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Some Inequalities for General Zeroth–Order Randić Index

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Abstract. Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be a simple connected graph with n vertices, m edges and vertex degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, $d_i = d(v_i)$. General zeroth–order Randić index of G is defined as ${}^0R_\alpha(G) = \sum_{i=1}^{n} d_i^\alpha$, where α is an arbitrary real number. In this paper we establish relationships between ${}^0R_\alpha(G)$ and ${}^0R_{\alpha-1}(G)$ and obtain new bounds for ${}^0R_\alpha(G)$. Also, we determine relationship between ${}^0R_\alpha(G)$, ${}^0R_\beta(G)$ and ${}^0R_{2\alpha-\beta}(G)$, where α and β are arbitrary real numbers. By the appropriate choice of parameters α and β, a number of old/new inequalities for different vertex–degree–based topological indices are obtained.

1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be a simple connected graph with n vertices, m edges and a sequence of vertex degrees $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$, $d_i = d(v_i)$. If vertices v_i and v_j are adjacent, we write $i \sim j$.

The topological indices form an important class of molecular structure descriptors used for quantifying information on molecules. Thousands of topological indices have been introduced in order to describe physical and chemical properties of molecules. Various mathematical properties of topological indices have been investigated, as well. As topological indices have been defined for quantifying information of graphs, this area could be classified into the so-called quantitative graph theory [8].

There is no doubt that the Randić index is the most studied, most often applied, and most popular topological index among all [21]. It was introduced by Milan Randić in 1975 [29] under the name *branching index* as a suitable measure of the extent of branching of the carbon–atom skeleton of saturated hydrocarbons. It is defined as

$$R(G) = \sum_{i\sim j} \frac{1}{\sqrt{d_i d_j}}.$$

After the Randić index was introduced, based on its definition various Randić–type invariants have been proposed and studied.

Bollobás and Erdős [3] generalized this index by replacing $-1/2$ with any real number α:

$$R_\alpha(G) = \sum_{i\sim j} (d_i d_j)^\alpha,$$

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and called it *general Randić index*.

The sum of the α-th powers of the degrees of a (molecular) graph G

$$^{0}R_{\alpha}(G) = \sum_{i=1}^{n} d_i^{\alpha},$$

is known as *general zeroth–order Randić index* [14]. It is also met under names *general first Zagreb index* [18] and *variable first Zagreb index* [23] (see also [1, 2]).

Here we are interested in the following special cases of $^{0}R_{\alpha}(G)$:

- *Modified first Zagreb index*, $^{m}M_1(G) = {}^{0}R_{-2}(G)$, introduced in [27];
- *Inverse degree* or *modified total adjacency* index, $ID(G) = {}^{0}R_{-1}$, [10, 27];
- *Zeroth–order connectivity index* or *zeroth–order Randić index*, $^{0}R(G) = {}^{0}R_{-1/2}(G)$, [17];
- *First Zagreb index*, $M_1(G) = {}^{0}R_2(G)$, [12];
- *Forgotten topological index*, $F(G) = {}^{0}R_3(G)$, [12].

In this paper we determine relations between $^{0}R_{\alpha}(G)$ and $^{0}R_{\alpha-1}(G)$ and obtain new bounds for $^{0}R_{\alpha}(G)$. Also, we establish relations between $^{0}R_{\alpha}(G)$, $^{0}R_{\beta}(G)$ and $^{0}R_{2\alpha-\beta}(G)$, where α and β are arbitrary real numbers. As special cases, for some specific values of parameters α and β, we obtain a number of old/new inequalities for different vertex–degree–based topological indices of graphs.

2. Preliminaries

In this section we recall some discrete inequalities for real number sequences that will be used later in the paper.

Let $p = (p_i)$, $i = 1, 2, \ldots, n$, be a nonnegative real number sequence and $a = (a_i)$, $i = 1, 2, \ldots, n$, positive real number sequence. Then for any real r, such that $r \geq 1$ or $r \leq 0$, holds (Jensen's inequality) [25]

$$\left(\sum_{i=1}^{n} p_i\right)^{r-1} \sum_{i=1}^{n} p_i a_i^r \geq \left(\sum_{i=1}^{n} p_i a_i\right)^r . \tag{1}$$

If $0 \leq r \leq 1$, then the sense of (1) reverses. Equality holds if and only if $a_1 = a_2 = \cdots = a_n$, or for some k, $1 \leq k \leq n-1$, such that $p_1 = p_2 = \cdots = p_k = 0$ and $p_{k+1} = p_{k+2} + \cdots + p_n$ and $a_{k+1} = a_{k+2} = \cdots = a_n$.

