambda \in \Sigma_{\varepsilon,\mathcal{S}}(T)$ if, and only if, $\|R\| < \varepsilon\|\lambda\mathcal{S} - T\|$ and $\lambda \in \sigma_{\mathcal{S}}(T + R)$.* $\diamond$

Proof. Assume that $\lambda \in \Sigma_{\varepsilon,\mathcal{S}}(T)$. There are two cases to consider:
First case: If $\lambda \in \sigma_{\mathcal{S}}(T)$, then it is sufficient to take $R = 0$.

Second case: If $\lambda \in \Sigma_{\varepsilon,\mathcal{S}}(T) \setminus \sigma_{\mathcal{S}}(T)$, then $\lambda\mathcal{S} - T$ is invertible. Hence, by property $(\mathcal{P})$, there exists a non invertible linear relation D such that

$$\|\lambda\mathcal{S} - T - D\| = \frac{1}{\|(\lambda\mathcal{S} - T)^{-1}\|}.$$

Putting $R = \lambda\mathcal{S} - T - D$. Then ,

$$\|R\| = \frac{1}{\|(\lambda\mathcal{S} - T)^{-1}\|} < \varepsilon\|\lambda\mathcal{S} - T\|.$$

Also $D = \lambda\mathcal{S} - T - R$ is non invertible, that is $\lambda \in \sigma_{\mathcal{S}}(T + R)$. For the reverse inclusion, we suppose $\lambda \in \sigma_{\mathcal{S}}(T + R)$. We derive a contradiction from the assumption that $\lambda \notin \Sigma_{\varepsilon,\mathcal{S}}(T)$, which is equivalent to

$$\lambda \in \rho_{\mathcal{S}}(T) \text{ and } \|\lambda\mathcal{S} - T\|\|(\lambda\mathcal{S} - T)^{-1}\| \leq \frac{1}{\varepsilon}.$$

Since $\lambda \in \rho_{\mathcal{S}}(T)$, then $\lambda\mathcal{S} - T$ is an invertible linear relation and we have $\lambda\mathcal{S} - T - R$ is not invertible. Or, we have $(\lambda\mathcal{S} - T)(0) \subset T(0) \subset R(0)$ and $\mathcal{D}(\lambda\mathcal{S} - T) \supset \mathcal{D}(T) \supset \mathcal{D}(R)$, hence taking into account Proposition 2.3 and in view of property $(\mathcal{P})$, it follows that

$$\|R\| = \|(\lambda\mathcal{S} - T - R) - (\lambda\mathcal{S} - T)\| = \frac{1}{\|(\lambda\mathcal{S} - T)^{-1}\|}.$$

Therefore,

$$\frac{1}{\|(\lambda\mathcal{S} - T)^{-1}\|} = \|R\| < \varepsilon\|\lambda\mathcal{S} - T\|.$$

Consequently,

$$\|\lambda\mathcal{S} - T\|\|(\lambda\mathcal{S} - T)^{-1}\| > \frac{1}{\varepsilon}.$$

Which is a contraction. $\square$

References

[1] A. Ammar, Some results on semi-Fredholm perturbations of multivalued linear operators, Linear Multilinear Algebra 66 (2018), 1311–1332.
[2] A. Ammar, A characterization of some subsets of essential spectra of a multivalued linear operator. Complex Anal. Oper. Theory, 11 (2017), 175–196.
[3] A. Ammar, B. Boukettaya and A. Jeribi, Stability of the $\mathcal{S}$-left and $\mathcal{S}$-right essential spectra of a linear operator, Acta. Math. Sci. Ser. B Engl. Ed. 34 (2014), 1922–1934.
[4] A. Ammar, H. Daoud and A. Jeribi, $\mathcal{S}$-pseudospectra and $\mathcal{S}$-essential pseudospectra of linear relations, preprint (2019).
[5] A. Ammar, H. Daoud and A. Jeribi, The stability of pseudospectra and essential pseudospectra of linear relations, J. Pseudo-Differ. Oper. Appl, 7(2016), 473–491.
[6] A. Ammar, H. Daoud and A. Jeribi, Pseudospecta and essential pseudospectra of multuvalued linear relations, Mediterr. J. Math, 12 (2015), 1377–1395.
[7] A. Ammar, K. Mahfoudhi and A. Jeribi, A characterization of the condition pseudospectrum on Banch space, Functional Analysis, Approximation and Computation. 10 (2018), 13–21.
[8] A. Ammar, A. Jeribi and M. Zayed, The condition pseudospectrum of multivalued linear operators, preprint(2019).
[9] A. Jeribi, Spectral theory and applications of linear operators and block operator matrices, Springer-Verlag, New-York, 2015.
[10] A. Jeribi, N. Moalla and S. Yengui, $\mathcal{S}$-essential spectra and application to an example of transport operators, Math. Meth. Appl. Sci, 13 (2011).
[11] F. Abdmouleh, A. Ammar and A. Jeribi, Stability of the $\mathcal{S}$-essential spectra on a Banach space, Math. Slovaca 63(2013), 299–320.

[12] F. Abdmouleh, A. Jeribi and T. Álvarez, On a characterization of the essential spectra of a linear relation , preprint, 2015.
[13] R. W. Cross, Multivalued Linear Operators, Marcel Dekker, New York, 1998.
[14] T. Álvarez, A. Ammar and A. Jeribi, On the essential spectra of some matrix of linear relations, Math. Methods Appl, 37 (2014) 620–644.
[15] T. Álvarez, A. Ammar and A. Jeribi, A characterization of some subsets of $\mathcal{S}$-essential spectra of a multivalued linear operator, Colloq. Math. 135 (2014), 171–186.