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Singular Value Inequalities for Sector Matrices

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Abstract. In this paper, we present two singular value inequalities for sector matrices. As a consequence, we prove unitarily invariant norm inequalities for sector matrices. Moreover, we present some determinant inequalities for accretive-dissipative matrices.

1. Introduction

As customary, let M_n represent the set of all $n \times n$ complex matrices. A matrix $T \in M_n$ is called accretive-dissipative if in its cartesian decomposition, $T = A + iB$, the matrices A and B are positive semidefinite, where $A = \mathrm{Re}(T) = \frac{T+T^*}{2}$ and $B = \mathrm{Im}(T) = \frac{T-T^*}{2i}$ (see [1]). If the eigenvalues of matrix $T \in M_n$ are all real, the jth largest eigenvalue of T is denoted by $\lambda_j(T)$, $j = 1, 2, \cdots, n$. The singular values $s_j(T)(j = 1, 2, \cdots, n)$ of T are the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arrange in a decreasing order.

The numerical range of $A \in M_n$ is described by

$$W(A) = \{x^*Ax | x \in C^n, x^*x = 1\}.$$

For $\alpha \in [0, \frac{\pi}{2})$, let S_α be the sector denoted in the complex plane by

$$S_\alpha = \{z \in C : \mathrm{Re}(z) > 0, |\mathrm{Im}(z)| \le \tan\alpha \mathrm{Re}(z)\}.$$

Clearly, for some $\alpha \in [0, \frac{\pi}{2})$, if $W(A), W(B) \subset S_\alpha$, then $W(A + B) \subset S_\alpha$. As $0 \notin S_\alpha$, if $W(A) \subset S_\alpha$, then A is nonsingular. A matrix $A \in M_n$ is said to be sector matrix if its numerical range is contained in S_α , for some $\alpha \in [0, \frac{\pi}{2})$ (see [2]). Recently, many interesting articles have been devoted to study the singular value inequalities and unitarily invariant norm inequalities for sector matrices, see [3–7] and references therein.

Let $A \in M_n$ be such that $W(A) \subset S_\alpha$ and U be the unitary part of A in the polar decomposition $A = U|A|$. Mohammad [8, Theorem 1.1] proved that

$$|A| \le \frac{\sec(\alpha)}{2}[\mathrm{Re}(A) + U^*(\mathrm{Re}(A))U], \tag{1}$$

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where $\sec(\alpha)$ is the secant of α.

Garg and Aujla [9, Theorem 2.8, 2.10] proved that if $A, B \in M_n$ and $1 \leq r \leq 2$, then

$$\prod_{j=1}^{k} s_j(|A+B|^r) \leq \prod_{j=1}^{k} s_j(I_n + |A|^r) \prod_{j=1}^{k} s_j(I_n + |B|^r) \tag{2}$$

and

$$\prod_{j=1}^{k} s_j(I_n + f(|A+B|)) \leq \prod_{j=1}^{k} s_j(I_n + f(|A|)) \prod_{j=1}^{k} s_j(I_n + f(|B|)). \tag{3}$$

where $f : [0, \infty) \to [0, \infty)$ is an operator concave function and $1 \leq k \leq n$.

Let $A, B \in M_n$ be positive semidefinite, $r = 1$ and $f(X) = X$ for any $X \in M_n$ in (2) and (3), we have

$$\prod_{j=1}^{k} s_j(A+B) \leq \prod_{j=1}^{k} s_j(I_n + A) \prod_{j=1}^{k} s_j(I_n + B), 1 \leq k \leq n \tag{4}$$

and

$$\prod_{j=1}^{k} s_j(I_n + A + B) \leq \prod_{j=1}^{k} s_j(I_n + A) \prod_{j=1}^{k} s_j(I_n + B), 1 \leq k \leq n. \tag{5}$$

Let $k = n$ in (4) and (5), we obtain

$$\det(A+B) \leq \det(I_n + A)\det(I_n + B) \tag{6}$$

and

$$\det(I_n + A + B) \leq \det(I_n + A)\det(I_n + B). \tag{7}$$

This paper firstly gives two singular value inequalities for sector matrices according to (1), (2) and (3). And then, we obtain unitarily invariant norm inequalities for sector matrices. Moreover, we present some determinant inequalities for accretive-dissipative matrices based on (6) and (7).