Let $x = (x_i)$, $i = 1, 2, \ldots, n$, be a nonnegative real number sequence and $a = (a_i)$, $i = 1, 2, \ldots, n$, positive real number sequence. In [28] it was proven that for any $r \geq 0$ holds

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n} a_i\right)^r}. \tag{2}$$

Equality is reached if and only if $r = 0$ or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

3. Main results

In the following theorem we establish relations between ${}^0R_\alpha(G)$ and ${}^0R_{\alpha-1}(G)$.

Theorem 3.1. *Let G be a simple connected graph with $n \ge 3$ vertices and m edges. Then for any real α, such that $\alpha \le 0$ or $\alpha \ge 1$, holds*

$$ {}^0R_\alpha(G) \ge \max\left\{ \frac{(2m-n)^\alpha}{(n-ID(G))^{\alpha-1}} + {}^0R_{\alpha-1}(G), \frac{(2m+n)^\alpha}{(n+ID(G))^{\alpha-1}} - {}^0R_{\alpha-1}(G) \right\}. \tag{3} $$

If $0 \le \alpha \le 1$, then

$$ {}^0R_\alpha(G) \le \min\left\{ \frac{(2m-n)^\alpha}{(n-ID(G))^{\alpha-1}} + {}^0R_{\alpha-1}(G), \frac{(2m+n)^\alpha}{(n+ID(G))^{\alpha-1}} - {}^0R_{\alpha-1}(G) \right\}. \tag{4} $$

Equalities hold if and only if either $\alpha = 0$, or $\alpha = 1$, or G is regular, or when $d_{k+1} = d_{k+2} = \cdots = d_n = 1$ and $d_1 = d_2 = \cdots = d_k > 1$, where $1 \le k \le n-1$.

Proof. For any real α we have that

$$ {}^0R_\alpha(G) - {}^0R_{\alpha-1}(G) = \sum_{i=1}^{n} (d_i - 1)\, d_i^{\alpha-1} \tag{5} $$

and

$$ {}^0R_\alpha(G) + {}^0R_{\alpha-1}(G) = \sum_{i=1}^{n} (d_i + 1)\, d_i^{\alpha-1}. \tag{6} $$

For $r = \alpha$, $p_i = 1 - \frac{1}{d_i}$, $a_i = d_i$, $i = 1, 2, \dots, n$, the inequality (1) becomes

$$ \left(\sum_{i=1}^{n} \left(1 - \frac{1}{d_i}\right) \right)^{\alpha-1} \sum_{i=1}^{n} (d_i - 1)\, d_i^{\alpha-1} \ge \left(\sum_{i=1}^{n} \left(1 - \frac{1}{d_i}\right) d_i \right)^{\alpha}, $$

that is

$$ \sum_{i=1}^{n} (d_i - 1)\, d_i^{\alpha-1} \ge \frac{(2m-n)^\alpha}{(n-ID(G))^{\alpha-1}}. \tag{7} $$

On the other hand, for $r = \alpha$, $p_i = 1 + \frac{1}{d_i}$, $a_i = d_i$, $i = 1, 2, \dots, n$, the inequality (1) transforms into

$$ \left(\sum_{i=1}^{n} \left(1 + \frac{1}{d_i}\right) \right)^{\alpha-1} \sum_{i=1}^{n} (d_i + 1)\, d_i^{\alpha-1} \ge \left(\sum_{i=1}^{n} \left(1 + \frac{1}{d_i}\right) d_i \right)^{\alpha}, $$

that is

$$ \sum_{i=1}^{n} (d_i + 1)\, d_i^{\alpha-1} \ge \frac{(2m+n)^\alpha}{(n+ID(G))^{\alpha-1}}. \tag{8} $$

Now, from (5) and (7), that is (6) and (8), we get

$$ {}^0R_\alpha(G) \ge {}^0R_{\alpha-1}(G) + \frac{(2m-n)^\alpha}{(n-ID(G))^{\alpha-1}} \tag{9} $$

and

$$^{0}R_{\alpha}(G) \geq \frac{(2m+n)^{\alpha}}{(n+ID(G))^{\alpha-1}} - {}^{0}R_{\alpha-1}(G). \tag{10}$$

The inequalities (3) and (4) are obtained from (9) and (10).

