Filomat 33:16 (2019), 5191–5198
https://doi.org/10.2298/FIL1916191A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Proper Semiconformal Symmetries of Spacetimes with Divergence-Free Semiconformal Curvature Tensor

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Abstract. In the present paper, the symmetries admitted by semiconformal curvature tensor in semiconformally symmetric spacetime have been studied and we show that a four-dimensional spacetime admitting a proper semiconformal symmetry is semiconformally flat or of the Petrov type N. It is also shown that a four-dimensional spacetime with divergence-free semiconformal curvature tensor admitting a proper semiconformal symmetry is locally of the Petrov type O or has four distinct principal null directions. In both the cases, we found that if the spacetime admits an infinitesimal semiconformal Killing vector field then the scalar curvature of the spacetime vanishes.

1. Introduction

The existence of one or more Killing vector fields or homothetic vector fields are required for solving Einstein's field equations. In fact, most of the familiar solutions depend on this hypothesis and these vector fields correspond to spacetime symmetries, commonly known as collineations. Katzin et al. ([5], [6]), pioneers of the concept of symmetries of spacetime, defined these symmetries through the vanishing of Lie derivatives of certain tensor with respect to a vector field and this vector may be time like, spacelike or null. For example, Lie derivative of metric tensor gives motion and vector field corresponding to this symmetry is called Killing vector field. The symmetries defined through semiconformal and conformal curvature tensors preserve the casual character of the spacetime manifold. The semiconformal curvature tensor is a special case of conformal curvature tensor and is invariant under the conharmonic transformation. Collinson and French ([2]) obtained that a vacuum spacetime with a proper conformal symmetry is either locally flat or of Petrov type N. Later on, Eardley et al. ([3]) investigated that asymptotically flat spacetimes with certain geometric and energy conditions with a proper conformal symmetry are of Petrov type O. Garfinkle and Tian ([4]) have studied that the four-dimentional Einstein spaces with proper conformal symmetry are of Petrov type O, which characterize de Sitter and anti-de Sitter cosmological models. For dimensions $n > 3$, Kerckhove ([12]) pointed out that a non-Ricci-flat Einstein manifold with proper conformal symmetry is locally warped product whose fiber is Einstein and base has constant sectional curvature, provided that the span of closed conformal vector fields is non–degenrate and has a constant dimension. In the two

2010 *Mathematics Subject Classification*. 53B20; 83C45; 53A45; 83C20
Keywords. curvature tensors; gravitational fields; collineations
Received: 06 August 2019; Accepted: 21 October 2019
Communicated by Ljubica Velimirović
The third author supported by the Grant Proj. No. NRF-2018-R1D1A1B-05040381 from National Research Foundation of Korea.
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different cases, viz. conformal symmetric spacetime and a four dimensional spacetime with divergence-free Weyl conformal curvature tensor, Sharma ([13], [14]) proved that such spacetimes with proper conformal symmetry are either locally flat or of the Petrov type N. Later on, Abdussattar and Babita Dwivedi [1], discussed the above cases for conharmonic curvature tensor. Recently, De and Suh [15] studied the Weakly semiconformally symmetric manifolds. Motivated by the above works, we have studied the symmetric and divergence-free semiconformal curvature tensor with proper semiconformal symmetry of the spacetime. We have proved the following results

Theorem 1.1. *If a semiconformally symmetric spacetime admits a proper semiconformal symmetry, then the scalar curvature of the spacetime vanishes and the spacetime is either semiconformally flat or of Petrov type N.*

Theorem 1.2. *If ξ is a proper semiconformal symmetry with the divergence-free semiconformal curvature tensor, then the scalar curvature of the spacetime vanishes and the spacetime is either semiconformally flat or of Petrov type N.*

