Filomat 33:16 (2019), 5273–5290
https://doi.org/10.2298/FIL1916273A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Classification of Warped Product Pointwise Semi-Slant Submanifolds in Complex Space Forms

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Abstract. The main principle of this paper is to show that, a warped product pointwise semi-slant submanifold of type $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ in a complex space form $\widetilde{M}^{2m}(c)$ admitting shrinking or steady gradient Ricci soliton, whose potential function is a well-define warped function, is an *Einstein* warped product pointwise semi-slant submanifold under extrinsic restrictions on the second fundamental form inequality attaining the equality in [4]. Moreover, under some geometric assumption, the connected and compactness with nonempty boundary are treated. In this case, we propose a necessary and sufficient condition in terms of *Dirichlet energy function* which show that a connected, compact warped product pointwise semi-slant submanifold of complex space forms must be a Riemannian product. As more applications, for the first one, we prove that M^n is a trivial compact warped product, when the warping function exist the solution of PDE such as *Euler-Lagrange equation*. In the second one, by imposing boundary conditions, we derive a necessary and sufficient condition in terms of Ricci curvature, and prove that, a compact warped product pointwise semi-slant submanifold M^n of a complex space form, is either a CR-warped product or just a usual Riemannian product manifold. We also discuss some obstructions to these constructions in more details.

1. Introduction and motivations results

At present, the clue of the warped product manifolds plays very important roles not only in differential geometry but also in general relativity theory in physics. For example, Robertson-Walker space-times, asymptotically flat spacetime, Schwarzschild spacetime, and Reissner-Nordstrom spacetime are warped product manifolds [31].

The idea of warped product manifolds with negative curvatures was introduced by Bishop and O'Neill [6]. Assume that $(N_1^{n_1}, g_1)$ and $(N_2^{n_2}, g_2)$ are two Riemannian manifolds and $f : N_1^{n_1} \to (0, \infty)$ is a positive differentiable function on $N_1^{n_1}$. Consider the product manifold $N_1^{n_1} \times N_2^{n_2}$ with canonical projections $\gamma_i : N_1^{n_1} \times N_2^{n_2} \to N_i^{n_i}$ via $\gamma_i(x_1, x_2) = x_i$ $(i = 1, 2)$ for every $p = (x_1, x_2) \in N_1^{n_1} \times N_2^{n_2}$. The warped product $M^n = N_1^{n_1} \times_f N_2^{n_2}$ is the product manifold $N_1^{n_1} \times N_2^{n_2}$ equipped with the Riemannian structure g such that

$$g(X, X) = g_1(\gamma_{1*}(X), \gamma_{1*}(X)) + f^2(\gamma_1(p)) g_2(\gamma_{2*}(X), \gamma_{2*}(X)) \tag{1.1}$$

2010 *Mathematics Subject Classification*. Primary 53C40; Secondary 53C42, 53B25

Keywords. Warped product submanifolds, complex space form, Hamiltonian, critical kinetic energy, gradient Ricci soliton, gradient Ricci curvature and Hessian Tensor.

Received: 22 March 2019; Revisd: 06 August 2019; Accepted: 17 September 2019

Communicated by Mića S. Stanković

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for any tangent vector $X \in \mathfrak{X}(T_pM)$, $p \in M$. Then we have $g = g_1 + f^2 g_2$. In this case, the function f is called a warping function on M^n. The following lemma is a direct consequence of the warped product manifolds:

Lemma 1.1. ([6]) *Let $M^n = N_1^{n_1} \times_f N_2^{n_2}$ be a warped product manifold. For $X, Y \in \mathfrak{X}(TN_1)$ and $Z, W \in \mathfrak{X}(TN_2)$, we have*

(i) $\nabla_X Y \in \mathfrak{X}(TN_1)$,
(ii) $\nabla_Z X = \nabla_X Z = (X \ln f)Z$,
(iii) $\nabla_Z W = \nabla'_Z W - g(Z, W)\nabla \ln f$,

where ∇ and ∇' denote Levi-Civita connections on M and N_2, respectively. Furthermore, $\nabla \ln f$ is the gradient of $\ln f$*, which is defined as:*

$$g(\nabla \ln f, X) = X(\ln f). \tag{1.2}$$

The following remarks are consequences of Lemma 1.1:

Remark 1.1. *A warped product manifold $M^n = N_1^{n_1} \times_f N_2^{n_2}$ is said to be trivial or simply if the warping function f is constant function along $N_1^{n_1}$.*

Remark 1.2. *If $M^n = N_1^{n_1} \times_f N_2^{n_2}$ is a warped product manifold, then $N_1^{n_1}$ is totally geodesic on M^n and $N_2^{n_2}$ is totally umbilical on M^n.*

In the last few decades, the theory of warped product submanifolds has been a magnificent field in almost Hermitian manifolds and almost contact metric manifolds. Specifically, Chen [10] introduced the notion of CR-warped product submanifolds in Kaehler manifolds. Many articles have been written on warped product submanifolds in the different type of structures (see [18] and references therein). Recently, Sahin [41] studied the warped product pointwise semi-slant submanifolds in a Kaehler manifold and obtained the following inequality:

$$||h||^2 \geq 2n_2\Big(\csc^2\theta + \cot^2\theta\Big)||\nabla \ln f||^2, \tag{1.3}$$

whose equality holds if $N_T^{n_1}$ is totally geodesic and $N_\theta^{n_1}$ is totally umbilical in a Kaehler manifold $\widetilde{M}^{2m}$. According to Nash's theorem [36], we know that in a sufficiently high co-dimension, every Riemannian manifold is isometrically immersed in a suitable Euclidean space. In particular, every warped product $N_1^{n_1} \times_f N_2^{n_2}$ can be immersed as a Riemannian submanifold in some Euclidean space. Based on these concepts, many geometers have obtained geometric obstructions for CR-warped product in different ambient space forms (for instance [5, 13, 22]). Motivated by previous studies, a sharp relationship between the warping function and the squared norm of the second fundamental form for a non-trivial warped product pointwise semi-slant submanifold $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ isometrically immersed in a complex space form $\widetilde{M}^{2m}(c)$ with constant holomorphic sectional curvature c satisfies the optimal inequality [4] was established as in the form

$$||h||^2 \geq 2n_2\bigg(||\nabla \ln f||^2 + \frac{n_1 c}{4} - \Delta(\ln f)\bigg). \tag{1.4}$$

The equality holds in (1.4) if and only if $N_T^{n_1}$ is totally geodesic, $N_\theta^{n_2}$ is totally umbilical and M^n is minimal into $\widehat{M}^{2m}(c)$. Some applications were also derived on compact Riemannian submanifold considering the equality case without boundaries. These types of results give a lot of influence to the analysis of physics, in terms of the second fundamental form (see [12, 14]). Therefore, we need to find some classification of such inequality when satisfying equality and Riemannian manifold is compact. Some classifications of different types of inequalities which attain equalities in various ambient space forms can be found in [2, 5, 13, 15, 17, 19, 42].

