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The (Signless Laplacian) Spectral Radius (Of Subgraphs) of Uniform Hypergraphs

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Abstract. Let $\lambda_1(G)$ and $q_1(G)$ be the spectral radius and the signless Laplacian spectral radius of a k-uniform hypergraph G, respectively. In this paper, we give the lower bounds of $d - \lambda_1(H)$ and $2d - q_1(H)$, where H is a proper subgraph of a f(-edge)-connected d-regular (linear) k-uniform hypergraph. Meanwhile, we also give the lower bounds of $2\Delta - q_1(G)$ and $\Delta - \lambda_1(G)$, where G is a nonregular f(-edge)-connected (linear) k-uniform hypergraph with maximum degree Δ.

1. Introduction

A hypergraph $G = (V, E)$ is a pair consisting of a vertex set $V = \{1, 2, \ldots, n\}$, and an edge set $E = \{e_1, e_2, \ldots, e_m\}$, where e_i $(1 \le i \le m)$ is a subset of V. A hypergraph is called k-uniform if every edge contains precisely k vertices. We will use the term k-graphs in place of k-uniform hypergraphs. A hypergraph G is called linear provided that each pair of the edges of G has at most one common vertex [1]. Given two k-graphs $G = (V, E)$ and $H = (V', E')$, if $V' \subseteq V$ and $E' \subseteq E$, then H is said to be a subgraph (sub-hypergraph) of G. If H is a subgraph of a k-graph G, and $H \neq G$, then H is called a proper subgraph of G [11]. A tensor $\mathcal{A}$ with order k and dimension n over the complex field $\mathbb{C}$ is a multidimensional array

$$\mathcal{A} = (a_{i_1 i_2 \ldots i_k}),\ 1 \le i_1, i_2, \ldots, i_k \le n.$$

The tensor $\mathcal{A}$ is called symmetric if its entries are invariant under any permutation of their indices. For a vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$, $\mathcal{A}x^{k-1}$ is a vector in $\mathbb{C}^n$ whose i-th component is the following

$$(\mathcal{A}x^{k-1})_i = \sum_{i_2, \ldots, i_k = 1}^{n} a_{i i_2 \ldots i_k} x_{i_2} \cdots x_{i_k},\ \forall\, i \in [n].$$

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Let $x^{[k-1]} = (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1})^T \in \mathbb{C}^n$. If $\mathcal{A}x^{k-1} = \lambda x^{[k-1]}$ has a solution $x \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, then λ is called an eigenvalue of $\mathcal{A}$ and x is an eigenvector associated with λ. And the spectral radius of $\mathcal{A}$ is defined as $\lambda_1(\mathcal{A}) = \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } \mathcal{A}\}$. Also, a tensor $\mathcal{A}$ of order k and dimension n uniquely determines a k-th degree homogeneous polynomial function $\mathcal{A}x^k$, which is

$$x^T(\mathcal{A}x^{k-1}) = \sum_{i_1,i_2,\ldots,i_k=1}^{n} a_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

The adjacency tensor [6] of a k-graph G with n vertices, denoted by $\mathcal{A}(G)$, is an order k dimension n symmetric tensor with entries $a_{i_1 i_2 \ldots i_k}$ such that

$$a_{i_1 i_2 \ldots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, i_2, \ldots, i_k\} \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Let λ be an eigenvalue of a k-graph G with eigenvector x. Since $\mathcal{A}(G)x^{k-1} = \lambda x^{[k-1]}$, we know that cx is also an eigenvector of λ for any nonzero constant c. So we can choose x such that $\sum_{i=1}^{n} x_i^k = 1$. In this case, we have [6, 9]

$$\lambda = x^T(\mathcal{A}(G)x^{k-1}) = k \sum_{e \in E(G)} x^e,$$

where $x^e = x_{i_1} x_{i_2} \cdots x_{i_k}$, $e = \{i_1, i_2, \ldots, i_k\}$.

For a k-graph G, we denote $N_G(v)$ as the set of neighbours of v in G, and $E_G(v)$ as the set of edges containing v in G. The degree of a vertex v in G, denoted by $d_v = d_G(v)$, is $|E_G(v)|$. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$ denote the minimum degree and the maximum degree of G, respectively. If all vertices of G have the same degree, then G is called regular. Let $\mathcal{D} = \mathcal{D}(G)$ be a k-th order n-dimensional diagonal tensor with its diagonal element $d_{ii\ldots i}$ being d_i, the degree of vertex i of G, for all $i \in [n]$. Then $\mathcal{Q}(G) = \mathcal{D}(G) + \mathcal{A}(G)$ is the signless Laplacian tensor of the hypergraph G [16]. The signless Laplacian eigenvalues refer to the eigenvalues of the signless Laplacian tensor. Let $q_1(G)$ be the signless Laplacian spectral radius of G.

In a k-graph G, a path of length l is defined to be an alternating sequence of vertices and edges $u_1, e_1, u_2, \ldots, u_l, e_l, u_{l+1}$, where $u_1, \ldots, u_{l+1}$ are distinct vertices of G, $e_1, \ldots, e_l$ are distinct edges of G and $u_i, u_{i+1} \in e_i$ for $i = 1, \ldots, l$. For any two vertices u and v of G, if there exists a path connecting u and v, then G is called connected. A hypergraph G is called f-edge-connected if $G - U$ is connected for any edge subset $U \subseteq E(G)$ satisfying $|U| < f$. A hypergraph G is called f-connected if there exist f paths connecting u and v in G, where no pair of them have any other elements in common except u and v, for any $u, v \in V(G)$ [27].