Equalities in (7) and (8) hold if and only if either $\alpha = 0$, or $\alpha = 1$, or $d_1 = d_2 = \cdots = d_n$, which implies that equalities in (3) and (4) hold if and only if either $\alpha = 0$, or $\alpha = 1$, or G is regular. $\square$

Based on (9) and (10) we have the following corollary of Theorem 3.1.

Corollary 3.2. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then for any real α, such that $\alpha \leq 0$ or $\alpha \geq 1$, holds*

$$^{0}R_{\alpha}(G) \geq \frac{1}{2}\left(\frac{(2m-n)^{\alpha}}{(n-ID(G))^{\alpha-1}} + \frac{(2m+n)^{\alpha}}{(n+ID(G))^{\alpha-1}}\right).$$

If $0 \leq \alpha \leq 1$, the sense of the above inequality reverses. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or G is a regular graph.

Corollary 3.3. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$\begin{aligned}
M_1(G) &\geq \max\left\{\frac{(2m-n)^2}{n-ID(G)} + 2m, \frac{(2m+n)^2}{n+ID(G)} - 2m\right\},\\
M_1(G) &\geq \frac{1}{2}\left(\frac{(2m-n)^2}{n-ID(G)} + \frac{(2m+n)^2}{n+ID(G)}\right),\\
F(G) &\geq \max\left\{\frac{(2m-n)^3}{(n-ID(G))^2} + M_1(G), \frac{(2m+n)^3}{(n+ID(G))^2} - M_1(G)\right\},\\
F(G) &\geq \frac{1}{2}\left(\frac{(2m-n)^3}{(n-ID(G))^2} + \frac{(2m+n)^3}{(n+ID(G))^2}\right)\\
{}^{m}M_1(G) &\geq \frac{1}{2}\left(\frac{(n-ID(G))^3}{(2m-n)^2} + \frac{(n+ID(G))^3}{(2m+n)^2}\right).
\end{aligned}$$

Equalities hold if and only if G is regular.

Corollary 3.4. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then for any real $\alpha \geq 1$ holds*

$$^{0}R_{\alpha}(G) \geq {}^{0}R_{\alpha-1}(G) + (2m-n)\left(\frac{2m}{n}\right)^{\alpha-1}. \tag{11}$$

Equality holds if and only if $\alpha = 1$ or G is regular graph.

Proof. According to arithmetic–harmonic mean inequality for real numbers (see e.g. [26]), we have that

$$\sum_{i=1}^{n}\frac{1}{d_i}\sum_{i=1}^{n} d_i \geq n^2,$$

i.e.

$$ID(G) \geq \frac{n^2}{2m}. \tag{12}$$

From the above inequality follows

$$\frac{2m-n}{n-ID(G)} \geq \frac{2m}{n},$$

therefore for any real $\alpha \geq 1$ holds

$$\left(\frac{2m-n}{n-ID(G)}\right)^{\alpha-1} \geq \left(\frac{2m}{n}\right)^{\alpha-1}.$$

Now from the above and inequality (9) we arrive at (11). □

Corollary 3.5. *Let G be a simple connected graph with n vertices and m edges. Then*

$$F(G) \geq M_1(G) + \frac{4m^2(2m-n)}{n^2} \tag{13}$$

and

$$M_1(G) \geq \frac{4m^2}{n}. \tag{14}$$

Equalities hold if and only if G is regular.

Remark 3.6. *Since*

$$M_1(G) + \frac{4m^2(2m-n)}{n^2} \geq \frac{8m^3}{n^2},$$

the inequality (13) *is stronger than inequality*

$$F(G) \geq \frac{8m^3}{n^2},$$

proven in [16].

The inequality (14) *was proven in* [9] (*see also* [5, 15, 16, 30]).