Therefore, we only consider a non-trivial warped product pointwise semi-slant submanifold of the type $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ isometrically immersed into a complex space form because other types of warped products are trivial in Kaehler manifold. We also consider connected, compact Riemannian submanifolds whose boundaries are non-empty and provided some new necessary and sufficient conditions for warped product pointwise semi-slant submanifolds, which can be reduced to Riemannian product manifolds. Assume that φ is a differential function on M^n. The gradient ∇f of f is given as

$$g(\nabla\varphi, X) = X\varphi, \quad and \;\; \nabla\varphi = \sum_{i=1}^{n} e_i(\varphi)e_i, \tag{1.5}$$

and the Laplacian $\Delta\varphi$ of φ is defined as.

$$\Delta\varphi = \sum_{i=1}^{n}\{(\nabla_{e_i}e_i)\varphi - e_i(e_i(\varphi))\} = -\sum_{i=1}^{n} g(\nabla_{e_i} grad\varphi, e_i). \tag{1.6}$$

Using the Laplacian, the Hessian $Hess(\varphi)$ of φ is defined as.

$$\Delta\varphi = -Trace(Hess(\varphi)) = -\sum_{i=1}^{n} Hess(\varphi)(e_i, e_i), \tag{1.7}$$

$$Hess_{\mathbb{C}}(\varphi)(X, Y) = Hess(\varphi)(X, Y) + Hess(\varphi)(JX, JX) \tag{1.8}$$

where $Hess_{\mathbb{C}}(\varphi)$ is the complex Hessian of φ. We observed that the definition triviality for a warped product implies M^n to be a simply Riemannian product manifold. If we use the warping function $\ln f$ in (1.8), we obtain the following theorem

Theorem 1.1. *Let $\phi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \longrightarrow \widetilde{M}^{2m}(c)$ be an isometric immersion from a warped product pointwise semi-slant manifold $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ into a complex space form $\widetilde{M}^{2m}(c)$. If the following is satisfies*

$$\|h\|^2 \geq 2n_2\left(\frac{c}{4}n_1 + \sum_{i=1}^{d_1} Hess_{\mathbb{C}}(\ln f)(e_i, e_i)\right), \tag{1.9}$$

where $Hess_{\mathbb{C}}(\ln f)$ is the complex Hessian of the warping function $\ln f$ which satisfies Eq. (1.8), then M^n is a trivial warped product pointwise semi-slant submanifold of a complex space form $\widetilde{M}^{2m}(c)$.

If the basic inequality (1.4) attaining equality case, we prove the following theorem.

Theorem 1.2. *Let $\phi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \longrightarrow \widetilde{M}^{2m}(c)$ be an isometric immersion from a warped product pointwise semi-slant $N_T^{n_1} \times_f N_\theta^{n_2}$ into a complex space form $\widetilde{M}^{2m}(c)$ such that the following equality is satisfied for the warped product submanifold M^n:*

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2 = n_2 \sum_{i=1}^{d_1} Hess_{\mathbb{C}}(\ln f)(e_i, e_i) + \frac{n_1 n_2 c}{4}. \tag{1.10}$$

Then, one of the following statements is satisfied for M^n:

(i) The non-trivial warped product pointwise semi-slant manifold $N_T^{n_1} \times_f N_\theta^{n_2}$ is trivial, or simply M^n is a Riemannian product manifold.

(ii) The pointwise slant function is given by $\theta = \cot^{-1}\left(\sqrt{n_2}\right)$ for the warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ in a complex space form $\widetilde{M}^{2m}(c)$.

In [20], Calin and Chang presented the geometric approach to Riemannian manifolds and found applications to partial differential equations which include the Lagrangian formalism on Riemannian manifolds; the energy-momentum tensor and conservation laws; the Hamiltonian formalism; Hamilton-Jacobi theory; harmonic functions and geodesics; and fundamental solutions for heat operators with potential. For a compact submanifold of a Riemannian manifold with a boundary, we have the following theorem:

Theorem 1.3. [20] *Let M^n be a connected and compact Riemannian manifold and f be a positive differentiable function defined on M^n such that $\Delta\varphi = 0$ on M and $\varphi = 0$ on the boundary ∂M of M. Then φ vanishes identically.*

We assume that M^n is a compact Riemannian manifold and f is a positive differentiable function on M^n. Then, the kinetic energy function on M^n is defined as [20]:

$$\mathbf{E}(\varphi) = \frac{1}{2}\int_M \|\nabla\varphi\|^2 \mathrm{d}V, \tag{1.11}$$

where dV denotes the volume element of M^n. Due to the impact of pointwise slant function $\theta : M^n \longrightarrow \mathbb{R}$ in a warped product pointwise semi-slant submanifold $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$, and taking into account of Theorem 1.3 and the kinetic energy formula (1.11), we have the following theorem:

Theorem 1.4. *Assume that $\phi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \longrightarrow \widetilde{M}^{2m}(c)$ is an isometric immersion from a connected and compact warped product pointwise semi-slant manifold $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ into a complex space form $\widetilde{M}^{2m}(c)$. Then $N_T^{n_1} \times_f N_\theta^{n_2}$ is simply a Riemannian product of $N_T^{n_1}$ and $N_\theta^{n_2}$ if and only if the kinetic energy of the warping function satisfies:*

$$\mathbf{E}(\ln f) = \frac{1}{4n_2}\tan^2\theta\left\{\int_M\left(\frac{n_2 n_1 c}{4} - \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\|h_\mu(e_i,e_j)\|^2\right)\mathrm{d}V - 8n_2\int_M\left(\csc^2\theta\cot\theta\Big(\frac{d\theta}{\mathrm{d}V}\Big)\mathbf{E}(\ln f)\right)\mathrm{d}V\right\}, \tag{1.12}$$

where $0 < \mathbf{E}(\ln f) < \infty$ and $\mathrm{d}V$ is the volume element of M^n.

Moreover, the Hamiltonian in a local orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ of T_pM^n, $p \in M$ is defined by

$$H(\mathrm{d}f, p) = \frac{1}{2}\sum_{j=1}^{n} df(e_j)^2 = \frac{1}{2}\sum_{j=1}^{n} e_j(f)^2 = \frac{1}{2}\|\nabla f\|^2. \tag{1.13}$$

Considering the Hamiltonian of the warping function and the Theorem 1.3, we have the following theorem:

Theorem 1.5. *An isometric immersion $\phi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \longrightarrow \widetilde{M}^{2m}(c)$ of a non-trivial connected, compact warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ into a complex space form $\widetilde{M}^{2m}(c)$ is trivial if and only if it satisfies, for a local orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ of T_pM^n, $p \in M$,*

$$H\Big(\mathrm{d}(\ln f), p\Big) = \frac{1}{4n_2}\tan^2\theta\left(\frac{n_2 n_1 c}{4} - \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\|h_\mu(e_i,e_j)\|^2\right). \tag{1.14}$$

Now, we give a motivation for our work which is related to the classification of the Ricci curvature. Many results on the Ricci curvature and gradient Ricci curvature have been obtained during the last decades (see [23, 25, 32, 35, 37] for more details). Furthermore, Ricci solitons are natural extensions of Einstein manifolds and they appear as self-similar solutions of the Ricci flow $\frac{\partial g_{ij}}{\partial(t)} = -2R_{ij}$. Ricci solitons are also important in understanding singularities of the Ricci flow. Since the notion of a Ricci soliton was introduced by Hamilton [32], the gradient expanding, steady or shrinking Ricci soliton have been studied by many geometers (see [8, 9, 21, 26]) in different geometric aspects. A Riemannian metric g on a Riemannian manifold M^n is called a Ricci soliton if a smooth vector field X exists, such that the Ricci curvature satisfies the following condition

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.15}$$

for any constant λ, where $\mathcal{L}$ is the Lie derivative. A Ricci soliton is called expanding, steady and shrinking if $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$, respectively. If we choose $X = \nabla\varphi$ for a smooth function φ defined on M^n, then

M^n is called a gradient Ricci soliton with respect to φ as a potential function. In this case, the fundamental equation can be written as:

$$Ric + \nabla^2\varphi = \lambda g, \tag{1.16}$$

where $\nabla^2\varphi$ denotes the Hessian of φ.

