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Perturbation Theory for Core and Core-EP Inverses of Tensor via Einstein Product

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Abstract. In this paper, for given tensors $\mathcal{A}$, $\mathcal{E}$ and $\mathcal{B} = \mathcal{A} + \mathcal{E}$, we investigate the perturbation bounds for the core inverse $\mathcal{A}^{\#}$ and core-EP inverse $\mathcal{A}^{\oplus}$ under some conditions via Einstein product.

1. Introduction

There are several papers on the core inverse and core-EP inverse [1–5, 7–9]. Recently, there are recent monographs [10–12] on the generalized inverse.

For convenience, we first adopt some of the terminologies which will be used in this paper. For a positive integer N, let $[N] = \{1, \ldots, N\}$. An order N tensor $\mathcal{A} = (\mathcal{A}_{i_1,i_2,\ldots,i_N})_{1\le i_j \le \mathbf{I}_j}$, $(j = 1, \ldots, N)$ is a multidimensional array with $\mathbf{I}_1\mathbf{I}_2\cdots\mathbf{I}_N$ entries. Let $\mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ and $\mathbb{R}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ be the sets of the order N dimension $\mathbf{I}_1\times\cdots\times\mathbf{I}_N$ tensors over the complex field $\mathbb{C}$ and the real field $\mathbb{R}$, respectively. Each entry of $\mathcal{A}$ is denoted by $a_{i_1\cdots i_N}$.

For a tensor $\mathcal{A} = (a_{i_1\cdots i_N j_1\cdots j_M}) \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$, let $\mathcal{B} = (b_{i_1\cdots i_M} j_1\cdots j_N) \in \mathbb{C}^{J_1\times\cdots\times J_M\times I_1\times\cdots\times I_N}$ be the conjugate transpose of $\mathcal{A}$, where $b_{i_1\cdots i_M j_1\cdots j_N} = \bar{a}_{j_1\cdots j_M i_1\cdots i_N}$. The tensor $\mathcal{B}$ is denoted by $\mathcal{A}^*$. When $b_{i_1\cdots i_M j_1\cdots j_N} = a_{j_1\cdots j_M i_1\cdots i_N}$, $\mathcal{B}$ is the transpose of $\mathcal{A}$, and is denoted by $\mathcal{A}^T$. A tensor $\mathcal{D} = (d_{i_1\cdots i_N i_1\cdots i_N}) \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ is called a diagonal tensor if all its entries are zero except for $d_{i_1\cdots i_N i_1\cdots i_N}$ In case of all the diagonal entries $d_{i_1\cdots i_N i_1\cdots i_N} = 1$, we call $\mathcal{D}$ as a unit tensor and is denoted by $\mathcal{I}$. Similarly, $\mathcal{O}$ denotes the zero tensor in case of all the entries are zero.

The Einstein product of tensors is defined in [13] by the operation $*_N$ via

$$(\mathcal{A} *_N \mathcal{B})_{i_1\ldots i_N j_1\ldots j_M} = \sum_{k_1\ldots k_N} \mathcal{A}_{i_1\ldots i_N k_1\ldots k_N} \mathcal{B}_{k_1\ldots k_N j_1\ldots j_M}, \tag{1}$$

where $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{K}_1\times\cdots\times\mathbf{K}_N}$, $\mathcal{B} \in \mathbb{C}^{\mathbf{K}_1\times\cdots\times\mathbf{K}_N\times\mathbf{J}_1\times\cdots\times\mathbf{J}_M}$ and $\mathcal{A} *_N \mathcal{B} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{J}_1\times\cdots\times\mathbf{J}_M}$. The associative law of this tensor product holds. In the above formula, when $\mathcal{B} \in \mathbb{C}^{\mathbf{K}_1\times\cdots\times\mathbf{K}_N}$, then

$$(\mathcal{A} *_N \mathcal{B})_{i_1 i_2\ldots i_N} = \sum_{k_1\ldots k_N} \mathcal{A}_{i_1\ldots i_N k_1\ldots k_N} \mathcal{B}_{k_1\ldots k_N},$$

where $\mathcal{A} *_N \mathcal{B} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$. For convenience, we denote $\mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ simply by $\mathbb{C}^{I(N)\times I(N)}$.

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Definition 1.1. [16] *For $\mathcal{A} \in \mathbb{C}^{I(N)\times K(N)}$, the range $\mathcal{R}(\mathcal{A})$ and the null space $\mathcal{N}(\mathcal{A})$ of $\mathcal{A}$ are defined by*

$$\mathcal{R}(\mathcal{A}) = \{\mathcal{Y} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N} : \mathcal{Y} = \mathcal{A} *_N \mathcal{X},\ \mathcal{X} \in \mathbb{C}^{\mathbf{K}_1\times\cdots\times\mathbf{K}_N}\}$$

$$\mathcal{N}(\mathcal{A}) = \{\mathcal{X} \in \mathbb{C}^{\mathbf{K}_1\times\cdots\times\mathbf{K}_N} : \mathcal{A} *_N \mathcal{X} = \mathcal{O}\},$$

where $\mathcal{O}$ is an appropriate order zero tensor.

Definition 1.2. [16] *The inner product on $\mathbb{C}^{\mathbf{N}_1\times\cdots\times\mathbf{N}_K}$ is defined by*

$$\langle \mathcal{X}, \mathcal{Y}\rangle = \sum_{n_r\in[N_r], r\in[k]} \overline{\mathcal{X}}_{n_1n_2\cdots n_k}\mathcal{Y}_{n_1n_2\cdots n_k},\ \ \forall \mathcal{X}, \mathcal{Y} \in \mathbb{C}^{\mathbf{N}_1\times\cdots\times\mathbf{N}_K},$$

and the spectral norm $\|\cdot\|_2$ is defined as [17, Lemma 2.1]

$$\|\mathcal{X}\|_2 = \sqrt{\lambda_{\max}(\mathcal{X}^* *_N \mathcal{X})},$$

where $\lambda_{\max}(\mathcal{X}^* *_N \mathcal{X})$ denotes the largest eigenvalue of $\mathcal{X}^* *_N \mathcal{X}$.