Theorem 3.7. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then for any real α, such that $\alpha \leq 1$ or $\alpha \geq 2$, holds*

$${}^0R_\alpha(G) \geq \max\left\{\frac{(M_1(G)-2m)^{\alpha-1}}{(2m-n)^{\alpha-2}} + {}^0R_{\alpha-1}(G), \frac{(M_1(G)+2m)^{\alpha-1}}{(2m+n)^{\alpha-2}} - {}^0R_{\alpha-1}(G)\right\}. \tag{15}$$

If $1 \leq \alpha \leq 2$, then

$${}^0R_\alpha(G) \leq \min\left\{\frac{(M_1(G)-2m)^{\alpha-1}}{(2m-n)^{\alpha-2}} + {}^0R_{\alpha-1}(G), \frac{(M_1(G)+2m)^{\alpha-1}}{(2m+n)^{\alpha-2}} - {}^0R_{\alpha-1}(G)\right\}. \tag{16}$$

Equalities hold if and only if either $\alpha = 1$, or $\alpha = 2$, or G is regular, or when $d_{k+1} = d_{k+2} = \cdots = d_n = 1$ and $d_1 = d_2 = \cdots = d_k > 1$, where $1 \leq k \leq n-1$.

Proof. For $r = \alpha - 1$, $p_i = d_i - 1$, $a_i = d_i$, $i = 1, 2, \ldots, n$, the inequality (1) transforms into

$$\left(\sum_{i=1}^{n}(d_i-1)\right)^{\alpha-2} \sum_{i=1}^{n}(d_i-1)\,d_i^{\alpha-1} \geq \left(\sum_{i=1}^{n}(d_i-1)\,d_i\right)^{\alpha-1},$$

i.e.

$$\sum_{i=1}^{n}(d_i-1)\,d_i^{\alpha-1} \geq \frac{(M_1(G)-2m)^{\alpha-1}}{(2m-n)^{\alpha-2}}. \tag{17}$$

On the other hand, for $r = \alpha - 1$, $p_i = d_i + 1$, $a_i = d_i$, $i = 1, 2, \ldots, n$, the inequality (1) becomes

$$\left(\sum_{i=1}^{n}(d_i+1)\right)^{\alpha-2} \sum_{i=1}^{n}(d_i+1)\,d_i^{\alpha-1} \geq \left(\sum_{i=1}^{n}(d_i+1)\,d_i\right)^{\alpha-1},$$

i.e.

$$\sum_{i=1}^{n}(d_i+1)\,d_i^{\alpha-1} \geq \frac{(M_1(G)+2m)^{\alpha-1}}{(2m+n)^{\alpha-2}}. \tag{18}$$

Combining (5) and (17), respectively (6) and (18), we obtain

$$^0R_\alpha(G) \geq {}^0R_{\alpha-1}(G) + \frac{(M_1(G)-2m)^{\alpha-1}}{(2m-n)^{\alpha-2}} \tag{19}$$

and

$$^0R_\alpha(G) \geq \frac{(M_1(G)+2m)^{\alpha-1}}{(2m+n)^{\alpha-2}} - {}^0R_{\alpha-1}(G). \tag{20}$$

The inequalities (15) and (16) are directly obtained from to (19) and (20).

Equalities in (17) and (18) hold if and only if either $\alpha = 1$, or $\alpha = 2$, or $d_1 = d_2 = \cdots = d_n$. This implies that equalities in (15) and (16) hold if and only if either $\alpha = 1$, or $\alpha = 2$, or G is a regular graph. □

Corollary 3.8. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then for any real α, such that $\alpha \leq 1$ or $\alpha \geq 2$, holds*

$$^0R_\alpha(G) \geq \frac{1}{2}\left(\frac{(M_1(G)-2m)^{\alpha-1}}{(2m-n)^{\alpha-2}} + \frac{(M_1(G)+2m)^{\alpha-1}}{(2m+n)^{\alpha-2}}\right).$$

If $1 \leq \alpha \leq 2$, then the sense of the above inequality reverses. Equality holds if and only if either $\alpha = 1$, or $\alpha = 2$, or G is regular.

Corollary 3.9. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$F(G) \geq \max\left\{\frac{(M_1(G)-2m)^2}{2m-n} + M_1(G), \frac{(M_1(G)+2m)^2}{2m+n} - M_1(G)\right\}$$

and

$$ID(G) \geq \max\left\{\frac{(2m-n)^3}{(M_1(G)-2m)^2} + {}^mM_1(G), \frac{(2m+n)^3}{(M_1(G)+2m)^2} - {}^mM_1(G)\right\}.$$

Equalities hold if and only if G is regular, or when $d_{k+1} = d_{k+2} = \cdots = d_n = 1$ and $d_1 = d_2 = \cdots = d_k > 1$, where $1 \leq k \leq n-1$.