Spectral graph theory has a long history behind its development [2, 7]. It is natural to generalize spectral theory of graphs to hypergraphs. Recently, there are many work about the spectral theory of hypergraphs [8, 12, 14, 16, 17, 21–23]. In [20], Stevanović proposed a question: How small $\Delta - \lambda_1(G)$ can be when G is an irregular graph with maximum degree Δ and spectral radius $\lambda_1(G)$? Cioabă et al. [5] gave a lower bound on $\Delta - \lambda_1(G)$ for irregular graphs, which improved previous bounds of Stevanović [20] and of Zhang [26]. Cioabă [4] obtained a lower bound on $\Delta - \lambda_1(G)$ for an irregular graph G with maximum degree Δ and diameter D. Nikiforov [13] presented a lower bound on $\lambda_1(G) - \lambda_1(H)$ for a proper subgraph H of a connected regular graph G. Shi [18] obtained a lower bound on $\Delta - \lambda_1(G)$ for a connected irregular graph G in terms of its diameter and average degree. Ning et al. [15] gave a lower bound on $2\Delta - q_1(G)$ for a connected irregular graph G in terms of the diameter. Shui et al. [19] gave a lower bound on $2\Delta - q_1(G)$ and $2\Delta - q_1(H)$ for a k-connected irregular graph G and a proper spanning subgraph H of a Δ-regular k-connected graph, respectively. Li et al. [10] obtianed the lower bounds on $\Delta - \lambda_1(G)$ for irregular connected k-graphs in terms of vertex degrees, the diameter, and the number of vertices and edges. Yuan et al. [25] gave some bounds on $\lambda_1(G)$ and $q_1(G)$ for a k-graph G in terms of its degrees of vertices. Chen et al. [3] presented several upper bounds on $\lambda_1(G)$ and $q_1(G)$ for a k-graph G in terms of degree sequences. We are inspired by two articles of Shui et al. [19] and Li et al. [10]. In this paper, we give the bounds of (signless Laplacian) spectral radius of subgraphs of f(-edge)-connected d-regular (linear) k-graphs. We also give the bounds of (signless Laplacian) spectral radius of connected nonregular (linear) k-graphs.

2. Preliminaries

In this section, we give some useful lemmas.

Let G be a connected k-graph. By Perron-Frobenius theorem of nonnegative tensors [24], $\lambda_1(G)$ (resp., $q_1(G)$) is an eigenvalue of $\mathcal{A}(G)$ (resp., $\mathcal{Q}(G)$), and there exists a unique positive eigenvector $x = (x_1, \ldots, x_n)^T$ corresponding to $\lambda_1(G)$ (resp., $q_1(G)$) with $\sum_{i=1}^{n} x_i^k = 1$, and x is called the principal eigenvector of $\mathcal{A}(G)$ (resp., $\mathcal{Q}(G)$).

The following Lemma 2.1 is from the proof of Theorem 4.1 in [10].

Lemma 2.1. *([10]) Let G be a connected k-graph with n vertices and $\lambda_1(G)$ be the spectral radius of G with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Let $x_u = \max_{i\in V(G)}\{x_i\}$ and $x_v = \min_{i\in V(G)}\{x_i\}$. Let $P : u = u_0, e_1, u_1, \ldots, u_r = v$ be a path from u to v, where e_i is an edge containing vertices u_{i-1} and u_i. Then*

$$\sum_{w_i, w_j \in e \in E(P)} (x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 \geq \frac{k}{2r}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2.$$

Lemma 2.2. *([8]) Let $a_1, \ldots, a_n$ be nonnegative real numbers. Then*

$$\frac{a_1 + \cdots + a_n}{n} - (a_1 \cdots a_n)^{\frac{1}{n}} \geq \frac{1}{n(n-1)} \sum_{1\leq i<j\leq n} (\sqrt{a_i} - \sqrt{a_j})^2,$$

equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

Lemma 2.3. *([18]) Let a, b, y_1, y_2 be positive numbers. Then*

$$a(y_1 - y_2)^2 + by_2^2 \geq \frac{ab}{a+b} y_1^2,$$

equality holds if and only if $y_2 = \frac{ay_1}{a+b}$.

Two paths P_1, P_2 are called edge-disjoint if the edges of P_1 have no common with the edges of P_2.

Lemma 2.4. *([27]) A hypergraph G is f-edge-connected if and only if there are f mutual edge-disjoint paths between each pair of vertices.*

Lemma 2.5. *([27]) If a hypergraph G is f-connected, then there are f mutual vertex-disjoint paths between each pair of vertices.*

Lemma 2.6. *([10]) Let G be a connected k-graph with n vertices, minimum degree δ and maximum degree Δ, and let $x = (x_1, \ldots, x_n)^T$ be the principal eigenvector of $\mathcal{A}(G)$. Then $x_{max} \geq ((\frac{\delta}{\Delta})^{\frac{k}{2(k-1)}} + n - 1)^{-\frac{1}{k}}$, where $x_{max} = \max_{1\leq i\leq n}\{x_i\}$.*

In fact, we can prove similarly that Lemma 2.6 also holds for the principal eigenvector of $\mathcal{Q}(G)$, where G is a connected k-graph with n vertices.

3. The (signless Laplacian) spectral radius of subgraphs of f(-edge)-connected d-regular k-graphs

In this section, we will give a bound of the spectral radius and the signless Laplacian spectral radius of a subgraph of a f-edge-connected d-regular k-graph G, respectively. And we will give a bound on the the spectral radius and the signless Laplacian spectral radius of a subgraph of a f-connected d-regular linear k-graph G, respectively.

Lemma 3.1. *Let H be a maximal proper subgraph of a f(-edge)-connected d-regular k-graph G such that $f \geq 2$, and $\lambda_1(H)$ be the spectral radius of H with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Then*

$$d - \lambda_1(H) = \sum_{i=1}^{n}(d - d_i)x_i^k + \sum_{e=\{w_1,w_2,\ldots,w_k\}\in E(H)} (x_{w_1}^k + \cdots + x_{w_k}^k - kx^e),$$

where d_i is the degree of the vertex i of H.