Definition 1.3. [14] *Let $\mathcal{A} \in \mathbb{C}^{I(N)\times K(N)}$. The tensor $\mathcal{X} \in \mathbb{C}^{K(N)\times I(N)}$ which satisfies*

$$(1)\quad \mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{A}; \qquad (2)\quad \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X};$$

$$(3)\quad (\mathcal{A} *_N \mathcal{X})^* = \mathcal{A} *_N \mathcal{X}; \qquad (4)\quad (\mathcal{X} *_N \mathcal{A})^* = \mathcal{X} *_N \mathcal{A}$$

is called the Moore-Penrose inverse of $\mathcal{A}$, abbreviated by M-P inverse, denoted by $\mathcal{A}^{\dagger}$. If the equation (i) *of the above equations* (1) − (4) *holds, then $\mathcal{X}$ is called an* (i)*-inverse of $\mathcal{A}$, denoted by $\mathcal{A}^{(\mathrm{i})}$.*

Definition 1.4. [17] *Assume that $\mathcal{A} \in \mathbb{C}^{I(N)\times I(N)}$. Define*

$$\mathcal{A}^0 = \mathcal{I} \text{ and } \mathcal{A}^p = \mathcal{A}^{p-1} *_N \mathcal{A}, \quad for\ p \geq 2.$$

It is easily seen that

$$\{0\} \subseteq \cdots \subseteq \mathcal{R}(\mathcal{A}^{p+1}) \subseteq \mathcal{R}(\mathcal{A}^p) \subseteq \cdots \subseteq \mathcal{R}(\mathcal{A}^2) \subseteq \mathcal{R}(\mathcal{A}) \subseteq \mathcal{R}(\mathcal{I}) = \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$$

and

$$\{0\} = \mathcal{N}(\mathcal{I}) \subseteq \mathcal{N}(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}^2) \subseteq \cdots \subseteq \mathcal{N}(\mathcal{A}^p) \subseteq \mathcal{N}(\mathcal{A}^{p+1}) \subseteq \cdots \subseteq \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N}.$$

The smallest non-negative integer p such that $\mathcal{R}(\mathcal{A}^{p+1}) = \mathcal{R}(\mathcal{A}^p)$ (or $\mathcal{N}(\mathcal{A}^{p+1}) = \mathcal{N}(\mathcal{A}^p)$), denoted by $\mathrm{Ind}(\mathcal{A})$, *is called the index of $\mathcal{A}$.*

Definition 1.5. [17] *Let $\mathcal{A} \in \mathbb{C}^{I(N)\times I(N)}$. The tensor $\mathcal{X} \in \mathbb{C}^{I(N)\times I(N)}$ which satisfies*

$$(2)\ \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}; \qquad (5)\ \mathcal{A} *_N \mathcal{X} = \mathcal{X} *_N \mathcal{A}; \qquad (1^k)\ \mathcal{A}^{k+1} *_N \mathcal{X} = \mathcal{A}^k$$

is called the Drazin inverse of $\mathcal{A}$, denoted by $\mathcal{A}_d$. Especially, if $\mathrm{Ind}(\mathcal{A}) = 1$, *$\mathcal{X}$ is called the group inverse of $\mathcal{A}$, denoted by $\mathcal{A}_g$.*

According to Hartwig and Spindelböck decomposition [18] of tensors, every tensor $\mathcal{A} \in \mathbb{C}^{I(N)\times I(N)}$ of rank r can be represented by

$$\mathcal{A} = \mathcal{U} *_N \begin{pmatrix} \Sigma *_N \mathcal{K} & \Sigma *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*, \tag{2}$$

where $\Sigma \in \mathbb{C}^{R(N)\times R(N)}$ is a diagonal tensor of singular values of $\mathcal{A}$, and the tensors $\mathcal{K} \in \mathbb{C}^{R(N)\times R(N)}$, $\mathcal{L} \in \mathbb{C}^{R(N)\times(I_N-R_N)}$ satisfy

$$\mathcal{K} *_N \mathcal{K}^* + \mathcal{L} *_N \mathcal{L}^* = \mathcal{I} \tag{3}$$

It follows from (2) that the Moore-Penrose inverse of $\mathcal{A}$ is given as follows:

$$\mathcal{A}^\dagger = \mathcal{U} *_N \begin{pmatrix} \mathcal{K}^* *_N \Sigma^{-1} & \mathcal{O} \\ \mathcal{L}^* *_N \Sigma^{-1} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.$$

If $\mathrm{Ind}(\mathcal{A}) \leq 1$, then the group inverse of $\mathcal{A}$ is

$$\mathcal{A}_g = \mathcal{U} *_N \begin{pmatrix} \mathcal{K}^{-1} *_N \Sigma^{-1} & \mathcal{K}^{-1} *_N \Sigma^{-1} *_N \mathcal{K}^{-1} *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.$$

Lemma 1.6. [15] *Let $\mathcal{E} \in \mathbb{C}^{I(1)\times I(N)}$ be a tensor of index k. If $\|\mathcal{E}\|_2 < 1$, then $\mathcal{I} + \mathcal{E}$ is nonsingular and*

$$\|(\mathcal{I} + \mathcal{E})^{-1}\|_2 \leq \frac{1}{1 - \|\mathcal{E}\|_2}.$$

Lemma 1.7. [15] *Let $\mathcal{E} \in \mathbb{C}^{I(K)\times I(K)}$. If $\|\mathcal{E}\|_2 < 1$, then*

$$(\mathcal{I} - \mathcal{E})^{-1} = \sum_{n=0}^{\infty} \mathcal{E}^n, \tag{4}$$

$$\|(\mathcal{I} - \mathcal{E})^{-1} - \mathcal{I}\|_2 \leq \frac{\|\mathcal{E}\|_2}{1 - \|\mathcal{E}\|_2}. \tag{5}$$

The recent results on the core inverse of tensor can be found in [19, 20].