2. Preliminaries

Let (M^n, g) denotes an $n-$dimensional Riemannian manifold with nondegenerate metric g. The conharmonic curvature tensor is given by Z. Ahsan ([17])

$$L^h_{bcd} = R^h_{bcd} + \frac{1}{n-2}(\delta^h_c R_{bd} - \delta^h_d R_{bc} + g_{bd} R^h_c - g_{bc} R^h_d), \tag{1}$$

which is invariant under conharmonic transformation introduced by Ishii([16]), where R^h_{bcd}, R_{ab}, respectively Riemann and Ricci curvature tensor. J. Kim ([7]) has introduced the notion of semiconformal curvature tensor as

$$P^h_{bcd} = -(n-2)BC^h_{bcd} + [A + (n-2)B]L^h_{bcd}, \tag{2}$$

This tensor is also invariant under the same transformation discussed above, defined by ([8]).
Here C^h_{bcd} is a Weyl conformal curvature tensor defined by

$$C^h_{bcd} = R^h_{bcd} + \frac{1}{n-2}(\delta^h_c R_{bd} - \delta^h_d R_{bc} + g_{bd} R^h_c - g_{bc} R^h_d) + \frac{R}{(n-1)(n-2)}(\delta^h_d g_{bc} - \delta^h_c g_{bd}), \tag{3}$$

the constants A and B are not simultaneously zero. In particular, for $A = 1$ and $B = -\frac{1}{n-2}$, the semi-conformal curvature tensor reduces to Weyl conformal curvature tensor, and also for $A = 1$ and $B = 0$, it reduces to conharmonic curvature tensor. Further in our paper we assume that $A \neq 0$, $B \neq 0$ and $A + 2B \neq 0$ because for the case $A + 2B = 0$ the two tensors P^h_{bcd} and C^h_{bcd} become equivalent provided $B \neq 0$.

The semiconformal curvature tensor satisfies the following properties.

$$P_{hbcd} = -P_{bhcd} = -P_{hbdc} = P_{cdhb}, \tag{4}$$

and

$$P_{hbcd} + P_{chbd} + P_{bchd} = 0. \tag{5}$$

In view of equations (1) and (3), equation (2) takes the following form

$$P^h_{bcd} = A[R^h_{bcd} + \frac{1}{2}(\delta^h_c R_{bd} - \delta^h_d R_{bc} + g_{bd} R^h_c - g_{bc} R^h_d)] - \frac{BR}{3}(\delta^h_d g_{bc} - \delta^h_c g_{bd}), \tag{6}$$

after contraction over h and d we get

$$P_{bc} = -\Big(\frac{A+2B}{2}\Big)Rg_{bc}. \tag{7}$$

This is also invariant under conharmonic transformation.

Using the equations (3) and (6), we get

$$P^h_{bcd} = AC^h_{bcd} - \frac{1}{3}\Big[\delta^h_d P_{bc} - \delta^h_c P_{bd}\Big] \tag{8}$$

We shall be using the following definitions in our investigations

Definition 2.1. *A four-dimensional spacetime is said to admit a conformal motion along a vector field ξ if*

$$\pounds_\xi g_{ab} = 2\psi g_{ab}, \tag{9}$$

where $\psi = \frac{1}{4}\xi^d_{;d}$.

Definition 2.2. *A four-dimensional spacetime is called conformal collineation (Conf C), if there exist a vector field ξ such that*

$$\pounds_\xi \Gamma^b_{cd} = \delta^b_c \psi_{;d} + \delta^b_d \psi_{;c} - g_{cd} g^{bm}\psi_{;m} \tag{10}$$

where $\psi = \frac{1}{4}\xi^d_{;d}$.

Recently, M. Ali et al. ([10]) introduced a new symmetry as

$$\pounds_\xi P^h_{bcd} = 0, \tag{11}$$

and called it a semiconformal curvature collineation. For more literature on symmetries see Katzin et al. ([5]).

3. Main Results

Here we shall prove one proposition and the theorems stated in section 1. We then also obtain the results, which are arranged as lemma and corollary in this section.