Remark 1.3. *If the potential function φ is a constant function in* (1.16), *then gradient Ricci soliton $(M^n, \lambda, \nabla\varphi)$ is called an Einstein manifold.*

Taking into account the relation between Laplacian Δ and gradient ∇^2 such that $\Delta = \nabla^2$, (1.16) can be modified as

$$Ric + Hess(\varphi) = \lambda g. \tag{1.17}$$

It is also called the fundamental equation of Ricci tensor or gradient soliton in terms of the Hessian tensor. For more classifications, we refer to ([8, 21, 23, 26, 28–30, 32, 33, 35, 45] and references therein). In particular, if the vector field X is the gradient of $(\ln f)$, i.e., $X = \nabla(\ln f)$ in (1.15), we present a very interesting result where the gradient Ricci soliton decomposed with warped product submanifolds in complex space form as follows:

Theorem 1.6. *Let $\phi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \longrightarrow \widetilde{M}^{2m}(c)$ be an isometric immersion from a warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ into a complex space form $\widetilde{M}^{2m}(c)$ admitting shrinking gradient Ricci soliton such that the warping function being a soliton function. If we assume that $\theta \neq arc\cot\sqrt{n_2}$, then a warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ is an Einstein warped product pointwise semi-slant submanifold if and only if the following equation is satisfied:*

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} ||h_\mu(e_i, e_j)||^2 + 2n_2R^T = \Big(\lambda + \frac{c}{4}\Big)n_1n_2, \tag{1.18}$$

for a positive constant $\lambda \in \mathbb{R}$ and R^T is scalar curvature of $N_T^{n_1}$.

Another goal of our inequality (1.4) is to provide potential applications to the gradient Ricci curvature for a compact Riemannian manifold. Taking into account the Green Theorem (see [44] for more detail), we obtain the following theorem:

Theorem 1.7. *Assume that $\phi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \longrightarrow \widetilde{M}^{2m}(c)$ is an isometric immersion of a compact warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ into a complex space form $\widetilde{M}^{2m}(c)$. If the following equality is satisfied for the warped product submanifold M^n*

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} ||h_\mu(e_i, e_j^*)||^2 = \frac{n_1n_2c}{4} + n_2\int_M \mathcal{R}ic(\nabla \ln f, \bullet)\mathrm{dV}, \tag{1.19}$$

then, at least one of the following statements is true for M^n:

(i) A warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ of a complex space form $\widetilde{M}^{2m}(c)$ is a CR-warped product submanifold.

(ii) A warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ of a complex space form $\widetilde{M}^{2m}(c)$ is simply a Riemannian product of $N_T^{n_1}$ and $N_\theta^{n_2}$.

Moreover, we will provide some new results related with the above theorems.

2. Preliminaries

Let $(\widetilde{M}, J, g)$ be a $2m$-dimensional Hermitian manifold with an almost complex structure J and a Riemannian metric g, which satisfy $J^2 = -I$ and $g(JU, JV) = g(U, V)$, for vector fields $U, V \in \mathfrak{X}(T\widetilde{M})$, where $T\widetilde{M}$ denotes the tangent bundle on $\widetilde{M}$. Let $\widetilde{\nabla}$ be the covariant differential operator on $\widetilde{M}^{2m}$ with respect to g. If the almost complex structure J satisfies $(\widetilde{\nabla}_U J)V = 0$, for any $U, V \in \mathfrak{X}(T\widetilde{M})$, then an almost Hermitian manifold $(\widetilde{M}, J, g)$ is called a Kaehler manifold according to Yano and Kon [43].

Assume that M^n be isometrically immersed into an almost Hermitian manifold $\widetilde{M}^{2m}$ with the induced metric g. If ∇ and $\nabla^{\perp}$ are the induced Riemannian connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M^n, respectively, then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.1}$$

$$\widetilde{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.2}$$

for each $U, V \in \mathfrak{X}(TM)$ and $N \in \mathfrak{X}(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively. We have $g(h(U, V), N) = g(A_N U, V)$, for each $U, V \in \mathfrak{X}(TM)$. Now, for any $U \in \mathfrak{X}(TM)$ and $N \in \mathfrak{X}(T^{\perp}M)$, we have

$$(i)\ JU = TU + FU, \quad (ii)\ JN = tN + fN, \tag{2.3}$$

where $TU(tN)$ and $FU(fN)$ are the tangential and normal components of $JU(JN)$, respectively. If T is identically zero, then the submanifold M^n is called a totally real submanifold. The Gauss equation for a submanifold M^n is following as:

$$\widetilde{R}\Big(X, Y, Z, W\Big) = R(X, Y, Z, W) + g\Big(h(X, Z), h(Y, W)\Big) - g\Big(h(X, W), h(Y, Z)\Big), \tag{2.4}$$

for any $X, Y, Z, W \in \mathfrak{X}(TM)$, where $\widetilde{R}$ and R are the curvature tensors on $\widetilde{M}^{2m}$ and M^n, respectively. Assuming that $\widetilde{M}^{2m}(c)$ is a complex space form of constant holomorphic sectional curvature c. Then the curvature tensor $\widetilde{R}$ of $\widetilde{M}^{2m}(c)$ can be expressed as

$$\widetilde{R}(X, Y)Z = \frac{c}{4}\Bigg(g(X, Z)Y - g(Y, Z)X + g(X, JZ)JY - g(Y, JZ)X + 2g(X, JY)JZ\Bigg). \tag{2.5}$$

The mean curvature vector H for an orthonormal frame $\{e_1, e_2, \cdots e_n\}$ of tangent space TM on M^n is defined by

$$H = \frac{1}{n} trace(h) = \frac{1}{n} \sum_{i=1}^{n} h\Big(e_i, e_i\Big), \tag{2.6}$$

where $n = dimM$. In addition, we set

$$h^r_{ij} = g(h(e_i, e_j), e_r) \ \ and \ \ ||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.7}$$

If J preserves any tangent space of M^n, that is, $J(T_x M) \subseteq T_x M$, for each $x \in M^n$, then M^n is called a *holomorphic* submanifold. Similarly, M is called a *totally real* submanifold if J maps any tangent space of M^n into normal space, that is, $J(T_x M) \subseteq T^{\perp}M$, for each $x \in M^n$. Now, we give the following definition:

Definition 2.1. [10] *A Riemannian submanifold M^n of a Kaehler manifold $\widetilde{M}^{2m}$ is said to be a CR-submanifold if a pair of orthogonal distributions $\mathcal{D}^T$ and $\mathcal{D}^{\perp}$ exists such that*

(i) $TM = \mathcal{D}^T \oplus \mathcal{D}^{\perp}$,

(ii) the distribution $\mathcal{D}^T$ is holomorphic, that is, $J(\mathcal{D}^T) \subseteq \mathcal{D}^T$, and
(iii) the distribution $\mathcal{D}^\perp$ is totally real, that is, $J\mathcal{D}^\perp \subseteq (T^\perp M)$.

If d_1 and d_2 are the dimensions of a holomorphic distribution $\mathcal{D}^T$ and a totally real distribution $\mathcal{D}^\perp$ of a CR-submanifold of an almost Hermitian manifold $\widetilde{M}^{2m}$, then M^n is holomorphic if $d_2 = 0$, and totally real if $d_1 = 0$. It is called a proper CR-submanifold if neither $d_1 = 0$ nor $d_2 = 0$. Moreover, if μ is a holomorphic subspace under J of a normal bundle $T^\perp M$, then, for a CR-submanifold, the normal bundle $T^\perp M$ can be decomposed as

$$T^\perp M = J\mathcal{D}^\perp \oplus \mu.$$

A pointwise slant submanifold has been studied in almost Hermitian manifolds by Chen-Gray [16]. They defined as follows:

Definition 2.2. *[16] Assume that the set T^*M consists of all non-zero tangent vectors on a submanifold M^n of an almost Hermitian manifold $\widetilde{M}^{2m}$. Then, for any non-zero vector $X \in \mathfrak{X}(T_xM)$, $x \in M^n$, the angle $\theta(X)$ between JX and tangent space T_xM is called the Wirtinger angle of X. The Wirtinger angle become a real-valued function as $\theta : T^*M \to \mathbb{R}$, called the* **Wirtinger function (slant function)**. *In this case, the submanifold M^n of almost Hermitian manifolds $\widetilde{M}^{2m}$ is called a pointwise slant submanifold.*

Definition 2.3. *A point x in a pointwise slant submanifold is called a totally real point if the pointwise slant function θ satisfies $\cos\theta = 0$, at x. In the same way, a point x is called a complex point if the pointwise slant function satisfies $\sin\theta = 0$ at x.*

Definition 2.4. *If every point on M^n of almost Hermitian manifold $\widetilde{M}^{2m}$ is a totally real point, then the pointwise slant submanifold M^n is called a totally real submanifold. Similarly, if every point on M^n is a complex point, then M^n is said to be a complex submanifold.*

The following characterization theorem was derived by Chen-Gray in [16]:

Theorem 2.1. *Let M^n be a submanifold of an almost Hermitian manifold $\widetilde{M}^{2m}$. Then M^n is a pointwise slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = -\lambda I. \tag{2.8}$$

Furthermore, θ is a real-valued function defined on the tangent bundle TM, and satisfies $\lambda = \cos^2\theta$.