Definition 1.8. [19, 20] *Let $\mathcal{A} \in \mathbb{C}^{I(N)\times I(N)}$ be a given core tensor. A tensor $\mathcal{X} \in \mathbb{C}^{I(N)\times I(N)}$ satisfying*

$$\mathcal{X} *_N \mathcal{A}^2 = \mathcal{A}; \quad \mathcal{A} *_N \mathcal{X}^2 = \mathcal{X}; \quad (\mathcal{A} *_N \mathcal{X})^* = \mathcal{A} *_N \mathcal{X}$$

is called the core inverse of $\mathcal{A}$ and denoted by $\mathcal{A}^{\circledast}$

Lemma 1.9. [19, 20] *Let $\mathcal{A} \in \mathbb{C}^{I(N)\times I(N)}$ be given. Then $\mathcal{A}^{\circledast}$ satisfies equations* (1) *and* (2) *in Definition 1.3.*

By the definition of core inverse, we have the following lemma.

Lemma 1.10. [19, 20] *Let $\mathcal{A} \in \mathbb{C}^{I(N)\times I(N)}$ be of the form (2) and* $\mathrm{Ind}(\mathcal{A}) \leq 1$. *Then*

$$\mathcal{A}^{\circledast} = \mathcal{U} *_N \begin{pmatrix} (\Sigma *_N \mathcal{K})^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.$$

Another important generalized inverse is the core-EP inverse.

Definition 1.11. [8, 19] *Let $\mathcal{A} \in \mathbb{C}^{I(N)\times I(N)}$ and* $\mathrm{Ind}(\mathcal{A}) = k$. *A tensor $\mathcal{X} \in \mathbb{C}^{I(N)\times I(N)}$ satisfying*

$$\mathcal{X} *_N \mathcal{A}^{k+1} = \mathcal{A}^k; \quad \mathcal{A} *_N \mathcal{X}^2 = \mathcal{X}; \quad (\mathcal{A} *_N \mathcal{X})^* = \mathcal{A} *_N \mathcal{X}$$

is called core-EP inverse of $\mathcal{A}$ and it is denoted as $\mathcal{A}^{\oplus}$.

Lemma 1.12. [8, 19] *Let $\mathcal{A} \in \mathbb{C}^{I(N)\times I(N)}$ and* $\mathrm{Ind}(\mathcal{A}) = k$. *There is a Schur form of $\mathcal{A}$,*

$$\mathcal{A} = \mathcal{U} *_N \begin{pmatrix} \mathcal{T}_1 & \mathcal{T}_2 \\ \mathcal{O} & \mathcal{T}_3 \end{pmatrix} *_N \mathcal{U}^*, \tag{6}$$

where $\mathcal{U} \in \mathbb{C}^{I(N)\times I(N)}$ is a unitary tensor, $\mathcal{T}_1$ is a upper triangular tensor and $\mathcal{T}_3$ is a nilpotent tensor with $\mathrm{Ind}(\mathcal{T}_3) = k$.

From Definition 1.11 and (6), we can obtain that

$$\mathcal{A}^{\oplus} = \mathcal{U} *_N \begin{pmatrix} \mathcal{T}_1^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*. \tag{7}$$

Definition 1.13. [15, 22] *Let $I_1, ..., I_M, K_1, ..., K_N$ be given integers and $\mathfrak{I}, \mathfrak{R}$ are the integers defined as*

$$\mathfrak{I} = I_1 I_2 \cdots I_M, \quad \mathfrak{R} = K_1 K_2 \cdots K_N.$$

The reshaping operation

$$\text{rsh} : \mathbb{C}^{I(M)\times K(N)} \mapsto \mathbb{C}^{\mathfrak{I}\times\mathfrak{R}}$$

transforms a tensor $\mathcal{A} \in \mathbb{C}^{I(M)\times K(N)}$ into the matrix $A \in \mathbb{C}^{\mathfrak{I}\times\mathfrak{R}}$ using the Matlab *function* **reshape** *as follows:*

$$\text{rsh}(\mathcal{A}) = A = \text{reshape}(\mathcal{A}, \mathfrak{I}, \mathfrak{R}), \quad \mathcal{A} \in \mathbb{C}^{I(M)\times K(N)}, \quad A \in \mathbb{C}^{\mathfrak{I}\times\mathfrak{R}}.$$

The inverse reshaping of $A \in \mathbb{C}^{\mathfrak{I}\times\mathfrak{R}}$ is the tensor $\mathcal{A} \in \mathbb{C}^{I(M)\times K(N)}$ defined by

$$\text{rsh}^{-1}(A) = \mathcal{A} = \text{reshape}(A, I_1, ..., I_M, K_1, ..., K_N).$$

Also, an appropriate definition of the tensor rank, arising from the reshaping operation, was proposed in [22].

Definition 1.14. [15, 22] *Let $\mathcal{A} \in \mathbb{C}^{I(N)\times K(N)}$ and $A = \text{reshape}(\mathcal{A}, \mathfrak{I}, \mathfrak{R}) = \text{rsh}(\mathcal{A}) \in \mathbb{C}^{\mathfrak{I}\times\mathfrak{R}}$. Then the tensor rank of $\mathcal{A}$, denoted by* rshrank($\mathcal{A}$), *is defined by* rshrank($\mathcal{A}$) = rank(A).

2. Perturbation for core inverse

In this section, we present the optimal perturbations for the core inverse of tensors via Einstein product under two-sided and one-sided conditions.