Proof. Let $V(H) = V(G)$ and H differs from G in a single edge $\{u_1, u_2, \ldots, u_k\}$. We know that H is connected since $f \geq 2$. Let $x_u = \max_{i\in V(H)}\{x_i\}$ and $x_v = \min_{i\in V(H)}\{x_i\}$. We claim $u \neq u_i$ for any $1 \leq i \leq k$. Indeed, if $u = u_i$ for some $1 \leq i \leq k$, then

$$\lambda_1(H)x_{u_i}^{k-1} = \sum_{e=\{u_i,w_2,\ldots,w_k\}\in E(H)} a_{u_i w_2 \ldots w_k} x_{w_2} \ldots x_{w_k} \leq (d-1)x_{u_i}^{k-1},$$

and thus $\lambda_1(H) \leq d - 1$, contradicting the fact that $\lambda_1(H) > \frac{k|E(H)|}{n} = d - \frac{k}{n} > d - 1$. We also find that

$$\begin{aligned}
d - \lambda_1(H) &= d\sum_{i=1}^{n} x_i^k - k\sum_{e\in E(H)} x^e \\
&= d\sum_{i=1}^{n} x_i^k - \sum_{i=1}^{n} d_i x_i^k + \sum_{i=1}^{n} d_i x_i^k - k\sum_{e\in E(H)} x^e \\
&= \sum_{i=1}^{n}(d - d_i)x_i^k + \sum_{e=\{w_1,w_2,\ldots,w_k\}\in E(H)} (x_{w_1}^k + \cdots + x_{w_k}^k - kx^e).
\end{aligned}$$

□

Lemma 3.2. *Let H be a maximal proper subgraph of a f(-edge)-connected d-regular k-graph G such that $f \geq 2$ and $q_1(H)$ be the signless Laplacian spectral radius of H with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Then*

$$2d - q_1(H) = 2\sum_{i=1}^{n}(d - d_i)x_i^k + \sum_{e=\{w_1,w_2,\ldots,w_k\}\in E(H)} (x_{w_1}^k + \cdots + x_{w_k}^k - kx^e),$$

where d_i is the degree of the vertex i of H.

Proof. Similarly, let $V(H) = V(G)$ and H differs from G in a single edge $\{u_1, u_2, \ldots, u_k\}$. We know that H is connected since $f \geq 2$. Let $x_u = \max_{i\in V(H)}\{x_i\}$ and $x_v = \min_{i\in V(H)}\{x_i\}$. We claim $u \neq u_i$ for any $1 \leq i \leq k$. Indeed, if $u = u_i$ for some $1 \leq i \leq k$, then

$$q_1(H)x_{u_i}^{k-1} = d_{u_i}x_{u_i}^{k-1} + \sum_{e=\{u_i,w_2,\ldots,w_k\}\in E(H)} a_{u_i w_2 \ldots w_k} x_{w_2} \ldots x_{w_k} \leq 2(d-1)x_{u_i}^{k-1},$$

and thus $q_1(H) \leq 2d - 2$, contradicting the fact that $q_1(H) \geq 2\lambda_1(H) > 2\frac{k|E(H)|}{n} = 2d - \frac{2k}{n} > 2d - 2$. We also find that

$$\begin{aligned}
2d - q_1(H) &= 2d\sum_{i=1}^{n} x_i^k - \sum_{i=1}^{n} d_i x_i^k - k\sum_{e\in E(H)} x^e \\
&= 2\sum_{i=1}^{n}(d - d_i)x_i^k + \sum_{i=1}^{n} d_i x_i^k - k\sum_{e\in E(H)} x^e \\
&= 2\sum_{i=1}^{n}(d - d_i)x_i^k + \sum_{e=\{w_1,w_2,\ldots,w_k\}\in E(H)} (x_{w_1}^k + \cdots + x_{w_k}^k - kx^e).
\end{aligned}$$

□

Theorem 3.3. *Let G be a f-edge-connected d-regular k-graph with n vertices and $m(=\frac{dn}{k})$ edges, and H' be a proper subgraph of G. If $f, k \geq 2$, then*

$$d-\lambda_1(H') > \frac{k(f-1)^2}{[2(k-1)(m-1)+(f-1)^2]((\frac{d-1}{d})^{\frac{k}{2(k-1)}}+n-1)}.$$

Proof. Let H be a maximal proper subgraph of G, i.e., $V(H)=V(G)$ and H differs from G in a single edge $\{u_1, u_2, \ldots, u_k\}$. Let $\lambda_1(H)$ be the spectral radius of H with the principal eigenvector $x=(x_1, x_2, \ldots, x_n)^T$. Let $x_u=\max_{i\in V(H)}\{x_i\}$ and $x_v=\min_{i\in V(H)}\{x_i\}$. By Lemmas 2.2 and 3.1, we have

$$\begin{aligned} d-\lambda_1(H) &> x_{u_1}^k + x_{u_2}^k + \cdots + x_{u_k}^k + \frac{1}{k-1}\sum_{w_i, w_j\in e\in E(H)}(x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 \\ &\geq kx_v^k + \frac{1}{k-1}\sum_{w_i, w_j\in e\in E(H)}(x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2. \end{aligned} \tag{3.1}$$

Since G is a f-edge-connected d-regular k-graph, there are at least $f-1$ edge disjoint paths $P_1, \ldots, P_{f-1}$ connecting u and v in H. Let $P_t: u=v_0, e_1, v_1, \ldots, v_{r_t}=v$ be a path from u to v. Then we have $\sum_{t=1}^{f-1} r_t \leq m-1$. In addition, by Lemma 2.1, we have

$$\sum_{w_i, w_j\in e\in E(P_t)}(x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 \geq \frac{k}{2r_t}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2.$$

Thus, we have

$$\begin{aligned} \sum_{w_i, w_j\in e\in E(H)}(x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 &\geq \sum_{t=1}^{f-1}\sum_{w_i, w_j\in e\in E(P_t)}(x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 \\ &\geq \sum_{t=1}^{f-1}\frac{k}{2r_t}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 \\ &\geq \frac{k(f-1)^2}{\sum_{t=1}^{f-1}2r_t}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 \\ &\geq \frac{k(f-1)^2}{2(m-1)}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2. \end{aligned} \tag{3.2}$$

By (3.1) and (3.2), we have

$$d-\lambda_1(H) > kx_v^k + \frac{k(f-1)^2}{2(k-1)(m-1)}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2.$$