Note that, for a pointwise slant submanifold M^n of an almost Hermitian manifold $\widetilde{M}^{2m}$, Theorem 2.1 gives the follwoing relations:

$$g(TU, TV) = \cos^2\theta g(U, V), \tag{2.9}$$

$$g(FU, FV) = \sin^2\theta g(U, V). \tag{2.10}$$

for any $U, V \in \mathfrak{X}(TM)$.

The idea of pointwise semi-slant submanifolds as a natural generalization of CR-submanifolds of an almost Hermitian manifold in terms of a semi-slant function was defined and studied by Sahin [41] as follows:

Definition 2.5. [41] *A Riemannian submanifold M^n of a Kaehler manifold $\widetilde{M}^{2m}$ is said to be a pointwise semi-slant submanifold if there exist two orthogonal distributions $\mathcal{D}^T$ and $\mathcal{D}^\theta$ such that*

(i) $TM = \mathcal{D}^T \oplus \mathcal{D}^\theta$,
(ii) the distribution $\mathcal{D}^T$ is a complex (holomorphic) distribution, that is, $J(\mathcal{D}^T) \subseteq \mathcal{D}^T$, and

(iii) the distribution $\mathcal{D}^\theta$ is a pointwise slant distribution with a pointwise slant function $\theta : M^n \to \mathbb{R}$.

For some examples of pointwise semi-slant submanifold in a Kaehler manifold and related problems, we refer to [7, 38, 41].

Let us denote p and q as dimensions of the *complex* distribution $\mathcal{D}^T$ and the *pointwise slant* distribution $\mathcal{D}^\theta$ of a pointwise semi-slant submanifold in a Kaehler manifold $\widetilde{M}^{2m}$, respectively. Then the following remarks hold.

Remark 2.1. *M^n is invariant if $q = 0$ and pointwise slant if $p = 0$.*

Remark 2.2. *If the slant function $\theta : M^n \to \mathbb{R}$ is globally constant on M^n and $\theta = \frac{\pi}{2}$, then M^n is called a CR-submanifold.*

Remark 2.3. *If the slant function $\theta : M^n \to (0, \frac{\pi}{2})$, then M^n is called a proper pointwise semi-slant submanifold.*

Remark 2.4. *If μ is an invariant subspace under J of the normal bundle $T^\perp M$, then the normal bundle $T^\perp M$ can be decomposed as $T^\perp M = F\mathcal{D}^\theta \oplus \mu$ in the case of a pointwise semi-slant submanifold.*

3. Non-trivial warped product pointwise semi-slant submanifolds $N_T^{n_1} \times_f N_\theta^{n_2}$ into a complex space form

It is well known that if two factors of a warped product submanifold are holomorphic and pointwise slant submanifolds, then it is called a warped product pointwise semi-slant submanifold of almost Hermitian manifolds. There are two types of warped product pointwise semi-slant submanifolds of a Kaehler manifold:

(i) $N_\theta^{n_2} \times_f N_T^{n_1}$, *and* (ii) $N_T^{n_1} \times_f N_\theta^{n_2}$.

For the first case, we recall the following theorem:

Theorem 3.1. [41] *There is a no proper warped product pointwise semi-slant submanifold $M^n = N_\theta^{n_2} \times_f N_T^{n_1}$ in a Kaehler manifold $\widetilde{M}^{2m}$ such that $N_\theta^{n_2}$ is a proper pointwise slant submanifold and $N_T^{n_1}$ is a holomorphic submanifold of $\widetilde{M}^{2m}$.*

Before proceeding to the second case, we recall that the following result:

Lemma 3.1. [41] *Let $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ be a warped product pointwise semi-slant submanifold of a Kaehler manifold $\widetilde{M}^{2m}$, where $N_T^{n_1}$ and $N_\theta^{n_2}$ are holomorphic and pointwise slant submanifolds of $\widetilde{M}^{2m}$, respectively. Then*

$$g(h(X,Z), FTW) = -(JX\ln f)g(Z,TW) - (X\ln f)\cos^2\theta g(Z,W), \tag{3.1}$$

$$g(h(Z,JX), FW) = (X\ln f)g(Z,W) - (JX\ln f)g(Z,TW), \tag{3.2}$$

$$g(h(X,Y), FZ) = 0, \tag{3.3}$$

for any $X, Y \in \mathfrak{X}(TN_T)$ and $Z, W \in \mathfrak{X}(TN_\theta)$.

The first author in [4] proved the following theorem:

Theorem 3.2. [4] *Let $\varphi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \longrightarrow \widetilde{M}^{2m}$ be isometrically immersed from a warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ into a Kaehler manifold $\widetilde{M}^{2m}$. Then $N_T^{n_1}$ is always a minimal submanifold of $\widetilde{M}^{2m}$.*

Theorem 3.3. [1] *Let φ be a $\mathcal{D}^\theta$−minimal isometric immersion of a warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ into a Kaehler manifold $\widetilde{M}^{2m}$. If $N_\theta^{n_2}$ is totally umbilical in $\widetilde{M}^{2m}$, then φ is a $N_\theta^{n_2}$−totally geodesic.*

Theorem 3.2 and Theorem 3.3 were extended into complex space forms, and also they briefly describe a method to prove the triviality for both the inequality and the equality in Eq.(1.4) holds for a compact Riemannian submanifold without the boundary.

Theorem 3.4. [4] *On a compact orientated warped product pointwise semi-slant submanifold $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ in a complex space form $\widetilde{M}^{2m}(c)$, if the following inequality holds:*

$$\|h\|^2 \geq \frac{n_1 n_2 c}{2}, \tag{3.4}$$

where n_1 and n_2 are dimensions of $N_T^{n_1}$ and $N_\theta^{n_2}$, respectively, then M^n is simply a Riemannian product manifold.

For the equality case of inequality (1.4), the following result was proven:

Theorem 3.5. [4] *Let $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ be a compact orientated warped product pointwise semi-slant submanifold in a complex space form $\widetilde{M}^{2m}(c)$. Then M^n is simply a Riemannian product manifold if and only if*

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2 = \frac{n_1 n_2 c}{4}, \tag{3.5}$$

where h_μ is a component of h in $\Gamma(\mu)$.

Now, we give direct consequences of the inequality (1.4), from Theorem 3.4 and Theorem 3.5, as follows:

Remark 3.1. *If we consider $\theta = \frac{\pi}{2}$ in Theorem 3.4 and Theorem 3.5, then these theorems generalize the results for CR-warped products into a complex space form $\widetilde{M}^{2m}(c)$.*

Corollary 3.1. *Let $M^n = N_T^{n_1} \times_f N_\perp^{n_2}$ be a compact orientated CR-warped product submanifold into a complex space form $\widetilde{M}^{2m}(c)$. Then, M^n is a CR-product submanifold if and only if*

$$\|h\|^2 \geq \frac{n_2 n_1 c}{2},$$

where n_1 and n_2 are dimensions of $N_T^{n_1}$ and $N_\perp^{n_2}$, respectively.