Theorem 2.1. *Let $\mathcal{A}, \mathcal{E} \in \mathbb{C}^{I(N)\times I(N)}$ be of the form (2) and $\text{Ind}(\mathcal{A}) \leq 1$, $\mathcal{B} = \mathcal{A} + \mathcal{E}$. If the perturbation $\mathcal{E}$ satisfies $\mathcal{A} *_N \mathcal{A}^{\#} *_N \mathcal{E} = \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\#} = \mathcal{E}$ and $\|\mathcal{A}^{\#} *_N \mathcal{E}\|_2 < 1$, then*

$$\mathcal{B}^{\#} = (\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\#} = \mathcal{A}^{\#} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{\#})^{-1},$$

and

$$\mathcal{B} *_N \mathcal{B}^{\#} = \mathcal{A} *_N \mathcal{A}^{\#}, \quad \mathcal{B}^{\#} *_N \mathcal{B} = \mathcal{A}^{\#} *_N \mathcal{A} + (\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\#} *_N \mathcal{E} *_N (\mathcal{I} - \mathcal{A}^{\#} *_N \mathcal{A}).$$

Furthermore,

$$\frac{\|\mathcal{A}^{\#}\|_2}{1 + \|\mathcal{A}^{\#} *_N \mathcal{E}\|_2} \leq \|\mathcal{B}^{\#}\|_2 \leq \frac{\|\mathcal{A}^{\#}\|_2}{1 - \|\mathcal{A}^{\#} *_N \mathcal{E}\|_2}$$

and

$$\frac{\|\mathcal{B}^{\#} *_N \mathcal{B} - \mathcal{A}^{\#} *_N \mathcal{A}\|_2}{\|\mathcal{A}^{\#} *_N \mathcal{A}\|_2} \leq \frac{\|\mathcal{A}^{\#} *_N \mathcal{E}\|_2}{1 - \|\mathcal{A}^{\#} *_N \mathcal{E}\|_2}.$$

Proof. We assume that the perturbation $\mathcal{E}$ is partitioned by

$$\mathcal{E} = \mathcal{U} *_N \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix} *_N \mathcal{U}^*.$$

Using the fact that $\mathcal{A} *_N \mathcal{A}^{\#} *_N \mathcal{E} = \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\#} = \mathcal{E}$, together with

$$\mathcal{A} *_N \mathcal{A}^{\#} = \mathcal{U} *_N \begin{pmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*$$

implies $\mathcal{E}_{12} = \mathcal{O}, \mathcal{E}_{21} = \mathcal{O}, \mathcal{E}_{22} = \mathcal{O}$. It is easy to see that the perturbation $\mathcal{E}$ has the form

$$\mathcal{E} = \mathcal{U} *_N \begin{pmatrix} \mathcal{E}_{11} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.$$

Furthermore, we obtain

$$\mathcal{B} = \mathcal{A} + \mathcal{E} = \mathcal{U} *_N \begin{pmatrix} \Sigma *_N \mathcal{K} + \mathcal{E}_{11} & \Sigma *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.$$

In view of Lemma 1.6, since $\|\mathcal{A}^{\#} *_N \mathcal{E}\|_2 < 1$, then $\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E}$ is invertible and

$$\|(\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E})^{-1}\|_2 \leq \frac{1}{1 - \|\mathcal{A}^{\#} *_N \mathcal{E}\|_2}.$$

Moreover,

$$\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E} = \mathcal{U} *_N \begin{pmatrix} \mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11} & \mathcal{O} \\ \mathcal{O} & \mathcal{I} \end{pmatrix} *_N \mathcal{U}^*,$$

and

$$(\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11})^{-1} = ((\Sigma *_N \mathcal{K})^{-1} *_N (\Sigma *_N \mathcal{K} + \mathcal{E}_{11}))^{-1} = (\Sigma *_N \mathcal{K} + \mathcal{E}_{11})^{-1} *_N \Sigma *_N \mathcal{K},$$

this implies that $(\Sigma *_N \mathcal{K} + \mathcal{E}_{11})^{-1}$ exists.
Then the core inverse of $\mathcal{B}$ exists and has the following expression,

$$\begin{aligned} \mathcal{B}^{\#} &= \mathcal{U} *_N \begin{pmatrix} (\Sigma *_N \mathcal{K} + \mathcal{E}_{11})^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\ &= \mathcal{U} *_N \begin{pmatrix} (\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11})^{-1} *_N (\Sigma *_N \mathcal{K})^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\ &= (\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\#}. \end{aligned}$$

By using (4) of Lemma 1.7, direct computation shows that

$$(\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\#} = \mathcal{A}^{\#} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{\#})^{-1}.$$

Next, the perturbation bounds of core inverse are estimated. It is easy to verify that

$$\begin{aligned} &\mathcal{B}^{\#} *_N \mathcal{B} - \mathcal{A}^{\#} *_N \mathcal{A} \\ &= \mathcal{U} *_N \begin{pmatrix} \mathcal{O} & (\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11})^{-1} *_N \mathcal{K}^{-1} *_N \mathcal{L} - \mathcal{K}^{-1} *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\ &= \mathcal{U} *_N \begin{pmatrix} \mathcal{O} & ([\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11}]^{-1} - \mathcal{I}) *_N \mathcal{K}^{-1} *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\ &= \mathcal{U} *_N \begin{pmatrix} \mathcal{O} & (\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11})^{-1} *_N [\mathcal{I} - (\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11})] *_N \mathcal{K}^{-1} *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\ &= \mathcal{U} *_N \begin{pmatrix} \mathcal{O} & -[(\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11}]^{-1} *_N (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11} *_N \mathcal{K}^{-1} *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\ &= (\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\#} *_N \mathcal{E} *_N (\mathcal{A} *_N \mathcal{A}^{\#} - \mathcal{A}^{\#} *_N \mathcal{A}) \\ &= (\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\#} *_N \mathcal{E} *_N (\mathcal{I} - \mathcal{A}^{\#} *_N \mathcal{A}). \end{aligned}$$

Taking forms of both sides, we obtain

$$\begin{aligned} \|\mathcal{B}^{\#} *_N \mathcal{B} - \mathcal{A}^{\#} *_N \mathcal{A}\|_2 &\leq \|(\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E})^{-1}\|_2 \|\mathcal{A}^{\#} *_N \mathcal{E}\|_2 \|\mathcal{I} - \mathcal{A}^{\#} *_N \mathcal{A}\|_2 \\ &= \|(\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E})^{-1}\|_2 \|\mathcal{A}^{\#} *_N \mathcal{E}\|_2 \|\mathcal{A}^{\#} *_N \mathcal{A}\|_2. \end{aligned}$$

That is

$$\frac{\|\mathcal{B}^{\text{#}} *_N \mathcal{B} - \mathcal{A}^{\text{#}} *_N \mathcal{A}\|_2}{\|\mathcal{A}^{\text{#}} *_N \mathcal{A}\|_2} \leq \frac{\|\mathcal{A}^{\text{#}} *_N \mathcal{E}\|_2}{1 - \|\mathcal{A}^{\text{#}} *_N \mathcal{E}\|_2}.$$

The proof is complete. □

Next, we provide a perturbation bound for the core inverse under one-sided condition.