2. Main results

In the following, we give five lemmas which will turn out to be useful in the proof of our results.

Lemma 2.1. [10, P.72 III.19] Let $A, B \in M_n$. Then

$$\prod_{j=1}^{k} s_j(AB) \leq \prod_{j=1}^{k} s_j(A)s_j(B), 1 \leq j \leq n.$$

Lemma 2.2. [10, Theorem III.5.6] Let $A, B \in M_n$. There exist unitary matrices $U, V \in M_n$ such that

$$|A+B| \leq U|A|U^* + V|B|V^*.$$

Lemma 2.3. [1, Theorem 3.2] Let $A, B \in M_n$ be accretive-dissipative. Then

$$\sqrt{2}|\det(A+B)|^{\frac{1}{n}} \geq |\det A|^{\frac{1}{n}} + |\det B|^{\frac{1}{n}}.$$

Lemma 2.4. [11, Lemma 6] Let $A, B \in M_n$ be positive semidefinite. Then

$$|\det(A+iB)| \leq \det(A+B) \leq 2^{\frac{n}{2}}|\det(A+iB)|.$$

Lemma 2.5. [1, Theorem 3.3] Let $A, B \in M_n$ be accretive-dissipative and $0 < \mu < 1$. Then

$$|\det A|^{\mu}|\det B|^{1-\mu} \leq 2^{\frac{n}{2}}|\det(\mu A + (1-\mu)B|.$$

Theorem 2.6. Let $A, B \in M_n$ be such that $W(A), W(B) \subset S_\alpha$. Then

$$\prod_{j=1}^{k} s_j(A+B) \le \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)) \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B)) \tag{8}$$

and

$$\prod_{j=1}^{k} s_j(I_n + A+B) \le \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)) \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B)), \tag{9}$$

where $1 \le k \le n$.

Proof . Let U_1, U_2, V_1 and V_2 be unitary matrices.

$$\begin{aligned}
\prod_{j=1}^{k} s_j(A+B) &= \prod_{j=1}^{k} \lambda_j(|A+B|) \\
&= \prod_{j=1}^{k} s_j(|A+B|) \\
&\le \prod_{j=1}^{k} s_j(I_n + |A|) \prod_{j=1}^{k} s_j(I_n + |B|) \quad (\text{by } (2)) \\
&\le \prod_{j=1}^{k} s_j[I_n + \frac{\sec(\alpha)}{2}(\mathrm{Re}(A) + U_1^*\mathrm{Re}(A)U_1)] s_j[I_n + \frac{\sec(\alpha)}{2}(\mathrm{Re}(B) + V_1^*\mathrm{Re}(B)V_1)] \quad (\text{by } (1)) \\
&\le \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)) \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B)) \quad (\text{by } (5), \text{Lemma } 2.1).
\end{aligned}$$

To prove (9), we compute

$$\begin{aligned}
\prod_{j=1}^{k} s_j(I_n + A + B) &\le \prod_{j=1}^{k} s_j(U_2|I_n|U_2^* + V_2|A+B|V_2^*) \quad (\text{by Lemma } 2.2) \\
&= \prod_{j=1}^{k} s_j(I_n + V_2|A+B|V_2^*) \\
&\le \prod_{j=1}^{k} s_j(I_n + |A+B|) \quad (\text{by Lemma } 2.1) \\
&\le \prod_{j=1}^{k} s_j(I_n + |A|) \prod_{j=1}^{k} s_j(I_n + |B|) \quad (\text{by } (3)) \\
&\le \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)) \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B)) \quad (\text{by } (1), (5), \text{Lemma } 2.1).
\end{aligned}$$