Corollary 3.2. *Assume that $M^n = N_T^{n_1} \times_f N_\perp^{n_2}$ is a compact orientated CR-warped product submanifold in a complex space form $\widetilde{M}^{2m}(c)$ such that $N_T^{n_1}$ is holomorphic and $N_\perp^{n_2}$ is totally real in $\widetilde{M}^{2m}(c)$. Then, M^n is simply a Riemannian product submanifold if and only if*

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2 = \frac{n_2 n_1 c}{4}.$$

Based on the Laplacian property of a positive differential function defined on any compact Riemannian manifold, we obtain the following corollary by using Eqs. (1.4):

Theorem 3.6. *Assume that $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ is a warped product pointwise semi-slant submanifold in a complex space form $\widetilde{M}^{2m}(c)$, and let $N_T^{n_1}$ be a compact invariant submanifold and λ be a non-zero eigenvalue of the Laplacian on $N_T^{n_1}$. Then*

$$\int_{N_T^{n_1}} \|h\|^2 dV_T \geq \int_{N_T^{n_1}} \Big(\frac{c}{2} n_2 n_1\Big) dV_T + 2n_2\lambda \int_{N_T^{n_1}} (\ln f)^2 dV_T, \tag{3.6}$$

where dV_T is the volume element on ${N_T}^{n_1}$. Moreover, the equality sign of (3.6) holds if and only if we have

(i) $\Delta \ln f = \lambda \ln f$.

(ii) In the warped product pointwise semi-slant submanifold, both $N_T^{n_1}$ and $N_\theta^{n_2}$ are totally geodesic.

Proof. Assume that f is a non-constant function. By using the minimum principle property on the eigenvalue λ, we have

$$\int_{N_T^{n_1}} \|\nabla \ln f\|^2 dV_T \geq \lambda \int_{N_T^{n_1}} (\ln f)^2 dV_T, \tag{3.7}$$

with the equality holding if and only if one has $\Delta \ln f = \lambda \ln f$. From (A) and (3.7), we get the required result (3.6), and the proof completes. □

Corollary 3.3. *Let $\ln f$ be a harmonic function on $N_T^{n_1}$. Then, any warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ into a complex space form $\widetilde{M}^{2m}(c)$ with $c \leq 0$ does not exist.*

Proof. Let $N_T^{n_1} \times_f N_\theta^{n_2}$ be a warped product pointwise semi-slant submanifold in a complex space form $\widetilde{M}^{2m}(c)$ such that $\ln f$ is a harmonic function on $N_T^{n_1}$. Then, from the inequality (A), one obtains $c > 0$ and this completes the proof of the corollary. □

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Corollary 3.4. *A warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ into a complex space form $\widetilde{M}(c)$ with $c \leq 0$ such that $\ln f$ be a positive eigenfunction of the Laplacian on $N_T^{n_1}$ corresponding to an eigenvalue $\lambda \geq 0$ does not exist.*

4. Necessary and sufficient conditions for a warped product pointwise semi-slant submanifold of a complex space form to be a Riemannian product manifold

The following study is devoted to proving that a warped product pointwise semi-slant submanifold isometrically immersed into a complex space form is a trivial warped product submanifold by using inequality (1.4). As immediate consequences, we obtained complete classifications of warped product submanifolds in terms of various mathematical tools.

4.1. Consequences to the Hessian of warping functions

In this section, we find some fundamental applications in terms of the Hessian of a positive differentiable function. In this direction, we derive some necessary and sufficient conditions under which a warped product pointwise semi-slant submanifold isometrically immersed submanifold into a complex space form becomes a Riemannian product manifold.

Proof of Theorem 1.1.

From (1.6), one obtains

$$\Delta(\ln f) = -\sum_{i=1}^{n} g\Big(\nabla_{e_i} grad \ln f, e_i\Big) = -\sum_{i=1}^{2d_1} g\Big(\nabla_{e_i} grad \ln f, e_i\Big) + \sum_{j=1}^{2d_2} g\Big(\nabla_{e_j} grad \ln f, e_j\Big).$$

Then, one obtains

$$\Delta(\ln f) = -\sum_{i=1}^{d_1} g\Big(\nabla_{e_i} grad \ln f, e_i\Big) - \sum_{i=1}^{d_1} g\Big(\nabla_{Je_i} grad \ln f, Je_i\Big) - \sum_{j=1}^{d_2} g\Big(\nabla_{e_j} grad \ln f, e_j\Big)$$
$$- \sec^2\theta \sum_{i=1}^{d_2} g\Big(\nabla_{Te_j} grad \ln f, Te_j\Big).$$

Taking into account that ∇ is a Levi-Civita connection on M^n and (1.7), we derive

$$\Delta(\ln f) = -\sum_{i=1}^{d_1}\Big(Hess(\ln f)(e_i,e_i) + Hess(\ln f)(Je_i,Je_i)\Big) - \sum_{j=1}^{d_2}\Big(e_j g\big(grad\ln f, e_j\big) - g\big(\nabla_{e_j}e_j, grad\ln f\big)\Big).$$
$$- \sec^2\theta\sum_{j=1}^{d_2}\Big(Te_j g\big(grad\ln f, Te_j\big) - g\big(\nabla_{Te_j}Te_j, grad\ln f\big)\Big).$$

From the property of the gradient function (1.5), we get

$$\Delta(\ln f) = -\sum_{i=1}^{d_1}\Big(Hess(\ln f)(e_i,e_i) + Hess(\ln f)(Je_i,Je_i)\Big) - \sum_{j=1}^{d_2}\Big(e_j(e_j\ln f) - (\nabla_{e_j}e_j\ln f)\Big)$$
$$- \sec^2\theta\sum_{j=1}^{d_2}\Big(Te_j(Te_j\ln f) - (\nabla_{Te_j}Te_j\ln f)\Big).$$
$$= -\sum_{i=1}^{d_1}\Big(Hess(\ln f)(e_i,e_i) + Hess(\ln f)(Je_i,Je_i)\Big) - \sum_{j=1}^{d_2}\Big(e_j\big(\frac{g(grad f, e_j)}{f}\big) - \frac{1}{f}g(\nabla_{e_j}e_j, grad f)\Big)$$
$$- \sec^2\theta\sum_{j=1}^{d_2}\Big(Te_j\big(\frac{g(grad f, Te_j)}{f}\big) - \frac{1}{f}g(\nabla_{Te_j}Te_j, grad f)\Big).$$

From the hypothesis of a warped product pointwise semi-slant submanifold, $N_T^{n_1}$ is totally geodesic in M^n. It implies that $grad f \in \mathfrak{X}(TN_T)$, and from Lemma 1.1(ii), we obtain

$$\Delta(\ln f) = -\sum_{i=1}^{d_1}\Big(Hess(\ln f)(e_i,e_i) + Hess(\ln f)(Je_i,Je_i)\Big) - \sum_{j=1}^{d_2}\Big(g(e_j,e_j)\|\nabla\ln f\|^2 + \sec^2\theta g(Te_j,Te_j)\|\nabla\ln f\|^2\Big).$$

Finally, from (2.9), we get:

$$\Delta(\ln f) = -\sum_{i=1}^{d_1}\Big(Hess(\ln f)(e_i,e_i) + Hess(\ln f)(Je_i,Je_i)\Big) - n_2\|\nabla\ln f\|^2. \tag{4.1}$$

Thus from (1.4), (4.1) and (1.8), it follows that

$$\|h\|^2 \geq 2n_2\Big(\frac{c}{4}n_1 + (n_2+1)\|\nabla\ln f\|^2 + \sum_{i=1}^{d_1}Hess_{\mathbb{C}}(\ln f)(e_i,e_i)\Big). \tag{4.2}$$

If the inequality (1.9) holds, then (4.2) implies that $\|\nabla\ln f\|^2 \leq 0$, which is impossible. Therefore, we can conclude that $grad\ln f = 0$, and so f is a constant function on M^n. Hence, M^n becomes a trivial warped product pointwise semi-slant submanifold.

Proof of Theorem 1.2.