Theorem 2.2. *Let $\mathcal{A}, \mathcal{E} \in \mathbb{C}^{I(N)\times I(N)}$ be of the form (2) and* $\operatorname{Ind}(\mathcal{A}) \leq 1$, *$\mathcal{B} = \mathcal{A} + \mathcal{E}$. If the perturbation $\mathcal{E}$ satisfies $\mathcal{A} *_N \mathcal{A}^{\text{#}} *_N \mathcal{E} = \mathcal{E}$ and $\|\mathcal{A}^{\text{#}} *_N \mathcal{E}\|_2 < 1$, then*

$$\mathcal{B}^{\text{#}} = (\mathcal{I} + \mathcal{A}^{\text{#}} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\text{#}} = \mathcal{A}^{\text{#}} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{\text{#}})^{-1},$$

and

$$\mathcal{B} *_N \mathcal{B}^{\text{#}} = \mathcal{A} *_N \mathcal{A}^{\text{#}}, \quad \mathcal{B}^{\text{#}} *_N \mathcal{B} = \mathcal{A}^{\text{#}} *_N \mathcal{A} + (\mathcal{I} + \mathcal{A}^{\text{#}} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\text{#}} *_N \mathcal{E} *_N (\mathcal{I} - \mathcal{A}^{\text{#}} *_N \mathcal{A}).$$

Furthermore,

$$\frac{\|\mathcal{A}^{\text{#}}\|_2}{1 + \|\mathcal{A}^{\text{#}} *_N \mathcal{E}\|_2} \leq \|\mathcal{B}^{\text{#}}\|_2 \leq \frac{\|\mathcal{A}^{\text{#}}\|_2}{1 - \|\mathcal{A}^{\text{#}} *_N \mathcal{E}\|_2}$$

and

$$\frac{\|\mathcal{B}^{\text{#}} *_N \mathcal{B} - \mathcal{A}^{\text{#}} *_N \mathcal{A}\|_2}{\|\mathcal{A}^{\text{#}} *_N \mathcal{A}\|_2} \leq \frac{\|\mathcal{A}^{\text{#}} *_N \mathcal{E}\|_2}{1 - \|\mathcal{A}^{\text{#}} *_N \mathcal{E}\|_2}.$$

Proof. We assume that the perturbation $\mathcal{E}$ is partitioned by

$$\mathcal{E} = \mathcal{U} *_N \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix} *_N \mathcal{U}^*.$$

Using the fact that $\mathcal{A} *_N \mathcal{A}^{\text{#}} *_N \mathcal{E} = \mathcal{E}$, it together with

$$\mathcal{A} *_N \mathcal{A}^{\text{#}} = \mathcal{U} *_N \begin{pmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*$$

implies $\mathcal{E}_{21} = \mathcal{O}$, $\mathcal{E}_{22} = \mathcal{O}$. It is easy to see that the perturbation $\mathcal{E}$ has the form

$$\mathcal{E} = \mathcal{U} *_N \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.$$

Furthermore, we obtain

$$\mathcal{B} = \mathcal{A} + \mathcal{E} = \mathcal{U} *_N \begin{pmatrix} \Sigma *_N \mathcal{K} + \mathcal{E}_{11} & \Sigma *_N \mathcal{L} + \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.$$

Since $\|\mathcal{A}^{\text{#}} *_N \mathcal{E}\|_2 < 1$ and $\mathcal{A} *_N \mathcal{A}^{\text{#}}$ is an orthogonal projection, we obtain

$$\|\mathcal{A}^{\text{#}} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\text{#}}\|_2 \leq \|\mathcal{A}^{\text{#}} *_N \mathcal{E}\|_2 \|\mathcal{A} *_N \mathcal{A}^{\text{#}}\|_2 = \|\mathcal{A}^{\text{#}} *_N \mathcal{E}\|_2 < 1.$$

Then $\mathcal{I} + \mathcal{A}^{\text{#}} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\text{#}}$ is invertible, so $[\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11}]^{-1}$ exists. Then the core inverse of $\mathcal{B}$ exists and has the following expression.

$$\begin{aligned}
\mathcal{B}^{\text{#}} &= \mathcal{U} *_N \begin{pmatrix} (\Sigma *_N \mathcal{K} + \mathcal{E}_{11})^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\
&= \mathcal{U} *_N \begin{pmatrix} [\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11}]^{-1} *_N (\Sigma *_N \mathcal{K})^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\
&= (\mathcal{I} + \mathcal{A}^{\text{#}} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\text{#}})^{-1} *_N \mathcal{A}^{\text{#}} \\
&= \mathcal{A}^{\text{#}} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{\text{#}})^{-1} \\
&= (\mathcal{I} + \mathcal{A}^{\text{#}} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\text{#}},
\end{aligned}$$

and

$$
\begin{aligned}
&\mathcal{B}^{\#} *_N \mathcal{B}\\
=&\mathcal{U} *_N \begin{pmatrix} \mathcal{I} & (\Sigma *_N \mathcal{K} + \mathcal{E}_{11})^{-1} *_N (\Sigma *_N \mathcal{L} + \mathcal{E}_{12}) \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*\\
=&\mathcal{U} *_N \begin{pmatrix} \mathcal{I} & (\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11})^{-1} *_N (\Sigma *_N \mathcal{K})^{-1} *_N (\Sigma *_N \mathcal{L} + \mathcal{E}_{12}) \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*\\
=&\mathcal{U} *_N \begin{pmatrix} \mathcal{I} & [\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11}]^{-1} *_N [(\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{12} + \mathcal{K}^{-1} *_N \mathcal{L}] \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.
\end{aligned}
$$