Corollary 3.10. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$ID(G) \geq \frac{1}{2}\left(\frac{(2m-n)^3}{(M_1(G)-2m)^2} + \frac{(2m+n)^3}{(M_1(G)+2m)^2}\right)$$

and

$$^mM_1(G) \geq \frac{1}{2}\left(\frac{(2m-n)^4}{(M_1(G)-2m)^3} + \frac{(2m+n)^4}{(M_1(G)+2m)^3}\right).$$

Equalities hold if and only if G is a regular graph.

The proof of the next result is fully analogous to that of Theorems 3.1 and 3.7, and hence omitted.

Theorem 3.11. *Let G be a simple connected graph with $m \geq 2$ edges. Then for any real α, such that $\alpha \leq 2$ or $\alpha \geq 3$, holds*

$$^0R_\alpha(G) \geq \max\left\{\frac{(F(G)-M_1(G))^{\alpha-2}}{(M_1(G)-2m)^{\alpha-3}} + {}^0R_{\alpha-1}(G), \frac{(F(G)+M_1(G))^{\alpha-2}}{(M_1(G)+2m)^{\alpha-3}} - {}^0R_{\alpha-1}(G)\right\}.$$

If $2 \leq \alpha \leq 3$, *then*

$$ {}^{0}\mathcal{R}_{\alpha}(G) \leq \min\left\{ \frac{(F(G)-M_1(G))^{\alpha-2}}{(M_1(G)-2m)^{\alpha-3}} + {}^{0}\mathcal{R}_{\alpha-1}(G), \frac{(F(G)+M_1(G))^{\alpha-2}}{(M_1(G)+2m)^{\alpha-3}} - {}^{0}\mathcal{R}_{\alpha-1}(G) \right\}. $$

Equalities hold if and only if either $\alpha = 2$, *or* $\alpha = 3$, *or* G *is regular, or when* $d_{k+1} = d_{k+2} = \cdots = d_n = 1$ *and* $d_1 = d_2 = \cdots = d_k > 1$, *where* $1 \leq k \leq n-1$.

Corollary 3.12. *Let* G *be a simple connected graph with* $m \geq 2$ *edges. Then for any real* α *such that* $\alpha \leq 2$ *or* $\alpha \geq 3$ *holds*

$$ {}^{0}\mathcal{R}_{\alpha}(G) \geq \frac{1}{2}\left(\frac{(F(G)-M_1(G))^{\alpha-2}}{(M_1(G)-2m)^{\alpha-3}} + \frac{(F(G)+M_1(G))^{\alpha-2}}{(M_1(G)+2m)^{\alpha-3}} \right). $$

If $2 \leq \alpha \leq 3$, *then the sense of the above inequality reverses. Equality holds if and only if* $\alpha = 2$ *or* $\alpha = 3$ *or* G *is a regular graph.*

In the next theorems we determine relations between invariants ${}^{0}\mathcal{R}_{\alpha}(G)$, ${}^{0}\mathcal{R}_{\beta}(G)$ and ${}^{0}\mathcal{R}_{2\alpha-\beta}(G)$, where α and β are arbitrary real numbers.

Theorem 3.13. *Let* G *be a simple connected graph with* $n \geq 3$ *vertices. Then for any real numbers* α *and* β *hold*

$$ {}^{0}\mathcal{R}_{\alpha}(G) \leq \Delta^{\alpha} + \delta^{\alpha} + \sqrt{\left({}^{0}\mathcal{R}_{2\alpha-\beta}(G) - \Delta^{2\alpha-\beta} - \delta^{2\alpha-\beta}\right)\left({}^{0}\mathcal{R}_{\beta}(G) - \Delta^{\beta} - \delta^{\beta}\right)}, \tag{21} $$

with equality if and only if $\alpha = \beta$ *or* $d_2 = d_3 = \cdots = d_{n-1}$.