The right hand side of the above inequality is a quadratic function of $x_v^{\frac{k}{2}}$. By Lemma 2.3, we have

$$d-\lambda_1(H) > \frac{k(f-1)^2}{2(k-1)(m-1)+(f-1)^2}x_u^k.$$

By Lemma 2.6, we have

$$\begin{aligned} d-\lambda_1(H) &> \frac{k(f-1)^2}{[2(k-1)(m-1)+(f-1)^2]((\frac{\delta(H)}{\Delta(H)})^{\frac{k}{2(k-1)}}+n-1)} \\ &= \frac{k(f-1)^2}{[2(k-1)(m-1)+(f-1)^2]((\frac{d-1}{d})^{\frac{k}{2(k-1)}}+n-1)}. \end{aligned}$$

Therefore, we have

$$d-\lambda_1(H') > \frac{k(f-1)^2}{[2(k-1)(m-1)+(f-1)^2]((\frac{d-1}{d})^{\frac{k}{2(k-1)}}+n-1)}.$$

□

Theorem 3.4. *Let G be a f-edge-connected d-regular k-graph with n vertices and* $m(=\frac{dn}{k})$ *edges, and H′ be a proper subgraph of G. If* $f,k\geq 2$, *then*

$$2d-q_1(H') > \frac{2k(f-1)^2}{[4(k-1)(m-1)+(f-1)^2]((\frac{d-1}{d})^{\frac{k}{2(k-1)}}+n-1)}.$$

Proof. Let H be a maximal proper subgraph of G, i.e., $V(H)=V(G)$ and H differs from G in a single edge $\{u_1,u_2,\dots,u_k\}$. Let $q_1(H)$ is the signless Laplacian spectral radius of H with a principal eigenvector x. Let $x_u=\max_{i\in V(H)}\{x_i\}$ and $x_v=\min_{i\in V(H)}\{x_i\}$. By Lemmas 2.2 and 3.2, we have

$$\begin{aligned}2d-q_1(H) &> 2(x_{u_1}^k+x_{u_2}^k+\cdots+x_{u_k}^k)+\frac{1}{k-1}\sum_{w_i,w_j\in e\in E(H)}(x_{w_i}^{\frac{k}{2}}-x_{w_j}^{\frac{k}{2}})^2\\ &\geq 2kx_v^k+\frac{1}{k-1}\sum_{w_i,w_j\in e\in E(H)}(x_{w_i}^{\frac{k}{2}}-x_{w_j}^{\frac{k}{2}})^2.\end{aligned} \tag{3.3}$$

Since G is a f-edge-connected d-regular k-graph, there are at least $f-1$ edge disjoint paths connecting u and v in H. By (3.2) and (3.3), then we have

$$2d-q_1(H) > 2kx_v^k+\frac{k(f-1)^2}{2(k-1)(m-1)}(x_u^{\frac{k}{2}}-x_v^{\frac{k}{2}})^2.$$

The right hand side of the above inequality is a quadratic function of $x_v^{\frac{k}{2}}$. By Lemma 2.3, we have

$$2d-q_1(H) > \frac{2k(f-1)^2}{4(k-1)(m-1)+(f-1)^2}x_u^k.$$

By Lemma 2.6, we have

$$\begin{aligned}2d-q_1(H) &> \frac{2k(f-1)^2}{[4(k-1)(m-1)+(f-1)^2]((\frac{\delta(H)}{\Delta(H)})^{\frac{k}{2(k-1)}}+n-1)}\\ &= \frac{2k(f-1)^2}{[4(k-1)(m-1)+(f-1)^2]((\frac{d-1}{d})^{\frac{k}{2(k-1)}}+n-1)}.\end{aligned}$$

Therefore, we have

$$2d-q_1(H') > \frac{2k(f-1)^2}{[4(k-1)(m-1)+(f-1)^2]((\frac{d-1}{d})^{\frac{k}{2(k-1)}}+n-1)}.$$

□

Theorem 3.5. *Let G be a f-connected d-regular linear k-graph with n vertices, and H′ be a proper subgraph of G. If* $f,k\geq 2$, *then*

$$d-\lambda_1(H') > \frac{2k(f-1)^2}{(2n+d(k^2-k-2)+4)(f-1)^2+h},$$

where $h=k(k-1)(n-k-d+2)((n+2(f-2))^2-(f-1))$.

Proof. Let H be a maximal proper subgraph of G, i.e., $V(H) = V(G)$ and H differs from G in a single edge $\{u_1, u_2, \ldots, u_k\}$. We know that H is connected since $f \geq 2$. Let $\lambda_1(H)$ be the spectral radius of H with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Let $x_u = \max_{i\in V(H)}\{x_i\}$ and $x_v = \min_{i\in V(H)}\{x_i\}$. By the proof of Lemma 3.1, we claim $u \neq u_i$ for $1 \leq i \leq k$. By Lemmas 2.2 and 3.1, we have

$$\begin{aligned} d - \lambda_1(H') &\geq d - \lambda_1(H) \\ &> x_{u_1}^k + x_{u_2}^k + \cdots + x_{u_k}^k + \frac{1}{k-1} \sum_{w_i,w_j\in e\in E(H)} (x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 \\ &\geq kx_v^k + \frac{1}{k-1} \sum_{w_i,w_j\in e\in E(H)} (x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2. \end{aligned} \tag{3.4}$$