Now we can estimate

$$
\begin{aligned}
&\mathcal{B}^{\#} *_N \mathcal{B} - \mathcal{A}^{\#} *_N \mathcal{A}\\
=&\mathcal{U} *_N \begin{pmatrix} \mathcal{O} & [\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11}]^{-1} *_N [(\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{12} + \mathcal{K}^{-1} *_N \mathcal{L}] - \mathcal{K}^{-1} *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*\\
=&\mathcal{U} *_N \begin{pmatrix} \mathcal{O} & ([\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11}]^{-1} - \mathcal{I}) *_N \mathcal{K}^{-1} *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*\\
+&\mathcal{U} *_N \begin{pmatrix} \mathcal{O} & [\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11}]^{-1} *_N (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*\\
=&\mathcal{U} *_N \begin{pmatrix} \mathcal{O} & -[\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11}]^{-1} *_N (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11} *_N \mathcal{K}^{-1} *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*\\
+&\mathcal{U} *_N \begin{pmatrix} \mathcal{O} & [\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11}]^{-1} *_N (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*\\
=&\mathcal{U} *_N \begin{pmatrix} [\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11}]^{-1} *_N (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} \begin{pmatrix} \mathcal{O} & -\mathcal{K}^{-1} *_N L \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*\\
+&\mathcal{U} *_N \begin{pmatrix} \mathcal{O} & [\mathcal{I} + (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{11}]^{-1} *_N (\Sigma *_N \mathcal{K})^{-1} *_N \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*\\
=&(\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\#})^{-1} *_N \mathcal{A}^{\#} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\#} *_N (\mathcal{A} *_N \mathcal{A}^{\#} - \mathcal{A}^{\#} *_N \mathcal{A})\\
+&(\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\#})^{-1} *_N \mathcal{A}^{\#} *_N \mathcal{E} *_N (\mathcal{I} - \mathcal{A} *_N \mathcal{A}^{\#})\\
=&(\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\#})^{-1} *_N \mathcal{A}^{\#} *_N \mathcal{E} *_N (\mathcal{I} - \mathcal{A}^{\#} *_N \mathcal{A})\\
=&\mathcal{A}^{\#} *_N \mathcal{E} *_N (\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E})^{-1} *_N (\mathcal{I} - \mathcal{A}^{\#} *_N \mathcal{A})\\
=&(\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\#} *_N \mathcal{E} *_N (\mathcal{I} - \mathcal{A}^{\#} *_N \mathcal{A}).
\end{aligned}
$$

Taking norms of both sides, we obtain

$$\|\mathcal{B}^{\#} *_N \mathcal{B} - \mathcal{A}^{\#} *_N \mathcal{A}\|_2 \le \|(\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E})^{-1}\|_2 \|\mathcal{A}^{\#} *_N \mathcal{E}\|_2 \|\mathcal{I} - \mathcal{A}^{\#} *_N \mathcal{A}\|_2,$$

i.e.,

$$\frac{\|\mathcal{B}^{\#} *_N \mathcal{B} - \mathcal{A}^{\#} *_N \mathcal{A}\|_2}{\|\mathcal{A}^{\#} *_N \mathcal{A}\|_2} \le \frac{\|\mathcal{A}^{\#} *_N \mathcal{E}\|_2}{1 - \|\mathcal{A}^{\#} *_N \mathcal{E}\|_2}.$$

This completes the proof of the theorem. □

In a similar way, we obtain another one-sided perturbation formula.

Theorem 2.3. *Let $\mathcal{A}, \mathcal{E} \in \mathbb{C}^{I(N)\times I(N)}$ be of the form (2) and* $\mathrm{Ind}(\mathcal{A}) \le 1$, $\mathcal{B} = \mathcal{A} + \mathcal{E}$. *If the perturbation $\mathcal{E}$ satisfies $\mathcal{A}^{\#} *_N \mathcal{A} *_N \mathcal{E} = \mathcal{E}$ and $\|\mathcal{A}^{\#} *_N \mathcal{E}\|_2 < 1$, then*

$$\mathcal{B}^{\#} = (\mathcal{I} + \mathcal{A}^{\#} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\#} = \mathcal{A}^{\#} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{\#})^{-1}.$$

and

$$\mathcal{B} *_N \mathcal{B}^{\circledast} = \mathcal{A} *_N \mathcal{A}^{\circledast}, \mathcal{B}^{\circledast} *_N \mathcal{B} = \mathcal{A}^{\circledast} *_N \mathcal{A} + (\mathcal{I} + \mathcal{A}^{\circledast} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\circledast} *_N \mathcal{E} *_N (\mathcal{I} - \mathcal{A}^{\circledast} *_N \mathcal{A}).$$

3. Perturbation for core-EP inverse

In this section, we investigate the optimal perturbations for the core-EP inverse of tensors via Einstein product under one-sided conditions which extends the matrix case [9].

Theorem 3.1. *Let $\mathcal{A}, \mathcal{E} \in \mathbb{C}^{\mathbf{I(N)}\times\mathbf{I(N)}}$ be of the form(6) and* $\mathrm{Ind}(\mathcal{A}) = k$, $\mathcal{B} = \mathcal{A} + \mathcal{E} \in \mathbb{C}^{\mathbf{I(N)}\times\mathbf{I(N)}}$. *If the perturbation $\mathcal{E}$ satisfies $\mathcal{A} *_N \mathcal{A}^{\oplus} *_N \mathcal{E} = \mathcal{E}$ and $\|\mathcal{A}^{\oplus} *_N \mathcal{E}\|_2 < 1$, then*

$$\mathcal{B}^{\oplus} = (\mathcal{I} + \mathcal{A}^{\oplus} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\oplus} = \mathcal{A}^{\oplus} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{\oplus})^{-1}, \tag{8}$$

and

$$\mathcal{B} *_N \mathcal{B}^{\oplus} = \mathcal{A} *_N \mathcal{A}^{\oplus}, \quad \mathcal{B}^{\oplus} *_N \mathcal{B} = \mathcal{A}^{\oplus} *_N \mathcal{A} + (\mathcal{I} + \mathcal{A}^{\oplus} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\oplus} *_N \mathcal{E} *_N (\mathcal{I} - \mathcal{A}^{\oplus} *_N \mathcal{A}).$$