Proof. The inequality (2) can be considered as

$$ \sum_{i=2}^{n-1} \frac{x_i^2}{a_i} \geq \frac{\left(\sum_{i=2}^{n-1} x_i\right)^2}{\sum_{i=2}^{n-1} a_i}. $$

For $x_i = d_i^{\alpha}$, $a_i = d_i^{\beta}$, $i = 2, 3, \ldots, n-1$, where α and β are arbitrary real numbers, this inequality becomes

$$ \sum_{i=2}^{n-1} d_i^{2\alpha-\beta} \geq \frac{\left(\sum_{i=2}^{n-1} d_i^{\alpha}\right)^2}{\sum_{i=2}^{n-1} d_i^{\beta}}, $$

i.e.

$$ {}^{0}\mathcal{R}_{2\alpha-\beta}(G) - \Delta^{2\alpha-\beta} - \delta^{2\alpha-\beta} \geq \frac{\left({}^{0}\mathcal{R}_{\alpha}(G) - \Delta^{\alpha} - \delta^{\alpha}\right)^2}{{}^{0}\mathcal{R}_{\beta}(G) - \Delta^{\beta} - \delta^{\beta}}, \tag{22} $$

wherefrom (21) is obtained.

Equality in (22), and hence in (21), is attained if and only if $\alpha = \beta$ or $d_2 = d_3 = \cdots = d_{n-1}$. □

By similar arguments as in case of Theorem 3.13, the following results can be proven.

Theorem 3.14. *Let* G *be a simple connected graph with* $n \geq 2$ *vertices. Then for any real numbers* α *and* β *hold*

$$ {}^{0}\mathcal{R}_{\alpha}(G) \leq \Delta^{\alpha} + \sqrt{\left({}^{0}\mathcal{R}_{2\alpha-\beta}(G) - \Delta^{2\alpha-\beta}\right)\left({}^{0}\mathcal{R}_{\beta}(G) - \Delta^{\beta}\right)}. $$

Equality holds if and only if $\alpha = \beta$ *or* $d_2 = d_3 = \cdots = d_n$.

Theorem 3.15. *Let G be a simple connected graph with n vertices. Then for any real numbers α and β hold*

$$^{0}R_{\alpha}(G) \leq \sqrt{^{0}R_{2\alpha-\beta}(G)\,^{0}R_{\beta}(G)}.$$

Equality holds if and only if $\alpha = \beta$ or G is a regular graph.

By the appropriate choices of parameters α and β, from Theorems 3.13, 3.14 and 3.15 a number of old/new inequalities for different vertex–degree–based topological indices of graphs can be obtained. In the next corollary we list some of them.

Corollary 3.16. *Let G be a simple connected graph with n vertices and m edges. Then*

$$\begin{aligned}
^{0}R_{\alpha}(G) &\leq \sqrt{2m\,^{0}R_{2\alpha-1}(G)}\,, \\
^{0}R_{\alpha}(G) &\leq \sqrt{ID(G)\,^{0}R_{2\alpha+1}(G)}\,, \\
^{0}R_{\alpha}(G)^{0}R_{-\alpha}(G) &\geq n^2\,, && (23)\\
^{0}R_{1/2}(G) &\leq \sqrt{2mn}\,, \\
ID(G) &\leq \sqrt{n\,^{m}M_1(G)}\,, && (24)\\
ID(G) &\geq \frac{1}{\Delta}+\frac{1}{\delta}+\frac{(n-2)^2}{2m-\Delta-\delta}\,, && (25)\\
M_1(G) &\geq \Delta^2+\frac{(2m-\Delta)^2}{n-1}\,, && (26)\\
M_1(G) &\geq \Delta^2+\delta^2+\frac{(2m-\Delta-\delta)^2}{n-2}\,, && (27)\\
F(G) &\geq \frac{M_1(G)^2}{2m}\,, && (28)\\
F(G) &\geq \Delta^3+\delta^3+\frac{(M_1(G)-\Delta^2-\delta^2)^2}{2m-\Delta-\delta}\,. && (29)
\end{aligned}$$

The inequality (23) *was proven in* [19], *the inequality* (24) *in* [20], (25) *in* [7], (26) *in* [22], (27) *in* [6], (28) *in* [11]. *The inequality* (23) *was proven in* [13] *for the case $\alpha = 2$.*

In the next theorem we obtain a relation between $^{0}R_{2\alpha}(G)$ and $^{0}R_{\alpha}(G)$.