Proposition 3.1. *If the divergence of semiconformal curvature vanishes and the scalar curvature is covariantly constant, then the semiconformal curvature satisfies the Bianchi's second identity.*

Proof. Taking covariant derivative of equation (6) with respect to l, we get

$$P^h_{bcd;l} = A[R^h_{bcd;l} + \frac{1}{2}(\delta^h_c R_{bd;l} - \delta^h_d R_{bc;l} + g_{bd}R^h_{c;l} - g_{bc}R^h_{d;l})] - \frac{BR_{;l}}{3}(\delta^h_d g_{bc} - \delta^h_c g_{bd}). \tag{12}$$

Permutting cyclically twice for the indices (c, d, l) equation (12) leads to

$$P^h_{bdl;c} = A[R^h_{bdl;c} + \frac{1}{2}(\delta^h_d R_{bl;c} - \delta^h_l R_{bd;c} + g_{bl}R^h_{d;c} - g_{bd}R^h_{l;c})] - \frac{BR_{;c}}{3}(\delta^h_l g_{bd} - \delta^h_d g_{bl}). \tag{13}$$

and

$$P^h_{blc;d} = A[R^h_{blc;d} + \frac{1}{2}(\delta^h_l R_{bc;d} - \delta^h_c R_{bl;d} + g_{bc}R^h_{l;d} - g_{bl}R^h_{c;d})] - \frac{BR_{;d}}{3}(\delta^h_c g_{bl} - \delta^h_l g_{bc}). \tag{14}$$

Adding the equations (12), (13), (14) and using the Bianchi's second identity satisfied by Riemann curvature tensor, we get

$$\begin{aligned} P^h_{bcd;l} + P^h_{bdl;c} + P^h_{blc;d} &= \frac{A}{2}\Big[\delta^h_c(R_{bd;l} - R_{bl;d}) + \delta^h_d(R_{bl;c} - R_{bc;l}) + \delta^h_l(R_{bc;d} - R_{bd;c})\Big] \\ &+ \frac{A}{2}\Big[g_{bd}(R^h_{c;l} - R^h_{l;c}) + g_{bc}(R^h_{l;d} - R^h_{d;l}) + g_{bl}(R^h_{d;c} - R^h_{c;d})\Big] \\ &+ \frac{B}{3}\Big[\delta^h_d(g_{bl}R_{;c} - g_{bc}R_{;l}) + \delta^h_c(g_{bd}R_{;l} - g_{bl}R_{;d}) \\ &+ \delta^h_l(g_{bc}R_{;d} - g_{bd}R_{;c})\Big]. \end{aligned} \tag{15}$$

Now contracting over h and l, equation (12) reduces to

$$P^h_{bcd;h} = A\Big[R^h_{bcd;h} + \frac{1}{2}\Big(R_{bd;c} - R_{bc;d} + g_{bd}R_{;c} - g_{bc}R_{;d}\Big)\Big] - \frac{B}{3}\Big(g_{bc}R_{;d} - g_{bd}R_{;c}\Big). \tag{16}$$

It is known that the Bianchi second identity after contraction is

$$R^h_{bcd;h} = (R_{bc;d} - R_{bd;c}). \tag{17}$$

Now from equations (16) and (17), we have

$$P^h_{bcd;h} = \frac{A}{2}\Big(R_{bc;d} - R_{bd;c}\Big) - \Big(\frac{3A + 2B}{6}\Big)\Big(g_{bc}R_{;d} - g_{bd}R_{;c}\Big) \tag{18}$$

But given that the divergence of semiconformal curvature tensor vanishes, equation (15) reduces to the following form by using equation (18)

$$\begin{aligned} P^h_{bcd;l} + P^h_{bdl;c} + P^h_{blc;d} &= (A + B)[\delta^h_c(g_{bd}R_{;l} - g_{bl}R_{;d}) + \delta^h_d(g_{bl}R_{;c} - g_{bc}R_{;l}) \\ &+ \delta^h_l(g_{bc}R_{;d} - g_{bd}R_{;c})]. \end{aligned} \tag{19}$$

Moreover, if the scalar curvature tensor is covariantly constant, then equation (19) leads to

$$P^h_{bcd;l} + P^h_{bdl;c} + P^h_{blc;d} = 0, \tag{20}$$

which implies that the semiconformal curvature tensor satisfies the Bianchi's second identity. □

Sharma investigated the proper conformal symmetries of conformal symmetric spacetimes using the following equation (c.f., [11])

$$C_{hbcd;l} = 0, \tag{21}$$

It has been proved that the conformal symmetric spacetimes, admitting an infinitesimal symmetry are conformally flat or of Petrov type N. Also, it is known that Petrov type N gravitational fields represent the plane gravitational waves with parallel rays, provided the Einstein tensor is invariant under the infinitesimal conformal symmetry. Thus, the symmetry of semiconformal symmetric spacetime may be defined as

$$P_{hbcd;l} = 0. \tag{22}$$

where P_{hbcd} is semiconformal curvature tensor. We shall now prove our first main theorem as follows:

Proof of Theorem 1.1 The commutation formula for the semiconformal curvature tensor (in an analogous manner as given by Yano [9]).

$$\begin{aligned} \pounds_\xi\left(P^h_{bcd;l}\right) - \left(\pounds_\xi P^h_{bcd}\right)_{;l} &= \left(\pounds_\xi \Gamma^h_{lm}\right)P^m_{bcd} - \left(\pounds_\xi \Gamma^m_{lb}\right)P^h_{mcd} - \left(\pounds_\xi \Gamma^m_{lc}\right)P^h_{bmd} \\ &- \left(\pounds_\xi \Gamma^m_{ld}\right)P^h_{bcm}. \end{aligned} \tag{23}$$

Making use of equations (10), (11) and (22), equation (23) may take the form

$$\begin{aligned} &\left(\delta^h_l \psi_{;m} + \delta^h_m \psi_{;l} - g_{lm}\psi^h_{;}\right)P^m_{bcd} - \left(\delta^m_l \psi_{;b} + \delta^m_b \psi_{;l} - g_{lb}\psi^m_{;}\right)P^h_{mcd} \\ - &\left(\delta^m_l \psi_{;c} + \delta^m_c \psi_{;l} - g_{lc}\psi^m_{;}\right)P^h_{bmd} - \left(\delta^m_l \psi_{;d} + \delta^m_d \psi_{;l} - g_{ld}\psi^m_{;}\right)P^h_{bcm} = 0. \end{aligned} \tag{24}$$

Contracting equation (24) over h and l, we get

$$4\psi_{;m}P^m_{bcd} - \psi_{;m}\left(P^m_{bcd} + P^m_{dbc} + P^m_{cdb}\right) + \psi_{;c}P_{bd} - \psi_{;d}P_{bc} = 0. \tag{25}$$

But from equation (6) it is seen that

$$P^m_{\{bcd\}} = 0. \tag{26}$$

In view of equation (26), equation (25) leads to

$$4\psi_{;m}P^m_{bcd} + \psi_{;c}P_{bd} - \psi_{;d}P_{bc} = 0. \tag{27}$$

Contracting this equation over b and c and using equation (7), we get

$$\frac{7(A+2B)}{2}R\psi_{;d} = 0. \tag{28}$$

Since ξ is a proper semiconformal vector field, $\psi_{;d} \neq 0$ and due to our suppositions (i.e., $A \neq 0$, $B \neq 0$ and $A + 2B \neq 0$.) equation (28) leads to $R = 0$. Consequenly equation (7) will give

$$P_{bc} = 0 \tag{29}$$

and from equation (8), the semiconformal and conformal curvature tensor become identical for $A = 1$. Equation (27), now takes the form

$$4\psi_{;m}P^h_{bcd} = 0. \tag{30}$$

Using equation (30) in equation (24) and then multiplying by $\psi^l_{;}$, we get

$$\psi_{;l}\psi^l_{;}P^h_{bcd} = 0. \tag{31}$$

From the equation (31), either $\psi_{;l}\psi^l_{;} = 0$ or $P^h_{bcd} = 0$, i.e., the four-dimensional spacetime is semiconformally flat. Further as we have a four-dimensional spacetime admits proper semiconformal vector field ξ, $\psi^l_{;} \neq 0$, therefore, if $\psi^l_{;}\psi_{;l} = 0$ then $\psi^l_{;}$ must be null. Thus from equation (30) the spacetime is of the Petrov type N and the four repeated principal null directions of the semiconformal curvature tensor are given by $\psi^l_{;}$.

This completes the proof. □

For the proof of Theorem 1.2, the following lemma is required

Lemma 3.2. *The necessay and sufficient condition for the semiconformal curvature collineation is that* $\pounds_\xi(R) = -2\psi R$, *where* $\psi = \frac{1}{4}\xi^d_{;d}$.