Furthermore,

$$\frac{\|\mathcal{A}^{\oplus}\|_2}{1 + \|\mathcal{A}^{\oplus} *_N \mathcal{E}\|_2} \le \|\mathcal{B}^{\oplus}\|_2 \le \frac{\|\mathcal{A}^{\oplus}\|_2}{1 - \|\mathcal{A}^{\oplus} *_N \mathcal{E}\|_2}$$

and

$$\frac{\|\mathcal{B}^{\oplus} *_N \mathcal{B} - \mathcal{A}^{\oplus} *_N \mathcal{A}\|_2}{\|\mathcal{A}^{\oplus} *_N \mathcal{A}\|_2} \le \frac{\|\mathcal{A}^{\oplus} *_N \mathcal{E}\|_2}{1 - \|\mathcal{A}^{\oplus} *_N \mathcal{E}\|_2}.$$

Proof. We assume that the perturbation $\mathcal{E}$ is partitioned by

$$\mathcal{E} = \mathcal{U} *_N \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix} *_N \mathcal{U}^*.$$

Since $\mathcal{E}$ satisfies $\mathcal{A} *_N \mathcal{A}^{\oplus} *_N \mathcal{E} = \mathcal{E}$, together with

$$\mathcal{A} *_N \mathcal{A}^{\oplus} = \mathcal{U} *_N \begin{pmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*$$

leads to $\mathcal{E}_{21} = \mathcal{O}$, $\mathcal{E}_{22} = \mathcal{O}$. It is straightforward to see that the perturbation $\mathcal{E}$ has the form

$$\mathcal{E} = \mathcal{U} *_N \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*,$$

and the tensor $\mathcal{B}$ keeps the Schur form

$$\mathcal{B} = \mathcal{A} + \mathcal{E} = \mathcal{U} *_N \begin{pmatrix} \mathcal{T}_1 + \mathcal{E}_{11} & \mathcal{T}_2 + \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{T}_3 \end{pmatrix} *_N \mathcal{U}^*.$$

Since $\|\mathcal{A}^{\oplus} *_N \mathcal{E}\|_2 < 1$, and $\mathcal{A} *_N \mathcal{A}^{\oplus}$ is an orthogonal projection, we obtain

$$\|\mathcal{A}^{\oplus} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\oplus}\|_2 \le \|\mathcal{A}^{\oplus} *_N \mathcal{E}\|_F \|\mathcal{A} *_N \mathcal{A}^{\oplus}\|_2 = \|\mathcal{A}^{\oplus} *_N \mathcal{E}\|_2 < 1.$$

Then $\mathcal{I} + \mathcal{A}^{\oplus} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\oplus}$ is invertible, so $(\mathcal{I} + \mathcal{T}_1^{-1} *_N \mathcal{E}_{11})^{-1}$ exists.

Then the core-EP inverse of $\mathcal{B}$ exists, and it has the form as follows

$$\begin{aligned}
\mathcal{B}^{\oplus} &= \mathcal{U} *_N \begin{pmatrix} (\mathcal{T}_1 + \mathcal{E}_{11})^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\
&= \mathcal{U} *_N \begin{pmatrix} (\mathcal{I} + \mathcal{T}_1^{-1} *_N \mathcal{E}_{11})^{-1} *_N \mathcal{T}_1^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\
&= (\mathcal{I} + \mathcal{A}^{\oplus} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\oplus})^{-1} *_N \mathcal{A}^{\oplus} \\
&= \mathcal{A}^{\oplus} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{\oplus})^{-1} \\
&= (\mathcal{I} + \mathcal{A}^{\oplus} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\oplus},
\end{aligned}$$

and $\mathcal{B}^{\oplus} *_N \mathcal{B}$ possesses the following representation

$$\mathcal{B}^{\oplus} *_N \mathcal{B} = \mathcal{U} *_N \begin{pmatrix} \mathcal{I} & (\mathcal{T}_1 + \mathcal{E}_{11})^{-1} *_N (\mathcal{T}_2 + \mathcal{E}_{12}) \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.$$

Further,

$$\mathcal{A}^{\oplus} *_N \mathcal{A} = \mathcal{U} *_N \begin{pmatrix} \mathcal{I} & \mathcal{T}_1^{-1} *_N \mathcal{T}_2 \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.$$