Since G is a f-connected d-regular k-graph, by Lemma 2.5, there are at least $f-1$ vertex disjoint paths $P_1, P_2, \ldots, P_{f-1}$ connecting u and v in H. Thus, we have

$$\sum_{t=1}^{f-1} \mid V(P_t) \mid \leq n + 2(f-2).$$

Since G is a linear k-graph, we have $\mid V(P_t) \mid \geq \mid E(P_t) \mid +1$. Hence, $\frac{2|E(P_t)|}{k} \leq \frac{|V(P_t)|(|V(P_t)|-1)}{2}$. By Lemma 2.1, we have

$$\begin{aligned} \sum_{w_i,w_j\in e\in E(H)} (x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 &\geq \sum_{t=1}^{f-1} \sum_{w_i,w_j\in e\in E(P_t)} (x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 \\ &\geq \sum_{t=1}^{f-1} \frac{k}{2\mid E(P_t)\mid} (x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 \\ &\geq \sum_{t=1}^{f-1} \frac{2}{\mid V(P_t)\mid(\mid V(P_t)\mid -1)} (x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 \\ &\geq \frac{2(f-1)^2}{\sum_{t=1}^{f-1} \mid V(P_t)\mid(\mid V(P_t)\mid -1)} (x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 \\ &> \frac{2(f-1)^2}{\sum_{t=1}^{f-1} ((\mid V(P_t)\mid)^2 - 1)} (x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 \\ &\geq \frac{2(f-1)^2}{(\sum_{t=1}^{f-1} \mid V(P_t)\mid)^2 - (f-1)} (x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 \\ &\geq \frac{2(f-1)^2}{(n+2(f-2))^2 - (f-1)} (x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2. \end{aligned} \tag{3.5}$$

So by (3.4) and (3.5), we have

$$d - \lambda_1(H') > kx_v^k + \frac{2(f-1)^2}{(k-1)((n+2(f-2))^2 - (f-1))} (x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2. \tag{3.6}$$

Define

$$C = \frac{2k(f-1)^2}{(2n + d(k^2-k-2)+4)(f-1)^2 + h},$$

where $h = k(k-1)(n-k-d+2)((n+2(f-2))^2 - (f-1))$.

Case 1. If $\sum\limits_{i=1}^{k} x_{u_i}^k > C$, then from (3.4), we have

$$\begin{aligned} d-\lambda_1(H') &> (x_{u_1}^k + x_{u_2}^k + \cdots + x_{u_k}^k) + \frac{1}{k-1}\sum_{w_i,w_j\in e\in E(H)}(x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 \\ &> C + \frac{1}{k-1}\sum_{w_i,w_j\in e\in E(H)}(x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 \\ &\geq C. \end{aligned}$$

Case 2. Let $x_{u_1} = \min_{1\leq i\leq k}\{x_{u_i}\}$. Since $d_H(u_1) = d-1$, it is possible to choose at least $d-2$ distinct vertices $\{v_1, v_2, \ldots, v_{d-2}\}$ from $N_H(u_1)$ such that $u \notin \{v_1, v_2, \ldots, v_{d-2}\}$. If $\sum_{t=1}^{d-2} x_{v_t}^k \geq \frac{d(k-1)}{2}C$, by (3.4) again and Lemma 2.3, then we have

$$\begin{aligned} d-\lambda_1(H') &> \frac{2}{k-1}x_{u_1}^k + \frac{1}{k-1}\sum_{t=1}^{d-2}(x_{v_t}^{\frac{k}{2}} - x_{u_1}^{\frac{k}{2}})^2 \\ &= \frac{1}{k-1}\sum_{t=1}^{d-2}(\frac{2}{d-2}x_{u_1}^k + (x_{v_t}^{\frac{k}{2}} - x_{u_1}^{\frac{k}{2}})^2) \\ &\geq \frac{1}{k-1}\sum_{t=1}^{d-2}\frac{\frac{2}{d-2}}{\frac{2}{d-2}+1}x_{v_t}^k \\ &\geq \frac{1}{k-1}\frac{2}{d}\frac{d(k-1)}{2}C \\ &= C. \end{aligned}$$

Case 3. Since G is a linear k-graph, we have $v_t \neq u_i$, for $1 \leq t \leq d-2,\ 2 \leq i \leq k$. If $\sum\limits_{i=1}^{k} x_{u_i}^k \leq C$ and $\sum_{t=1}^{d-2} x_{v_t}^k < \frac{d(k-1)}{2}C$, then

$$x_u^k \geq \frac{1-\sum_{i=1}^{k}x_{u_i}^k - \sum_{t=1}^{d-2}x_{v_t}^k}{n-k-(d-2)} > \frac{1}{n-k-d+2}(1-C-\frac{d(k-1)}{2}C) = \frac{1}{n-k-d+2}(1-\frac{dk-d+2}{2}C),$$

and from (3.6) and Lemma 2.3, we obtain

$$d-\lambda_1(H') > \frac{2k(f-1)^2}{k(k-1)((n+2(f-2))^2-(f-1))+2(f-1)^2}x_u^k = C.$$

□

Theorem 3.6. *Let G be a f-connected d-regular linear k-graph with n vertices, and H' be a proper subgraph of G. If $f, k \geq 2$, then*

$$2d - q_1(H') > \frac{2k(f-1)^2}{(n+d(k^2-k-1)+2)(f-1)^2+h},$$

where $h = k(k-1)(n-k-d+2)((n+2(f-2))^2-(f-1))$.

Proof. Let H be a maximal proper subgraph of G, i.e., $V(H) = V(G)$ and H differs from G in a single edge $\{u_1, u_2, \ldots, u_k\}$. We know that H is connected since $f \geq 2$. Let $q_1(H)$ be the signless Laplacian spectral radius of H with a principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Let $x_u = \max_{i\in V(H)}\{x_i\}$ and $x_v = \min_{i\in V(H)}\{x_i\}$. By Lemmas 2.2 and 3.2, we have

$$2d - q_1(H') \geq 2d - q_1(H) > 2(x_{u_1}^k + x_{u_2}^k + \cdots + x_{u_k}^k) + \frac{1}{k-1}\sum_{w_i,w_j\in e\in E(H)}(x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2. \tag{3.7}$$

By (3.5) and (3.7), similarly, we have

$$2d - q_1(H') > 2kx_v^k + \frac{2(f-1)^2}{(k-1)((n+2(f-2))^2-(f-1))}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2. \tag{3.8}$$

Define

$$C = \frac{2k(f-1)^2}{(n+d(k^2-k-1)+2)(f-1)^2+h},$$

where $h = k(k-1)(n-k-d+2)((n+2(f-2))^2-(f-1))$.