Proof. Equation (6) may be expressed as

$$P^h_{bcd} = \Big(\frac{30A+64B}{204}\Big)\delta^h_c g_{bd} R - \Big(\frac{33A+64B}{204}\Big)\delta^h_d g_{bc} R. \tag{32}$$

Now taking the Lie derivative on both sides of this equation with respect to ξ, equation (32) leads to

$$\begin{aligned} \pounds_\xi P^h_{bcd} &= \Big(\frac{30A+64B}{204}\Big)\delta^h_c[R\pounds_\xi(g_{bd}) + g_{bd}\pounds_\xi(R)] \\ &- \Big(\frac{33A+64B}{204}\Big)\delta^h_d[R\pounds_\xi(g_{bc}) + g_{bc}\pounds_\xi(R)]. \end{aligned} \tag{33}$$

which on using equations (9) and (11) leads to

$$\Big(\frac{30A+64B}{204}\Big)\delta^h_c[2\psi g_{bd}R + g_{bd}\pounds_\xi(R)] - \Big(\frac{33A+64B}{204}\Big)\delta^h_d[2\psi g_{bc} + g_{bc}\pounds_\xi(R)] = 0, \tag{34}$$

which implies that

$$\pounds_\xi(R) = -2\psi R. \tag{35}$$

This completes the proof. □

Further Sharma in [13] has shown that the four-dimensional spacetime will be either conformally flat or of type N, if a vector field ξ is associated to the conformal symmetry of a spacetime with divergence-free Weyl conformal curvature tensor and proper conformal symmetry. We extend the similar idea for the semiconformal case and the result is obtained (Theorem 1.2). We have

Proof of Theorem 1.2. Taking Lie derivative on both sides of equation (19)

$$\begin{aligned} \pounds_\xi(P^h_{bcd;l} + P^h_{bdl;c} + P^h_{blc;d}) &= (A+B)[\delta^h_c\{(\pounds_\xi g_{bd})R_{;l} - (\pounds_\xi g_{bl})R_{;d}\} + \delta^h_c\{g_{bd}(\pounds_\xi R)_{;l} \\ &- g_{bl}(\pounds_\xi R)_{;d}\} + \delta^h_d\{(\pounds_\xi g_{bl})R_{;c} - (\pounds_\xi g_{bc})R_{;l}\} \\ &+ \delta^h_d\{g_{bl}(\pounds_\xi R)_{;c} - g_{bc}(\pounds_\xi R)_{;l}\} + \delta^h_l\{(\pounds_\xi g_{bc})R_{;d} \\ &- (\pounds_\xi g_{bd})R_{;c}\} + \delta^h_l\{g_{bc}(\pounds_\xi R)_{;d} - g_{bd}(\pounds_\xi R)_{;c}\}] \end{aligned} \tag{36}$$

Using equation (9) and Lemma 3.2, equation (36) leads to

$$\begin{aligned} \pounds_\xi(P^h_{bcd;l} + P^h_{bdl;c} + P^h_{blc;d}) &= -2(A+B)R[\delta^h_c(g_{bd}\psi_{;l} - g_{bl}\psi_{;d}) + \delta^h_d(g_{bl}\psi_{;c} - g_{bc}\psi_{;l}) \\ &+ \delta^h_l(g_{bc}\psi_{;d} - g_{bd}\psi_{;c})] \end{aligned} \tag{37}$$

similar to equation (23), we have two more equations

$$\begin{aligned} \pounds_\xi\Big(P^h_{bdl;c}\Big) - \Big(\pounds_\xi P^h_{bdl}\Big)_{;c} &= \Big(\pounds_\xi\Gamma^h_{cm}\Big)P^m_{bdl} - \Big(\pounds_\xi\Gamma^m_{cb}\Big)P^h_{mdl} - \Big(\pounds_\xi\Gamma^m_{cd}\Big)P^h_{bml} \\ &- \Big(\pounds_\xi\Gamma^m_{cl}\Big)P^h_{bdm}. \end{aligned} \tag{38}$$

and

$$\pounds_\xi\left(P^h_{blc;d}\right) - \left(\pounds_\xi P^h_{blc}\right)_{;d} = \left(\pounds_\xi \Gamma^h_{dm}\right)P^m_{blc} - \left(\pounds_\xi \Gamma^m_{db}\right)P^h_{mlc} - \left(\pounds_\xi \Gamma^m_{dl}\right)P^h_{bmc} - \left(\pounds_\xi \Gamma^m_{dc}\right)P^h_{blm}. \tag{39}$$