Theorem 3.17. *Let G be a simple connected graph with $n \geq 3$ vertices. Then for any real number α holds*

$$n\,^{0}R_{2\alpha}(G) - {^{0}R_{\alpha}(G)^2} \geq \frac{n}{2}\,(\Delta^{\alpha}-\delta^{\alpha})^2\,. \tag{30}$$

Equality is attained if and only if $d_2^{\alpha} = d_3^{\alpha} = \cdots = d_{n-1}^{\alpha} = \frac{\Delta^{\alpha}+\delta^{\alpha}}{2}$.

Proof. According to the Lagrange's identity (see e.g. [26]), we have that

$$\begin{aligned}
n\,^{0}R_{2\alpha}(G) - {^{0}R_{\alpha}(G)^2} &= n\sum_{i=1}^{n} d_i^{2\alpha} - \left(\sum_{i=1}^{n} d_i^{\alpha}\right)^2 = \sum_{1\leq i<j\leq n}\left(d_i^{\alpha}-d_j^{\alpha}\right)^2 \\
&= (\Delta^{\alpha}-\delta^{\alpha})^2 + \sum_{\substack{1\leq i<j\leq n\\ (i,j)\neq(1,n)}}\left(d_i^{\alpha}-d_j^{\alpha}\right)^2 && (31)\\
&\geq (\Delta^{\alpha}-\delta^{\alpha})^2 + \sum_{i=2}^{n-1}\left(\left(d_1^{\alpha}-d_i^{\alpha}\right)^2+\left(d_i^{\alpha}-d_n^{\alpha}\right)^2\right).
\end{aligned}$$

Now, setting $n = 2$, $r = 2$, $p_1 = p_2 = 1$, $a_1 = d_1^\alpha - d_i^\alpha$, $a_2 = d_i^\alpha - d_n^\alpha$ in (1) we get

$$(\Delta^\alpha - \delta^\alpha)^2 + \sum_{i=2}^{n-1}\left(\left(d_1^\alpha - d_i^\alpha\right)^2 + \left(d_i^\alpha - d_n^\alpha\right)^2\right) \geq (\Delta^\alpha - \delta^\alpha)^2 + \frac{1}{2}\sum_{i=2}^{n-1}\left(d_1^\alpha - d_n^\alpha\right)^2. \tag{32}$$

The inequality (30) immediately follows from (31) and (32).

Equality in (31) is attained if and only if $d_2^\alpha = d_3^\alpha = \cdots = d_{n-1}^\alpha$. Equality in (32) holds if and only if $d_i^\alpha = \frac{d_1^\alpha + d_n^\alpha}{2}$, for every $i = 2, \ldots, n-1$, therefore equality in (30) is attained if and only if $d_2^\alpha = d_3^\alpha = \cdots = d_{n-1}^\alpha = \frac{\Delta^\alpha + \delta^\alpha}{2}$. □

Corollary 3.18. *Let G be a simple connected graph with $n \geq 3$ vertices. Then*

$$M_1(G) \geq \frac{4m^2}{n} + \frac{1}{2}(\Delta - \delta)^2, \tag{33}$$

$$^mM_1(G) \geq \frac{ID(G)^2}{n} + \frac{1}{2}\left(\frac{1}{\delta} - \frac{1}{\Delta}\right)^2.$$

The inequality (33) *was proven in* [24] (*see also* [22] *and* [4]). *This inequality is stronger than* (14).