Case 1. If $\sum\limits_{i=1}^{k} x_{u_i}^k > \frac{C}{2}$, then from (3.7), we have

$$\begin{aligned} 2d - q_1(H') &> 2(x_{u_1}^k + x_{u_2}^k + \cdots + x_{u_k}^k) + \frac{1}{k-1}\sum_{w_i,w_j\in e\in E(H)}(x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 \\ &> 2\frac{C}{2} + \frac{1}{k-1}\sum_{w_i,w_j\in e\in E(H)}(x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 \\ &\geq C. \end{aligned}$$

Case 2. Let $x_{u_1} = \min_{1\leq i\leq k}\{x_{u_i}\}$. Since $d_H(u_1) = d-1$, it is possible to choose at least $d-2$ distance vertices $\{v_1, v_2, \ldots, v_{d-2}\}$ from $N_H(u_1)$ such that $u \notin \{v_1, v_2, \ldots, v_{d-2}\}$. If $\sum_{t=1}^{d-2} x_{v_t}^k \geq \frac{d(k-1)}{2}C$, by (3.7) again and Lemma 2.3, then we have

$$\begin{aligned} 2d - q_1(H') &> \frac{2}{k-1}x_{u_1}^k + \frac{1}{k-1}\sum_{t=1}^{d-2}(x_{v_t}^{\frac{k}{2}} - x_{u_1}^{\frac{k}{2}})^2 \\ &= \frac{1}{k-1}\sum_{t=1}^{d-2}(\frac{2}{d-2}x_{u_1}^k + (x_{v_t}^{\frac{k}{2}} - x_{u_1}^{\frac{k}{2}})^2) \\ &\geq \frac{1}{k-1}\sum_{t=1}^{d-2}\frac{\frac{2}{d-2}}{1+\frac{2}{d-2}}x_{v_t}^k \\ &\geq \frac{1}{k-1}\frac{2}{d}\frac{d(k-1)}{2}C \\ &= C. \end{aligned}$$

Case 3. Since G is a linear k-graph, we have $v_t \neq u_i$, for $1 \leq t \leq d-2$, $2 \leq i \leq k$. If $\sum\limits_{i=1}^{k} x_{u_i}^k \leq \frac{C}{2}$ and $\sum_{t=1}^{d-2} x_{v_t}^k < \frac{d(k-1)}{2}C$, then

$$x_u^k \geq \frac{1 - \sum_{i=1}^{k} x_{u_i}^k - \sum_{t=1}^{d-2} x_{v_t}^k}{n-k-(d-2)} > \frac{1}{n-k-d+2}(1 - \frac{C}{2} - \frac{d(k-1)}{2}C) = \frac{1}{n-k-d+2}(1 - \frac{dk-d+1}{2}C),$$

and from (3.8) and Lemma 2.3, we obtain

$$2d - q_1(H') > \frac{2k(f-1)^2}{k(k-1)((n+2(f-2))^2-(f-1))+(f-1)^2}x_u^k = C.$$

□

4. The signless Laplacian spectral radius of connected nonregular (linear) k-graphs

In this section, we mainly study the upper bounds of the (signless Laplacian) spectral radius of a f(-edge)-connected nonregular k-graph G with maximum degree Δ, respectively.

Theorem 4.1. *Let G be a nonregular f-edge-connected k-graph with n vertices, m edges, minimum degree δ and maximum degree Δ. Then*

$$2\Delta - q_1(G) > \frac{2k(n\Delta - km)f^2}{[4m(k-1)(n\Delta - km) + kf^2]((\frac{\delta}{\Delta})^{\frac{k}{2(k-1)}} + n - 1)}.$$

Proof. Let $q_1(G)$ be the signless Laplacian spectral radius of G with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Let $x_u = \max_{i\in V(G)}\{x_i\}$ and $x_v = \min_{i\in V(G)}\{x_i\}$. We also find that

$$\begin{aligned}2\Delta - q_1(G) &= 2\Delta\sum_{i=1}^{n} x_i^k - \sum_{i=1}^{n} d_i x_i^k - k\sum_{e\in E(G)} x^e\\ &= 2\sum_{i=1}^{n}(\Delta - d_i)x_i^k + \sum_{i=1}^{n} d_i x_i^k - k\sum_{e\in E(G)} x^e\\ &= 2\sum_{i=1}^{n}(\Delta - d_i)x_i^k + \sum_{e=\{w_1w_2\ldots w_k\}\in E(G)}(x_{w_1}^k + \ldots + x_{w_k}^k - kx^e),\end{aligned}$$

where d_i is the degree of the vertex i. By Lemma 2.2, we have

$$2\Delta - q_1(G) > 2(n\Delta - km)x_v^k + \frac{1}{k-1}\sum_{w_i,w_j\in e\in E(G)}(x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2. \tag{4.1}$$

Let $P_t : u = u_0, e_1, u_1, \ldots, u_{r_t} = v$ be a path from u to v. By Lemma 2.1, we have

$$\sum_{w_i,w_j\in e\in E(P_t)}(x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 \geq \frac{k}{2r_t}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2.$$

Since G is f-edge-connected, similar to (3.2), we have

$$\sum_{w_i,w_j\in e\in E(G)}(x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 \geq \sum_{t=1}^{f}\frac{k}{2r_t}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 \geq \frac{kf^2}{2m}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2. \tag{4.2}$$

By (4.1) and (4.2), we have

$$2\Delta - q_1(G) > 2(n\Delta - km)x_v^k + \frac{kf^2}{2m(k-1)}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2.$$

The right hand side of the above inequality is a quadratic function of $x_v^{\frac{k}{2}}$. By Lemma 2.3, we have

$$2\Delta - q_1(G) > \frac{2k(n\Delta - km)f^2}{4m(k-1)(n\Delta - km) + kf^2}x_u^k.$$

By Lemma 2.6, we have

$$2\Delta - q_1(G) > \frac{2k(n\Delta - km)f^2}{[4m(k-1)(n\Delta - km) + kf^2]((\frac{\delta}{\Delta})^{\frac{k}{2(k-1)}} + n - 1)}.$$

□

Theorem 4.2. *Let G be a nonregular f-connected linear k-graph with n vertices, m edges and maximum degree Δ. Then*

$$2\Delta - q_1(G) > \frac{2(n\Delta - km)f^2}{(n + 2(k-1)(n\Delta - km) + (k-2)(f-1))f^2 + h},$$

where $h = (n-f)(k-1)(n\Delta - km)((n+2f-2)^2 - f)$.