Adding equations (23), (38), (39) and making use of equations (10), (11) and (37), we get

$$\begin{aligned}
- \quad & 2(A+B)R[\delta^h_c(g_{bd}\psi_{;l} - g_{bl}\psi_{;d}) + \delta^h_d(g_{bl}\psi_{;c} - g_{bc}\psi_{;l}) + \delta^h_l(g_{bc}\psi_{;d} - g_{bd}\psi_{;c})] \\
= \quad & (\delta^h_l\psi_{;m} + \delta^h_m\psi_{;l} - g_{lm}\psi^{h}_{;})P^m_{bcd} - (\delta^m_l\psi_{;b} + \delta^m_b\psi_{;l} - g_{lb}\psi^{m}_{;})P^h_{mcd} \\
+ \quad & (\delta^h_c\psi_{;m} + \delta^h_m\psi_{;c} - g_{cm}\psi^{h}_{;})P^m_{bdl} - (\delta^m_c\psi_{;b} + \delta^m_b\psi_{;c} - g_{cb}\psi^{m}_{;})P^h_{mdl} \\
+ \quad & (\delta^h_d\psi_{;m} + \delta^h_m\psi_{;d} - g_{dm}\psi^{h}_{;})P^m_{blc} - (\delta^m_d\psi_{;b} + \delta^m_b\psi_{;d} - g_{lb}\psi^{m}_{;})P^h_{mlc}.
\end{aligned} \tag{40}$$

Now, contracting this equation over h and l and using equation (26), we get

$$4(A+B)R(g_{bd}\psi_{;c} - g_{bc}\psi_{;d}) = 3\psi_{;m}P^m_{bcd} + g_{cb}\psi^{m}_{;}P_{md} - g_{db}\psi^{m}_{;}P_{mc}. \tag{41}$$

Multiplying this equation by g^{bc} and using equation (7), we obtain $R = 0$, since $\psi_{;l} \neq 0$ and $(A+B) \neq 0$ (For $A + B = 0$ we are not getting any fruitful result). Hence equation (7) gives $P_{bc} = 0$. Using these facts in equation (41), we get equation (30) and consequently equation (40) leads to

$$\psi^{;h}P_{lbcd} + \psi_{;b}P^h_{lcd} = 0. \tag{42}$$

Multiplying this equation by $\psi^{l}_{;}$ to the above equation, we get equation (31). Thus theorem is established. □

Corollary 3.3. *If a four-dimensional spacetime has a divergence free semiconformal curvature tensor and vector field ξ is associated to a proper semiconformal symmetry of the spacetime, then the spacetime must have the divergence-free Riemann curvature tensor.*

Proof. Making use the fact that $R = 0$ in equation (16) and using the divergence-free condition for semiconformal curvature tensor, we get

$$R_{bc;d} - R_{bd;c} = 0. \tag{43}$$

In view of equation (43), equation (17) may takes the form

$$R^h_{bcd;h} = 0. \tag{44}$$

This completes the proof. □

4. Discussion

Finding the exact solutions of Einstein field equations is the prime motive in gravitational physics. These equations are highly non-linear PDEs, so some simplifying assumptions has to be made (compatible with dynamics of chosen distribution of matter) on the geometry of spacetime. These are geometrical/physical symmetries of spacetime manifold and also known as collineations. In this paper, we have obtained results on semi-conformal curvature collineation and the types of gravitational field. With the condition of divergence-free spacetime, we find an ease to explore an already established literature of spacetime symmetry. Our results can be used to find the solutions of field equations, to discuss the curvature flow and soliton.

Acknowledgement

Authors would like to thank the learned referee(s) and editor for the valuable suggestions and detailed comments that improved our paper. We are also thankful to Professor Zafar Ahsan and Dr. Tahir Hussain for the discussion on the contents as well as on the language of the paper.

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