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References

[1] A. Ali, L. Zhong, I. Gutman, Harmonic index and its generalizations: extremal results and bounds, MATCH Commun. Math. Comput. Chem., 81 (2019), 249–311.
[2] A. Ali, I. Gutman, E. Milovanović, I. Milovanović, Sum of powers of the degrees of graphs: extremal results and bounds, MATCH Commun. Math. Comput. Chem., 80 (2018), 5–84.
[3] B. Bollobás, P. Erdős, Graphs of extremal weights, Ars Comb., 50 (1998), 225–233.
[4] B. Borovićanin, K. C. Das, B. Furtula, I. Gutman, Zagreb indices: Bounds and extremal graphs, in: I. Gutman, B. Furtula, K. C. Das, E. Milovanović, I. Milovanović (Eds.), Bounds in Chemical Graph Theory – Basics, Univ. Kragujevac, Kragujevac, 2017, pp. 67–153.
[5] D. de Caen, An upper bound on the sum of squares of degrees in a graph, Discr. Math., 185 (1998), 245–248.
[6] K. C. Das, Sharp bounds for the sum of the squares of the degrees of a graph, Kragujevac J. Math., 25 (2003), 31–49.
[7] K. C. Das, K. Xu, J. Wang, On Inverse degree and topological indices of graphs, Filomat, 30 (2016), 2111–2120.
[8] M. Dehmer, F. Emmert–Streib, Y. Shi, Quantitative graph theory: A new branch of graph theory and network science, Inform. Sci., 418 (2017), 575–580.
[9] C. S. Edwards, The largest vertex degree sum for a triangle in a graph, Bull. London Math. Soc., 9 (1977), 203–208.
[10] S. Fajtlowicz, On conjectures of Graffiti-II, Congr. Numer., 60 (1987), 187–197.
[11] B. Furtula, I. Gutman, A forgotten topological index, J. Math. Chem., 53 (2015), 1184–1190.
[12] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, Chem. Phys. Lett., 17 (1972), 535–538.
[13] J. Hao, Theorems about Zagreb indices and modified Zagreb indices, MATCH Commun. Math. Comput. Chem., 65 (2011), 659–670.
[14] Y. Hu, X. Li, Y. Shi, T. Xu, I. Gutman, On molecular graphs with smallest and greatest zeroth–order general Randić index, MATCH Commun. Math. Comput. Chem., 54 (2005), 425–434.
[15] A. Ilić, D. Stevanović, On comparing Zagreb indices, MATCH Commun. Math. Comput. Chem., 62 (2009), 681–687.
[16] A. Khaksari, M. Ghorbani, On the forgotten topological index, Iranian J. Math. Chem., 8(3) (2017), 327–338.
[17] L. B. Kier, L. H. Hall, Molecular connectivity in chemistry and drug research, Academic Press, New York, 1976.
[18] X. Li, J. Zheng, A unified approach to the extremal trees for different indices, MATCH Commun. Math. Comput. Chem., 54 (2005), 195–208.
[19] B. Liu, I. Gutman, On general Randić indices, MATCH Commun. Math. Comput. Chem., 58 (2007), 147–154.
[20] B. Liu, I. Gutman, Estimating the Zagreb and the general Randić indices, MATCH Commun. Math. Comput. Chem., 57 (2007), 617–632.
[21] Y. Ma, S. Cao, Y. Shi, I. Gutman, M. Dehmer, B. Furtula, From the connectivity index to various Randić-type descriptors, MATCH Commun. Math. Comput. Chem., 80 (2018), 85–106.
[22] T. Mansour, M. A. Rostami, E. Suresh, G. B. A. Xavier, New sharp lower bounds for the first Zagreb index, Sci. Publ. State Univ. Novi Pazar, Ser. A: Appl. Math. Inform. Mech., 8(1) (2016), 11–19.

[23] A. Miličević, S. Nikolić, On variable Zagreb indices, Croat. Chem. Acta, 77 (2004), 97–101.
[24] E. I. Milovanović, I. Ž. Milovanović, Sharp bounds for the first Zagreb index and first Zagreb coindex, Miskolc Math. Notes, 16 (2015), 1017–1024.
[25] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, Classical and new inequalities in analysis, Kluwer Academic Publishers, Dordrecht, 1993.
[26] D. S. Mitrinović, P. M. Vasić, Analytic inequalities, Springer Verlag, Berlin-Heidelberg-New York, 1970.
[27] S. Nikolić, G. Kovačević, A. Miličević, N. Trinajstić, The Zagreb indices 30 years after, Croat. Chem. Acta, 76 (2003), 113–124.
[28] J. Radon, Über die absolut additiven Mengenfunktionen, Wiener Sitzungsber., 122 (1913), 1295–1438.
[29] M. Randić, On characterization of molecular branching, J. Am. Chem. Soc., 97 (1975), 6609–6615.
[30] Y. S. Yoon, J. K. Kim, A relationship between bounds on the sum of squares of degrees of a graph, J. Appl. Math. Comput., 21 (2006), 233–238.