Proof. Let $q_1(G)$ be the signless Laplacian spectral radius of G with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Let $x_u = \max_{i\in V(G)}\{x_i\}$ and $x_v = \min_{i\in V(G)}\{x_i\}$. Consider the following two cases:

Case 1. Suppose $d_u \leq \Delta - 1$. Since $\mathcal{Q}x^{k-1} = q_1 x^{[k-1]}$, we have

$$q_1(G)x_u^{k-1} = d_u x_u^{k-1} + \sum_{e=\{u,u_1,\ldots,u_{k-1}\}\in E(G)} x_{u_1}x_{u_2}\ldots x_{u_{k-1}} \leq 2(\Delta-1)x_u^{k-1}.$$

Thus, we have $q_1(G) \leq 2\Delta - 2$. Consequently,

$$2\Delta - q_1(G) \geq 2 > \frac{2(n\Delta - km)f^2}{(2(n\Delta - km) + n)f^2 + h},$$

where $h = (n-f)(k-1)(n\Delta - km)((n+2f-2)^2 - f)$.

Case 2. Suppose $d_u = \Delta$. Since G is a f-connected k-graph, there are at least f vertex disjoint paths $P_1, P_2, \ldots, P_f$ connecting u and v in G. By Lemma 2.5, we have

$$\sum_{t=1}^{f} | V(P_t) | \leq n + 2(f-1). \tag{4.5}$$

Thus, we have

$$\begin{aligned}
2\Delta - q_1(G) &= 2\Delta\sum_{i=1}^{n} x_i^k - \sum_{i=1}^{n} d_i x_i^k - k\sum_{e\in E(G)} x^e \\
&= 2\sum_{i=1}^{n}(\Delta - d_i)x_i^k + \sum_{i=1}^{n} d_i x_i^k - k\sum_{e\in E(G)} x^e \\
&= 2\sum_{i=1}^{n}(\Delta - d_i)x_i^k + \sum_{e=\{w_1w_2\ldots w_k\}\in E(G)} (x_{w_1}^k + \cdots + x_{w_k}^k - kx^e),
\end{aligned}$$

where d_i is the degree of the vertex i. By Lemma 2.2, we have

$$2\Delta - q_1(G) > 2(n\Delta - km)x_v^k + \frac{1}{k-1}\sum_{w_i,w_j\in e\in E(G)} (x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2. \tag{4.6}$$

Similar to the proof of (3.5), we have

$$\sum_{w_i,w_j\in e\in E(G)} (x_{w_i}^{\frac{k}{2}} - x_{w_j}^{\frac{k}{2}})^2 > \frac{2f^2}{(n+2f-2)^2 - f}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2. \tag{4.7}$$

By (4.6), (4.7) and Lemma 2.3, we have

$$\begin{aligned}
2\Delta - q_1(G) &> 2(n\Delta - km)x_v^k + \frac{2f^2}{(k-1)((n+2f-2)^2 - f)}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 \\
&\geq \frac{2(n\Delta - km)f^2}{(k-1)(n\Delta - km)((n+2f-2)^2 - f) + f^2}x_u^k.
\end{aligned} \tag{4.8}$$

Define

$$C = \frac{2(n\Delta - km)f^2}{(n + 2(k-1)(n\Delta - km) + (k-2)(f-1))f^2 + h},$$

where $h = (n-f)(k-1)(n\Delta - km)((n+2f-2)^2 - f)$.

Case 2.1. Suppose $f = 1$, we have

$$2\Delta - q_1(G) > \frac{2(n\Delta - km)}{(k-1)(n\Delta - km)(n^2-1) + 1} x_u^k \tag{4.9}$$

and

$$C = \frac{2(n\Delta - km)}{(n + 2(k-1)(n\Delta - km)) + h},$$

where $h = (n-1)(k-1)(n\Delta - km)(n^2-1)$.

Case 2.1.1. If $x_v^k \geq \frac{C}{2(n\Delta - km)}$, then from (4.6) and (4.7), we obtain

$$2\Delta - q_1(G) > 2(n\Delta - km)\frac{C}{2(n\Delta - km)} + \frac{2f^2}{(k-1)((n+2f-2)^2 - f)}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 > C.$$

Case 2.1.2. If $x_v^k < \frac{C}{2(n\Delta - km)}$, then since $\sum\limits_{i=1}^{n} x_i^k = 1$, we have

$$x_u^k \geq \frac{1 - x_v^k}{n-1} > \frac{1}{n-1}\left(1 - \frac{C}{2(n\Delta - km)}\right).$$

Thus, by (4.9), we have

$$2\Delta - q_1(G) > C.$$

Case 2.2. Suppose $f \geq 2$.

Case 2.2.1. If $x_v^k \geq \frac{C}{2(n\Delta - km)}$, then the result can be obtained using a similar argument of the case 2.1.1.