Assume that the equality holds in the inequality (1.4). Then, we have

$$\frac{n_1n_2c}{2} - 2n_2\Delta(\ln f) + 2n_2\|\nabla\ln f\|^2 = \|h\|^2.$$

By the definition of the components $\mathcal{D}$ and $\mathcal{D}^\theta$, the above equation can be expressed as:

$$\frac{n_1n_2c}{2} - 2n_2\Delta(\ln f) + 2n_2\|\nabla\ln f\|^2 = \|h(\mathcal{D},\mathcal{D})\|^2 + \|h(\mathcal{D}^\theta,\mathcal{D}^\theta)\|^2 + 2\|h(\mathcal{D},\mathcal{D}^\theta)\|^2. \tag{4.3}$$

Assume that $M = N_T^{n_1} \times_f N_\theta^{n_2}$ be an $n = n_1 + n_2$ dimensional warped product pointwise semi-slant submanifold of $2m$-dimensional Kaehler manifold $\widetilde{M}$ with $\dim(N_T^{n_1}) = n_1 = 2d_1$ and $\dim(N_\theta^{n_2}) = n_2 = 2d_2$, where $N_\theta^{n_2}$ and $N_T^{n_1}$ are integral manifolds of $\mathcal{D}^\theta$ and $\mathcal{D}$, respectively. Thus, we consider the $\{e_1, e_2, \cdots e_{d_1}, e_{d_1+1} = Je_1, \cdots e_{2d_1} = Je_{d_1}\}$ and $\{e_{2d_1+1} = e_1^*, \cdots e_{2d_1+d_2} = e_{d_2}^*, e_{2d_1+d_2+1} = e_{d_2+1}^* = \sec\theta Te_1^*, \cdots e_{n_1+n_2} = e_{n_2}^* = \sec\theta Te_{d_2}^*\}$ to be orthonormal frames of TN_T and TN_θ, respectively. Thus the orthonormal frames of the normal subbundles $F\mathcal{D}^\theta$ and μ are $\{e_{n+1} = \bar{e}_1 = \csc\theta Fe_1^*, \cdots e_{n+d_2} = \bar{e}_{d_2} = \csc\theta Fe_1^*, e_{n+d_2+1} = \bar{e}_{d_2+1} = \csc\theta\sec\theta FTe_1^*, \cdots, e_{n+2d_2} = \bar{e}_{2d_2} = \csc\theta\sec\theta FTe_{d_2}^*\}$ and $\{e_{n+2d_2+1}, \cdots e_{2m}\}$, respectively. According to the above orthonormal frames and using (2.7), Eq. (4.3) takes the new form

$$\frac{n_1n_2c}{2} - 2n_2\Delta(\ln f) + 2n_2\|\nabla\ln f\|^2 = \sum_{r=1}^{2m}\sum_{i,j=1}^{2d_1} g\Big(h(e_i, e_j), e_r\Big)^2 + \sum_{r=1}^{2m}\sum_{i,j=1}^{2d_2} g\Big(h(e_i^*, e_j^*), e_r\Big)^2 + 2\sum_{r=1}^{2m}\sum_{i=1}^{2d_1}\sum_{j=1}^{2d_2} g\Big(h(e_i, e_j^*), e_r\Big)^2. \tag{4.4}$$

On the other hand, the equality of inequality (1.4), implies that M^n is a minimal submanifold of $\widetilde{M}^{2m}(c)$. Theorem 3.2 and Theorem 3.3 imply $h(e_i, e_j) = h(e_t^*, e_s^*) = 0$, for $1 \le i, j \le 2d_1, 1 \le t, s \le 2d_2$. Applying these facts in Eq. (4.4), we get

$$\frac{n_1n_2c}{2} + 2n_2\|\nabla\ln f\|^2 = 2\sum_{r=1}^{2d_2}\sum_{i=1}^{2d_1}\sum_{j=1}^{2d_2} g\Big(h(e_i, e_j^*), \bar{e}_r\Big)^2 + 2\sum_{r=n+2d_2+1}^{2m}\sum_{i=1}^{2d_1}\sum_{j=1}^{2d_2} g\Big(h(e_i, e_j^*), e_r\Big)^2 + 2n_2\Delta(\ln f). \tag{4.5}$$

Using Lemma 3.1 (for example see [41]) in the first term of the right hand side of the Eq. (4.5), we derive

$$\frac{n_1n_2c}{4} + n_2\|\nabla\ln f\|^2 = n_2\Big(1 + 2\cot^2\theta\Big)\|\nabla\ln f\|^2 + \sum_{i=1}^{2d_1}\sum_{j=1}^{2d_2}\|h_\mu(e_i, e_j^*)\|^2 + n_2\Delta(\ln f),$$

which implies that

$$\frac{n_1n_2c}{4} = 2n_2\cot^2\theta\|\nabla\ln f\|^2 + n_2\Delta(\ln f) + \sum_{i=1}^{2d_1}\sum_{j=1}^{2d_2}\|h_\mu(e_i, e_j^*)\|^2. \tag{4.6}$$

From (4.6) and (4.1), we derive

$$\frac{1}{n_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\|h_\mu(e_i, e_j)\|^2 = \sum_{i=1}^{d_1}\Big\{Hess(\ln f)(e_i, e_i) + Hess(\ln f)(Je_i, Je_i)\Big\} + 2\Big(n_2 - \cot^2\theta\Big)\|\nabla\ln f\|^2 + \frac{n_1c}{4}. \tag{4.7}$$

By the hypothesis, if (1.10) holds, then (4.7) indicate that $\Big(n_2 - 2\cot^2\theta\Big)\|\nabla\ln f\|^2 = 0$, which implies that either $\|\nabla\ln f\|^2 = 0$ or $\Big(n_2 - 2\cot^2\theta\Big) = 0$. For the first case if $\|\nabla\ln f\|^2 = 0$, then f is a constant function on M^n (that is, M^n is simply a Riemannian product of $N_T^{n_1}$), and hence $N_\theta^{n_2}$ (M^n is trivial) proves the first statement (i). Similarly, the condition $\Big(n_2 - 2\cot^2\theta\Big) = 0$, proves the second statement (ii) of the Theorem. This completes the proof of the theorem.

Theorem 4.1. *There is no a warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ into a complex space form $\widetilde{M}^{2m}(c)$ with $c \le 0$ such that $N_T^{n_1}$ is a compact holomorphic submanifold and $N_\theta^{n_2}$ is a pointwise slant submanifold of $\widetilde{M}^{2m}(c)$.*

Proof. Assume that a warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ in a complex space form $\widetilde{M}^{2m}(c)$ with $c \leq 0$ exists such that $N_T^{n_1}$ is compact. Then, the function $\ln f$ has an absolute maximum at some point $p \in N_T^{n_1}$. At this critical point, the Hessian $Hess(\ln f)$ is non-positive definite. Thus, (1.9) leads to a contradiction. This completes the proof of the theorem. □

Theorem 4.2. *Assume that $M^n = N_T^{n_1} \times_f N_\theta^{n_2}$ is a warped product pointwise semi-slant submanifold in a complex space form $\widetilde{M}^{2m}(c)$. If $N_T^{n_1}$ is a compact invariant submanifold and λ is a non-zero eigenvalue of the Laplacian on $N_T^{n_1}$, then*

$$\int_{N_T} \left(\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_\mu(e_i, e_j)||^2 \right) dV_T \geq \int_{N_T} \Big(\frac{n_1 n_2 c}{4} \Big) dV_T - 2n_2\lambda \int_{N_T} \left(\cot^2 \theta (\ln f)^2 \right) dV_T,$$

where dV_T is the volume element on ${N_T}^{n_1}$. Moreover, the equality sign holds if and only if we have

(i) $\Delta \ln f = \lambda \ln f$.

(ii) In the warped product pointwise semi-slant submanifold, both $N_T^{n_1}$ and $N_\theta^{n_2}$ are totally geodesic.

Proof. Integrating Eq.(4.6) and using Green lemma

$$\int_{N_T} \frac{n_1 n_2 c}{4} dV_T = 2n_2 \int_{N_T} \cot^2 \theta ||\nabla \ln f||^2 dV_T + \int_{N_T} \sum_{i=1}^{2d_1} \sum_{j=1}^{2d_2} ||h_\mu(e_i, e_j^*)||^2 dV_T. \tag{4.8}$$

The minimum principle properties for the first eigenvalue λ and the warping function, we have

$$\lambda \int_{N_T} (\ln f)^2 dV \leq \int_{N_T} ||\nabla \ln f||^2 dV. \tag{4.9}$$

Combining above two equations, we get required proof. □

5. Classifications of the gradient Ricci solitons

Proof of Theorem 1.6.