This completes the proof. □

Corollary 2.7. Let $A, B \in M_n$ be such that $W(A), W(B) \subset S_\alpha$. Then

$$\|A+B\| \le \|I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)\|^2 \|I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B)\|^2 \tag{10}$$

and

$$\|I_n + A + B\| \le \|I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)\|^2 \|I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B)\|^2. \tag{11}$$

Proof . By (8), we obtain

$$\prod_{j=1}^{k} s_j^{\frac{1}{4}}(A+B) \leq \prod_{j=1}^{k} s_j^{\frac{1}{2}}(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)) \prod_{j=1}^{k} s_j^{\frac{1}{2}}(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B))$$

for $1 \leq k \leq n$.

By the property that weak log-majorization implies weak majorization and Cauchy-Schwarz inequality, we get

$$\sum_{j=1}^{k} s_j^{\frac{1}{4}}(A+B) \leq (\sum_{j=1}^{k} s_j(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)))^{\frac{1}{2}} (\sum_{j=1}^{k} s_j(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B)))^{\frac{1}{2}} \tag{12}$$

for $1 \leq k \leq n$.

Inequality (12) is equivalent to the following inequality:

$$\||A+B|^{\frac{1}{4}}\|_{(k)}^2 \leq \|I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)\|_{(k)} \|I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B)\|_{(k)}$$

for $1 \leq k \leq n$.

By Fan's dominance principle [10, P.93], we have

$$\||A+B|^{\frac{1}{4}}\|^2 \leq \|I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)\| \|I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B)\|.$$

Let $A + B = U|A + B|$ be the polar decomposition of $A + B$ and U be an unitary matrix. Thus, we have

$$\|A+B\| = \|U|A+B|\| = \|[|A+B|^{\frac{1}{4}}]^4\| \leq \||A+B|^{\frac{1}{4}}\|^4 \leq \|I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)\|^2 \|I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B)\|^2.$$

Similarly, we can obtain (11).
This completes the proof. □

Corollary 2.8. Let $A, B \in M_n$ be such that $W(A), W(B) \subset S_\alpha$. Then

$$|\det(A+B)| \leq [\det(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A))]^2 [\det(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B))]^2 \tag{13}$$

and

$$|\det(I_n + A + B)| \leq [\det(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A))]^2 [\det(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B))]^2. \tag{14}$$

Example 2.9. Let

$$A = B = \begin{bmatrix} \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}i & 0 \\ 0 & \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}i \end{bmatrix}.$$

We compute the right side of the inequality [3, Theorem 2.14 (13)]

$$\sec^2(\frac{\pi}{4})|\det(I_2 + A)||\det(I_2 + B)| \approx 7.6605.$$

Similarly, we have right side of the inequality (13)

$$[\det(I_2 + \frac{\sec(\frac{\pi}{4})}{2}\mathrm{Re}(A))]^2 [\det(I_2 + \frac{\sec(\frac{\pi}{4})}{2}\mathrm{Re}(B))]^2 \approx 5.9605.$$

This example shows that the inequality (13) is stronger than the inequality [3, Theorem 2.14 (13)].
Example 2.10. Let

$$A = B = \begin{bmatrix} 1-i & 0 \\ 0 & 1-i \end{bmatrix}.$$

For the right side of [3, Theorem 2.14 (13)] and (13), we have

$$\sec^2(\frac{\pi}{4})|\det(I_2 + A)||\det(I_2 + B)| = 50$$

and

$$[\det(I_2 + \frac{\sec(\frac{\pi}{4})}{2}\mathrm{Re}(A))]^2[\det(I_2 + \frac{\sec(\frac{\pi}{4})}{2}\mathrm{Re}(B))]^2 \approx 72.1248,$$

respectively. This shows that the inequality (13) is weaker than the inequality [3, Theorem 2.14 (13)].