Case 2.2.2. Since G is a f-connected linear k-graph, we have $d_v \geq f$. We can choose at least $f-1$ vertices from $N_G(v)$, denoted by $\{v_1, v_2, \ldots, v_{f-1}\}$, such that $u \notin \{v_1, v_2, \ldots, v_{f-1}\}$. If $\sum_{t=1}^{f-1} x_{v_t}^k > C(k-1)(1 + \frac{f-1}{2(n\Delta - km)})$, by (4.6), we have

$$\begin{aligned}
2\Delta - q_1(G) &> 2(n\Delta - km)x_v^k + \frac{1}{k-1}\sum_{t=1}^{f-1}(x_{v_t}^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 \\
&\geq \frac{2(n\Delta - km)}{k-1}x_v^k + \frac{1}{k-1}\sum_{t=1}^{f-1}(x_{v_t}^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2.
\end{aligned}$$

Similar to the proof of the case 2 of Theorem 3.6, we have

$$2\Delta - q_1(G) > C.$$

Case 2.2.3. If $x_v^k < \frac{C}{2(n\Delta - km)}$ and $\sum_{t=1}^{f-1} x_{v_t}^k \leq C(k-1)(1 + \frac{f-1}{2(n\Delta - km)})$, by $\sum\limits_{i=1}^{n} x_i^k = 1$, then we have

$$x_u^k \geq \frac{1}{n-f}\left(1 - x_v^k - \sum_{t=1}^{f-1} x_{v_t}^k\right) > \frac{1}{n-f}\left(1 - \frac{2(k-1)(n\Delta - km) + (k-1)(f-1) + 1}{2(n\Delta - km)}C\right).$$

Thus, by (4.8), we have

$$2\Delta - q_1(G) > C.$$

□

Theorem 4.3. *Let G be a nonregular f-connected linear k-graph with n vertices, m edges and maximum degree Δ. Then*

$$\Delta - \lambda_1(G) > \frac{2(n\Delta - km)f^2}{2(n + (k-1)(n\Delta - km) + (k-2)(f-1))f^2 + h},$$

which $h = (n - f)(k-1)(n\Delta - km)((n + 2f - 2)^2 - f)$.

Proof. The result can be obtained by using a similar argument of Theorem 4.2. □

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References

[1] A. Bretto, Hypergraph Theory: An Introduction, Springer, 2013.
[2] A.E. Brouwer, W.H. Haemers, Spectra of Graphs, Universitext, Springer, New York, 2012, http://www.win.tue.nl/ aeb/2WF02/spectra.pdf.
[3] D.M. Chen, Z.B. Chen, X.D. Zhang, Spectral radius of uniform hypergraphs and degree sequences, Front. Math. China 12 (2017) 1279-1288.
[4] S.M. Cioabă, The spectral radius and the maximum degree of irregular graphs, Electron. J. Combin. 14 (2007) #R38.
[5] S.M. Cioabă, D.A. Gregory, V. Nikiforov, Extreme eigenvalues of nonregular graphs, J. Commbin. Theory, Ser. B 97 (2007) 483-486.
[6] J. Cooper, A. Dutle, Spectra of uniform hypergraphs, Linear Algebra Appl. 436 (2012) 3268-3292.
[7] D. Cvetković, P. Rowlinson, S. Simić, An Introduction to the Theory of Graph Spectra, Cambridge University Press, Cambridge, 2010.
[8] W. Li, J. Cooper, A. Chang, Analytic connectivity of k-uniform hypergraphs, Linear Multilinear Algebra 65 (6) (2017) 1247-1259.
[9] H.H. Li, J.Y. Shao, L.Q. Qi, The extremal spectral radii of k-uniform supertrees, J. Comb. Optim. 32 (2016) 741-764.
[10] H.F. Li, J. Zhou, C.J. Bu, Principal eigenvectors and spectral radii of uniform hypergraphs, Linear Algebra Appl. 544 (2018) 273-285.
[11] L.L. Liu, L.Y. Kang, X.Y. Yuan, On the principal eigenvectors of uniform hypergraphs, Linear Algebra Appl. 511 (2016) 430-446.
[12] L.Y. Lu, S.D. Man, Connected hypergraphs with small spectral radius, Linear Algebra Appl. 509 (2016) 206-227.
[13] V. Nikiforov, The spectral radius of subgraphs of regular graphs, Electron. J. Combin. 14 (2007) #N20.
[14] V. Nikiforov, Analytic methods for uniform hypergraphs, Linear Alegebra Appl. 457 (2014) 455-535.
[15] W.J. Ning, H. Li, M. Lu, On the signless Laplacian spectral radius of irregular graphs, Linear Algebra Appl. 438 (2013) 2280-2288.
[16] L.Q. Qi, H^+-eigenvalues of Laplacian and signless Laplacian tensors, Commun. Math. Sci. 12 (2014) 1045-1064.
[17] J.Y. Shao, H.Y. Shan, B.F. Wu, Some spectral properties and characterizations of connected odd-bipartite uniform hypergraphs, Linear Multilinear Algebra 63 (2015) 2359-2372.
[18] L.S. Shi, The spectral radius of irregular graphs, Linear Algebra Appl. 431 (2009) 189-196.
[19] W.C. Shiu, P. Huang, P.K. Sun, The signless Laplacian spectral radius of k-connected irregular graphs, Linear Multilinear Algebra 65 (2017) 830-839.
[20] D. Stevanović, The largest eigenvalue of nonregular graphs, J. Combin. Theory, Ser. B 91 (2004) 143-146.
[21] P. Xiao, L.G. Wang, The maximum spectral radius of uniform hypergraphs with given number of pendant edges, Linear Multilinear Algebra 67 (2019) 1392-1403.
[22] P. Xiao, L.G. Wang, Y.F. Du, The first two largest spectral radii of uniform supertrees with given diameter, Linear Algebra Appl. 536 (2018) 103-119.
[23] P. Xiao, L.G. Wang, Y. Lu, The maximum spectral radii of uniform supertrees with given degree sequences, Linear Algebra Appl. 523 (2017) 33-45.
[24] Y.N. Yang, Q.Z. Yang, On some properties of nonnegative weakly irreducible tensors, arXiv:1111.0713.
[25] X.Y. Yuan, M. Zhang, M. Lu, Some upper bounds on the eigenvalues of uniform hypergraphs, Linear Algebra Appl. 484 (2015) 540-549.
[26] X.D. Zhang, Eigenvectors and eigenvalues of non-regular graphs, Linear Algebra Appl. 409 (2005) 79-86.
[27] A.A. Zykov, Hypergraphs, Uspehi Mat. Nauk 29 (1974) 89-154. (in Russian)