Assume that M^n is a warped product pointwise semi-slant submanifold of a complex space form $\widetilde{M}^{2m}(c)$. Suppose that a vector field X is equal to the gradient of a warping function $\ln f$, that is, $X = \nabla(\ln f)$. Then, the fundamental Eq. (1.17) of the gradient Ricci soliton takes a new form such as

$$Ric_{\ln f} + Hess(\ln f) = \lambda g, \tag{5.1}$$

for a positive constant $\lambda \in \mathbb{R}$ and Hessian tensor $Hess(\ln f)$ for the warping function $\ln f$. As we know, the Ricci tensor and Hessian tensor are symmetric $(0, 2)$ tensor fields. Thus, for any vector fields X and Y tangent to $N_T^{n_1}$, (5.1) implies that

$$Ric_{\ln f}(X, Y) + Hess(\ln f)(X, Y) = \lambda g(X, Y). \tag{5.2}$$

Assume that $\{e_1, \cdots e_{n_1}\}$ is an orthonormal frame for $N_T^{n_1}$ such that $\{e_1, e_2, \cdots, e_{d_1}, e_{d_1+1} = Je_1, \cdots e_{2d_1} = Je_{d_1}\}$. Taking $X = Y = e_i$, for $1 \leq i \leq d_1$ in (5.2) with the summation over the vector fields on $N_T^{n_1}$, we get

$$\sum_{i=1}^{d_1} Ric_{\ln f}(e_i, e_i) + \sum_{i=1}^{d_1} Hess(\ln f)\Big(e_i, e_i\Big) = \lambda d_1. \tag{5.3}$$

Replacing e_i by Je_i in the above equation, we get

$$\sum_{i=1}^{d_1} Ric_{\ln f}(Je_i, Je_i) + \sum_{i=1}^{d_1} Hess(\ln f)(Je_i, Je_i) = \lambda d_1. \tag{5.4}$$

Thus, from (5.3) and (5.4), it is easy to obtain

$$\sum_{i=1}^{d_1}\Big(Ric_{\ln f}(e_i, e_i) + Ric_{\ln f}(Je_i, Je_i)\Big) + \sum_{i=1}^{d_1} Hess(\ln f)(e_i, e_i) + \sum_{i=1}^{d_1} Hess(\ln f)(Je_i, Je_i) = 2d_1\lambda. \tag{5.5}$$

Again for the equality case of the inequality (1.4), we have the following equation from (4.7)

$$\frac{1}{n_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2 = \sum_{i=1}^{d_1}\Big\{Hess(\ln f)(e_i, e_i) + Hess(\ln f)(Je_i, Je_i)\Big\} + 2\Big(n_2 - \cot^2\theta\Big)\|\nabla \ln f\|^2 + \frac{n_1 c}{4}. \tag{5.6}$$

Using (5.5) and (5.6), and the fact that for Kaehler manifold, $Ric(X, X) = Ric(JX, JX)$, we have

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2 + 2n_2 R^T = n_1 n_2 \lambda + \frac{n_1 n_2 c}{4} + n_2\Big(n_2 - \cot^2\theta\Big)\|\nabla \ln f\|^2. \tag{5.7}$$

As if M^n is a warped product submanifold and satisfies Eq. (1.18), then, we find the following from (5.7)

$$n_2\Big(n_2 - \cot^2\theta\Big)\|\nabla \ln f\|^2 = 0$$

From the hypothesis, $\theta \neq \cot^{-1}\sqrt{n_2}$. Therefore, $\nabla(\ln f) = 0$, which implies that f is a constant function of M^n. Hence, M^n is an Einstein warped product pointwise semi-slant submanifold of $N_T^{n_1}$ and $N_\theta^{n_2}$ by Remark 1.3.

Conversely, assume that M^n is a Einstein warped product pointwise semi-slant submanifold. Then a warping function or a potential function $\ln f$ must be constant. This implies that $\|\nabla(\ln f)\|^2 = 0$. Thus, we get the required result (1.18) from Eq. (5.7). This completes the proof of the theorem.

Similarly, for steady gradient Ricci soliton, that is, $\lambda = 0$, we immediately obtain the following theorem

Theorem 5.1. *Assume that $\varphi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \longrightarrow \widetilde{M}^{2m}(c)$ is an isometric immersion of a warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ admitting steady gradient Ricci soliton into a complex space form $\widetilde{M}^{2m}(c)$ with such that $\theta \neq arc\cot\sqrt{n_2}$. Then the non-trivial warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ is an Einstein warped product submanifold if and only if*

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2 + n_2 R^T = \frac{n_1 n_2 c}{4}. \tag{5.8}$$

where R^T is scalar curvature of $N_T^{n_1}$.

Proof. The proof follows from Theorem 1.6 with $\lambda = 0$ in (5.1). □

6. Classifications of the Ricci curvature and divergence of the Hessian tensor

In this section, we studied some applications of the derived inequality with equality cases. Let identify any $(0, 2)$-tensor T on M with a $(1, 1)$-tensor by equation

$$g(T(Z), Y) = T(Z, Y).$$

for all $Y, Z \in \Gamma(TM)$. Thus , we get

$$div(\phi T) = \phi div T + T(\nabla\phi, \bullet) \quad \text{and} \quad \nabla(\phi T) = \phi\nabla T + d\phi \otimes T,$$

for all $\phi \in C^\infty(M)$. In particular, we have $div(\phi g) = d\phi$. Moreover, the following general facts are well known in the literature

$$(1)\ div\nabla^2\phi = Ric(\nabla\phi, \bullet) + d\Delta\phi \ \text{ and } (ii)\ \frac{1}{2}d\|\nabla\|^2 = \nabla^2\phi(\nabla, \bullet). \tag{6.1}$$

We consider M^n to be a compact Riemannian manifold with boundary and obtain following classification resultsWe considered M^n to be a compact Riemannian manifold with a compact boundary, and obtain some classifications.

6.1. Proof of Theorem 1.7

We use the Ricci identity (6.1). Applying these Ricci identity on a warping function $\phi = \ln f$, which implies that

$$div(Hess(\ln f)) = d\Big(\Delta(\ln f)\Big) + \mathcal{R}ic(\nabla \ln f, \cdot). \tag{6.2}$$

From the hypothesis, M^n is a compact warped product submanifold with a boundary, and then taking integration along the volume element dV, we get

$$\Delta(\ln f) = \int_M \Big(div(Hess(\ln f))\Big)\mathrm{dV} - \int_M \mathcal{R}ic(\nabla \ln f, \cdot)\mathrm{dV}. \tag{6.3}$$

Using the Green theorem on a compact manifold M^n, one gets $\int_M \Delta f dV = 0$. Using the results of Yano and Kon from (see [43]), it follows $\Delta f = -div(\nabla f)$ and from the Green lemma $\int_M div(X)dV = 0$ for an arbitrary vector field X on M^n. Thus, we get $\int_M div(Hess(\ln f))dV = 0$. Therefore, (6.3) implies that

$$\Delta(\ln f) = -\int_M \mathcal{R}ic(\nabla \ln f, \cdot)\mathrm{dV}. \tag{6.4}$$

On the other hand, assuming that the equality holds in the inequality (1.4), and from (4.6) we have

$$n_2\Delta(\ln f) + 2n_2\cot^2\theta\|\nabla \ln f\|^2 = \frac{n_1n_2c}{4} - \sum_{i=1}^{2d_1}\sum_{j=1}^{2d_2}\|h_\mu(e_i, e_j^*)\|^2. \tag{6.5}$$