We present the following determinant inequalities for accretive-dissipative matrices.
Theorem 2.11. Let $A, B \in M_n$ be accretive-dissipative. Then for $\mu \in [0, 1]$,

$$|\det A|^{\frac{1}{n}} + |\det B|^{\frac{1}{n}} \leq 2\sqrt{2}|\det(I_n + A)|^{\frac{1}{n}}|\det(I_n + B)|^{\frac{1}{n}} \tag{15}$$

and

$$|\det(\mu I_n + A)|^{\frac{1}{n}} + |\det((1-\mu)I_n + B)|^{\frac{1}{n}} \leq 2\sqrt{2}|\det(I_n + A)|^{\frac{1}{n}}|\det(I_n + B)|^{\frac{1}{n}}. \tag{16}$$

Proof. Let $A = A_1 + iA_2$ and $B = B_1 + iB_2$ be the cartesian decompositions of A and B. We have

$$\begin{aligned}
|\det A|^{\frac{1}{n}} + |\det B|^{\frac{1}{n}} &\leq \sqrt{2}|\det(A + B)|^{\frac{1}{n}} \quad (\text{by Lemma 2.3}) \\
&= \sqrt{2}|\det[(A_1 + B_1) + i(A_2 + B_2)]|^{\frac{1}{n}} \\
&\leq \sqrt{2}|\det(A_1 + A_2 + B_1 + B_2)|^{\frac{1}{n}} \quad (\text{by Lemma 2.4}) \\
&\leq \sqrt{2}[\det(I_n + A_1 + A_2)]^{\frac{1}{n}}[\det(I_n + B_1 + B_2)]^{\frac{1}{n}} \quad (\text{by (6)}) \\
&\leq 2\sqrt{2}|\det(I_n + A_1 + iA_2)|^{\frac{1}{n}}|\det(I_n + B_1 + iB_2)|^{\frac{1}{n}} \quad (\text{by Lemma 2.4}) \\
&= 2\sqrt{2}|\det(I_n + A)|^{\frac{1}{n}}|\det(I_n + B)|^{\frac{1}{n}}.
\end{aligned}$$

Similarly, by Lemmas 2.3, 2.4 and inequality (7), we can obtain (16).
This completes the proof. □
Theorem 2.12. Let $A, B \in M_n$ be accretive-dissipative. Then for $\mu \in (0, 1)$,

$$|\det A|^{\mu}|\det B|^{1-\mu} \leq 2^{\frac{3n}{2}}|\det(I_n + \mu A)||\det(I_n + (1-\mu)B)| \tag{17}$$

and

$$|\det(I_n + A)|^{\mu}|\det(I_n + B)|^{1-\mu} \leq 2^{\frac{3n}{2}}|\det(I_n + \mu A)||\det(I_n + (1-\mu)B)|. \tag{18}$$

Proof. Let $A = A_1 + iA_2$ and $B = B_1 + iB_2$ be the cartesian decompositions of A and B. We have

$$\begin{aligned}
|\det A|^{\mu}|\det B|^{1-\mu} &\leq 2^{\frac{n}{2}}|\det(\mu A + (1-\mu)B)| \quad (\text{by Lemma } 2.5) \\
&= 2^{\frac{n}{2}}|\det[\mu(A_1 + iA_2) + (1-\mu)(B_1 + iB_2)]| \\
&\leq 2^{\frac{n}{2}} \det[\mu(A_1 + A_2) + (1-\mu)(B_1 + B_2)] \quad (\text{by Lemma } 2.4) \\
&\leq 2^{\frac{n}{2}} \det(I_n + \mu(A_1 + A_2)) \det(I_n + (1-\mu)(B_1 + B_2)) \quad (\text{by } (6)) \\
&\leq 2^{\frac{3n}{2}}|\det(I_n + \mu A)||\det(I_n + (1-\mu)B)| \quad (\text{by Lemma } 2.4).
\end{aligned}$$

Similarly, by Lemmas 2.4, 2.5 and inequality (7), we can obtain (18).
This completes the proof. □

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