Now we can estimate

$$\begin{aligned}
&\mathcal{B}^{\oplus} *_N \mathcal{B} - \mathcal{A}^{\oplus} *_N \mathcal{A} \\
=&\mathcal{U} *_N \begin{pmatrix} \mathcal{O} & [(\mathcal{T}_1 + \mathcal{E}_{11})^{-1} - \mathcal{T}_1^{-1}] *_N \mathcal{T}_2 + (\mathcal{T}_1 + \mathcal{E}_{11})^{-1} *_N \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\
=&\mathcal{U} *_N \begin{pmatrix} \mathcal{O} & -(\mathcal{T}_1 + \mathcal{E}_{11})^{-1} *_N \mathcal{E}_{11} *_N \mathcal{T}_1^{-1} *_N \mathcal{T}_2 + (\mathcal{I} + \mathcal{T}_1^{-1} *_N \mathcal{E}_{11})^{-1} *_N \mathcal{T}_1^{-1} *_N \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\
=&\mathcal{U} *_N \begin{pmatrix} \mathcal{O} & -(\mathcal{I} + \mathcal{T}_1^{-1} *_N \mathcal{E}_{11})^{-1} *_N \mathcal{T}_1^{-1} *_N \mathcal{E}_{11} *_N \mathcal{T}_1^{-1} *_N \mathcal{T}_2 + (\mathcal{I} + \mathcal{T}_1^{-1} *_N \mathcal{E}_{11})^{-1} *_N \mathcal{T}_1^{-1} *_N \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\
=&-(\mathcal{I} + \mathcal{A}^{\oplus} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\oplus})^{-1} *_N \mathcal{A}^{\oplus} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\oplus} *_N (\mathcal{A}^{\oplus} *_N \mathcal{A} - \mathcal{I}) \\
&+(\mathcal{I} + \mathcal{A}^{\oplus} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\oplus})^{-1} *_N \mathcal{A}^{\oplus} *_N \mathcal{A} *_N \mathcal{A}^{\oplus} *_N \mathcal{E} *_N (\mathcal{I} - \mathcal{A} *_N \mathcal{A}^{\oplus}) \\
=&(\mathcal{I} + \mathcal{A}^{\oplus} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\oplus})^{-1} *_N \mathcal{A}^{\oplus} *_N \mathcal{E} *_N (\mathcal{A} *_N \mathcal{A}^{\oplus} - \mathcal{A}^{\oplus} *_N \mathcal{A}) \\
&+(\mathcal{I} + \mathcal{A}^{\oplus} *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\oplus})^{-1} *_N \mathcal{A}^{\oplus} *_N \mathcal{E} *_N (\mathcal{I} - \mathcal{A} *_N \mathcal{A}^{\oplus}) \\
=&(\mathcal{I} + \mathcal{A}^{\oplus} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\oplus} *_N \mathcal{E} *_N (\mathcal{I} - \mathcal{A}^{\oplus} *_N \mathcal{A}).
\end{aligned}$$

The proof is complete. □

In the same way, we obtain the similar perturbation formula.

Theorem 3.2. *Let* $\mathcal{A}, \mathcal{E} \in \mathbb{C}^{\mathbf{I(N)} \times \mathbf{I(N)}}$ *be of the form(6) and* $\mathrm{Ind}(\mathcal{A}) = k$, $\mathcal{B} = \mathcal{A} + \mathcal{E} \in \mathbb{C}^{\mathbf{I(N)} \times \mathbf{I(N)}}$. *If the perturbation* $\mathcal{E}$ *satisfies* $\mathcal{A}^{\oplus} *_N \mathcal{A} *_N \mathcal{E} = \mathcal{E}$ *and* $\|\mathcal{A}^{\oplus} *_N \mathcal{E}\|_2 < 1$, *then*

$$\mathcal{B}^{\oplus} = (\mathcal{I} + \mathcal{A}^{\oplus} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\oplus} = \mathcal{A}^{\oplus} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{\oplus})^{-1},$$

$$\mathcal{B} *_N \mathcal{B}^{\oplus} = \mathcal{A} *_N \mathcal{A}^{\oplus}.$$

Now we consider another perturbation formula with the weaker condition $(\mathcal{I} - \mathcal{A} *_N \mathcal{A}^{\oplus}) *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\oplus} = \mathcal{O}$.

Theorem 3.3. *Let $\mathcal{A}, \mathcal{E} \in \mathbb{C}^{\mathbf{I(N)}\times\mathbf{I(N)}}$ be of the form(6) and* $\mathrm{Ind}(\mathcal{A}) = k$, $\mathcal{B} = \mathcal{A} + \mathcal{E} \in \mathbb{C}^{\mathbf{I(N)}\times\mathbf{I(N)}}$. *If the perturbation $\mathcal{E}$ satisfies $(\mathcal{I} - \mathcal{A} *_N \mathcal{A}^{\oplus}) *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\oplus} = \mathcal{O}$ and $rank(\mathcal{A}^k) = rank(\mathcal{B}^k)$ with $\|\mathcal{A}^{\oplus} *_N \mathcal{E}\|_2 < 1$, then*

$$\mathcal{B}^{\oplus} = (\mathcal{I} + \mathcal{A}^{\oplus} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\oplus} = \mathcal{A}^{\oplus} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{\oplus})^{-1},$$

$$\mathcal{B} *_N \mathcal{B}^{\oplus} = \mathcal{A} *_N \mathcal{A}^{\oplus}.$$

Proof. We assume that the perturbation $\mathcal{E}$ is partitioned by

$$\mathcal{E} = \mathcal{U} *_N \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix} *_N \mathcal{U}^*.$$

Since $\mathcal{E}$ satisfies $(\mathcal{I} - \mathcal{A} *_N \mathcal{A}^{\oplus}) *_N \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^{\oplus} = \mathcal{O}$, together with

$$\mathcal{A} *_N \mathcal{A}^{\oplus} = \mathcal{U} *_N \begin{pmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*$$

implies $\mathcal{E}_{21} = \mathcal{O}$, and then $\mathcal{B}$ has the following expression

$$\mathcal{B} = \mathcal{A} + \mathcal{E} = \mathcal{U} *_N \begin{pmatrix} \mathcal{T}_1 + \mathcal{E}_{11} & \mathcal{T}_2 + \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{T}_3 + \mathcal{E}_{22} \end{pmatrix} *_N \mathcal{U}^*.$$

Now, from $\|\mathcal{A}^{\oplus} *_N \mathcal{E}\|_2 < 1$ and $rank(\mathcal{A}^k) = rank(\mathcal{B}^k)$, we can obtain that $\mathcal{T}_1 + \mathcal{E}_{11}$ is invertible and $rank[(\mathcal{T}_3 + \mathcal{E}_{22})^k] = \mathcal{O}$, $(\mathcal{T}_2 + \mathcal{E}_{22})^k = \mathcal{O}$. Moreover,

$$\mathcal{B}^{\oplus} = \mathcal{U} *_N \begin{pmatrix} (\mathcal{T}_1 + \mathcal{E}_{11})^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.$$

Similar to the proof of Theorem 3.1, we obtain

$$\mathcal{B}^{\oplus} = (\mathcal{I} + \mathcal{A}^{\oplus} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{\oplus} = \mathcal{A}^{\oplus} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{\oplus})^{-1},$$

and

$$\mathcal{B} *_N \mathcal{B}^{\oplus} = \mathcal{A} *_N \mathcal{A}^{\oplus}.$$

The proof is complete. □

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