From (6.4) and (6.5), we obtain

$$2\cot^2\theta\|\nabla \ln f\|^2 - \int_M \mathcal{R}ic(\nabla \ln f, \cdot)\mathrm{dV} = \frac{n_1c}{4} - \frac{1}{n_2}\sum_{i=1}^{2d_1}\sum_{j=1}^{2d_2}\|h_\mu(e_i, e_j^*)\|^2.$$

Further simplifications give

$$\frac{n_1c}{4} = -\int_M \mathcal{R}ic(\nabla \ln f, \cdot)\mathrm{dV} + \frac{1}{n_2}\sum_{i=1}^{2d_1}\sum_{j=1}^{2d_2}\|h_\mu(e_i, e_j^*)\|^2 + 2\cot^2\theta\|\nabla \ln f\|^2. \tag{6.6}$$

If the equality (1.19) is satisfied, then from (6.6) we get the following condition

$$2\cot^2\theta\|\nabla \ln f\|^2 = 0,$$

which implies

$$\cot^2\theta = 0, \quad or \quad \|\nabla \ln f\|^2 = 0.$$

Case 6.1. *If we choose* $\cot^2\theta = 0$, $\frac{\cos^2\theta}{\sin^2\theta} = 0$, *which implies that* $\cos\theta = 0$. *From the Definitions 2.3 and 2.4, we conclude that a pointwise slant submanifold* $N_\theta^{n_2}$ *becomes a totally real submanifold; hence,* M^n *becomes a CR-warped product submanifold of a complex space form* $\widetilde{M}^{2m}(c)$. *This completes the proof of (i) from Theorem 1.7.*

Case 6.2. *When* $\|\nabla \ln f\|^2 = 0$, $\nabla \ln f = 0$, *which implies that grad* $\ln f = 0$. *it shows that* f *is a constant function on* M^n. *Hence, from Remark 1.1, we conclude that* M^n *is a trivial warped product pointwise semi-slant submanifold of a complex space form* $\widetilde{M}^{2m}(c)$. *This is the second part (ii) of Theorem 1.7.*

7. Applications in Physical sciences

In this section, we considered a warped product pointwise semi-slant submanifold as a connected, compact warped product pointwise semi-slant submanifold with a nonempty boundary $\partial M \neq \emptyset$. Thus, we construct some necessary and sufficient conditions in terms of kinetic energy and Hamiltonian, whose positive differentiable function is a warping function, and classify non-trivial warped product submanifolds turning into trivial warped product submanifolds of a complex space form.

7.1. **Proof of Theorem 1.4.**

From Eq. (4.6) for the equality case of inequality (1.4), we have

$$\frac{n_1 n_2 c}{4} + n_2\Delta(\ln f) + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2 + 2n_2\cot^2\theta\|\nabla \ln f\|^2. \tag{7.1}$$

Taking integration on M^n over the volume element $\mathrm{d}V$ with nonempty boundary, we get

$$\int_M \Big(\frac{n_1 n_2 c}{4}\Big)\mathrm{d}V = \int_M \bigg(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2\bigg)\mathrm{d}V + n_2\int_M \Big(\Delta(\ln f)\Big)\mathrm{d}V + 2n_2\int_M \bigg(\cot^2\theta\Big(\|\nabla \ln f\|^2\Big)\bigg)\mathrm{d}V. \tag{7.2}$$

In the last term of the above equation, using the property of partial integration because θ is a slant function, we get

$$\int_M \Big(\frac{n_1 n_2 c}{4}\Big)\mathrm{d}V = \int_M \bigg(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2\bigg)\mathrm{d}V + n_2\int_M \Big(\Delta(\ln f)\Big)\mathrm{d}V + 2n_2\cot^2\theta\int_M \Big(\|\nabla \ln f\|^2\Big)dV$$
$$+ 4n_2\int_M \bigg(\csc^3\theta\cos\theta\Big(\frac{d\theta}{\mathrm{d}V}\Big)\Big(\int_M \|\nabla \ln f\|^2\Big)\mathrm{d}V\bigg)\mathrm{d}V.$$

From (1.11) and (7.2), it follows that

$$\int_M \Big(\frac{n_1 c}{4}\Big)\mathrm{d}V = \frac{1}{n_2}\int_M \bigg(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2\bigg)\mathrm{d}V + \int_M \Delta(\ln f)\mathrm{d}V + 4\cot^2\theta E(\ln f)$$
$$+ 8\int_M \bigg(\csc^3\theta\cos\theta\Big(\frac{d\theta}{\mathrm{d}V}\Big)E(\ln f)\bigg)\mathrm{d}V. \tag{7.3}$$

If the equality condition is satisfied in Eq. (1.12), then we get the following condition from (7.3)

$$\int_M \Delta(\ln f)\mathrm{d}V = 0 \quad \textit{on } M,$$

which implies that

$$\Delta(\ln f) = 0. \tag{7.4}$$

As we assumed that M^n is a connected and compact warped product pointwise semi-slant submanifold, from (7.4) and Theorem 7.1 it implies that $\ln f = 0 \Longrightarrow f = 1$, that is, f is constant on M^n. Hence, following Remark 1.1, the warped product submanifold M^n is simply a Riemannian product manifold. This completes the proof of the theorem.

7.2. **Proof of Theorem 1.5.**

Using Eq. (1.13) in Eq.(4.6), the equality of inequality (1.4) gives

$$\frac{n_2 n_1 c}{4} = 4n_2 \cot^2\theta H\Big(\mathrm{d}(\ln f), p\Big) + n_2\Delta(\ln f) + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2. \tag{7.5}$$

From Eq. (1.14) holds if and only if the Eq. (7.5) gives $\Delta(\ln f) = 0$ on M^n. Therefore, from Theorem 7.1, f is a constant function; hence, we conclude that M^n is a trivial warped product submanifold. This completes the proof of the theorem.

Furthermore, the Lagrangian for the positive differentiable function f of a compact Riemannian manifold is defined as

$$L = \frac{1}{2}\|\nabla f\|^2. \tag{7.6}$$

Theorem 7.1. *[20] The Euler-Lagrange equation for the Lagrangian* (7.6) *is*

$$\Delta f = 0. \tag{7.7}$$

Theorem 7.2. *Assume that $\phi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \longrightarrow \widetilde{M}^{2m}(c)$ is an isometric immersion of a compact warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ into complex space form $\widetilde{M}^{2m}(c)$. If the warping function satisfies the Euler-Lagrange equation, then the necessary condition for M^n to be a trivial warped product is*

$$\|h\|^2 \geq \frac{n_2 n_1 c}{2}. \tag{7.8}$$

Proof. If the warping function satisfies the Euler-Lagrange equation (7.7) for the Lagrangian (7.6), then from Theorem 7.1, we obtain

$$\Delta(\ln f) = 0, \tag{7.9}$$

Thus from inequality (1.4) and (7.9), we derive

$$\|h\|^2 \geq \frac{c}{2} n_2 n_1 + n_2\|\nabla \ln f\|^2. \tag{7.10}$$

If the inequality (7.8) holds, then from Eq. (7.10) we get a constant warping function $\ln f$ on M^n. □

Theorem 7.3. *Assume that $\phi : M^n = N_T^{n_1} \times_f N_\theta^{n_2} \longrightarrow \widetilde{M}^{2m}(c)$ is an isometric immersion of a compact warped product pointwise semi-slant submanifold $N_T^{n_1} \times_f N_\theta^{n_2}$ into a complex space form $\widetilde{M}^{2m}(c)$. If the warping function satisfies the Euler-Lagrange equation, then the necessary and sufficient condition for the warped product $N_T^{n_1} \times_f N_\theta^{n_2}$ to be a trivial warped product submanifold is*

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2 = \frac{n_2 n_1 c}{4}. \tag{7.11}$$

Proof. The proof can be done in a similar way as the proof of Theorem 7.2 by using (7.11), (4.6) and Theorem 7.1. □

Acknowledgements

The authors would like to thank the referees for their stick criticism and suggestions on this paper to improve the quality. The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through research groups program under grant number R. G. P. 2/57/40.

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