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Solvability of Infinite Systems of Nonlinear Integral Equations in Two Variables by Using Semi-Analytic Method

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Abstract. In this article, we generalize and investigate existence of solution for infinite systems of nonlinear integral equations with two variables in a given Banach sequence space $BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$ using Meir-Keeler condensing and noncompactness. Validity of results are shown with the help of an illustrative example. We also introduce a coupled semi-analytic method in the case of two variables in order to construct an iteration algorithm to find a numerical solution for above-mentioned problem. The numerical results show that the produced sequence for approximating the solution in the examples is in the Banach sequence space $BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$ itself.

1. Introduction, Definitions and Preliminaries

The measure of noncompactness was introduced by Kuratowski [17] in the year 1930. Subsequently, it was used by Darbo [9] to generalize Schauder's fixed point theorem as well as Banach's contraction principle for condensing operators. The Hausdorff measure χ of noncompactness was introduced by Goldenstein et al. [14] in 1957. It was further studied by Goldenstein and Markus [15] (see [8] for different types of measure of noncompactness).

By applying measures of noncompactness, many authors studied fixed point theory, differential equations, functional equations, integral and integro-differential equations, optimization problems, and so on (see [8]). For example, Mursaleen and Mohiuddine [21] proved existence theorems for infinite systems of differential equations in the space ℓ_p. On the other hand, existence theorems for the infinite systems of linear equations in the spaces ℓ_1 and ℓ_p were given by Alotaibi et al. [5]. Arab et al. [6] proved the existence of solutions of systems of integral equations in two variables. Das et al. [10] proved the existence of solution for infinite systems of integral equations in two variables in the spaces c_0 and ℓ_1. In a sequel, Das et al. [11]

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studied the existence of solution of infinite systems of integral equations in two variables in the space ℓ_p (see also the recent works [26]).

Consider a real Banach space E with the norm $\|\,.\,\|$. Let $B(a_0, d)$ be a closed ball in E centered at a_0 and with radius d. If X is a nonempty subset of E, then by $\bar{X}$ and Conv$X$ we denote the closure and convex closure of X. Moreover, we let $\mathcal{M}_E$ denote the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E$ we denote its subfamily consisting of all relatively compact sets. The following definition of a measure of noncompactness found in [8].

Definition 1.1. A function $\mu : \mathcal{M}_E \to \mathbb{R}_+(= [0, \infty))$ is called a measure of noncompactness if it satisfies the following conditions:

(i) the family $\ker \mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathcal{N}_E$.
(ii) $X \subset Y \implies \mu(X) \leqq \mu(Y)$.
(iii) $\mu(\bar{X}) = \mu(X)$.
(iv) $\mu(\text{Conv}X) = \mu(X)$.
(v) $\mu(\lambda X + (1-\lambda) Y) \leqq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
(vi) if $X_n \in \mathcal{M}_E,\ X_n = \bar{X}_n,\ X_{n+1} \subset X_n$ for $n = 1, 2, 3, \cdots$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then $\bigcap_{n=1}^{\infty} X_n \neq \phi$.

The family $\ker \mu$ is said to be the *kernel of measure* μ. A measure μ is said to be the sublinear if it satisfies the following conditions:

(1) $\mu(\lambda X) = |\lambda|\,\mu(X)$ for $\lambda \in \mathbb{R}$.
(2) $\mu(X + Y) \leqq \mu(Y) + \mu(Y)$.

A sublinear measure of noncompactness μ satisfies the following condition:

$$\mu(X \cup Y) = \max\{\mu(\lambda X), \mu(\lambda Y)\}$$

and such that $\ker \mu = \mathcal{N}_E$ is said to be regular.

For a bounded subset $\bar{S}$ of a metric space X, the Kuratowski measure of noncompactness is defined as follows:

$$\alpha(\bar{S}) = \inf\left\{\delta > 0 : \bar{S} = \bigcup_{i=1}^{n} \bar{S}_i \ \text{ and }\ \operatorname{diam}(\bar{S}_i) \leqq \delta \quad \left(1 \leqq i \leqq n \quad (n \in \mathbb{N})\right)\right\},$$

where $\operatorname{diam}(\bar{S}_i)$ denotes the diameter of the set $\bar{S}_i$, that is,

$$\operatorname{diam}(\bar{S}_i) = \sup\{d(x, y) : x, y \in \bar{S}_i\}$$

and $\mathbb{N}$ denotes the set of positive integers.

Definition 1.2. (see [4]) Let E_1 and E_2 be two Banach spaces and let μ_1 and μ_2 be two arbitrary measures of noncompactness on E_1 and E_2, respectively. An operator f from E_1 to E_2 is called a (μ_1, μ_2)-condensing operator if it is continuous and $\mu_2(f(D)) < \mu_1(D)$ for every set $D \subset E_1$ with compact closure.

Remark 1.3. If $E_1 = E_2$ and $\mu_1 = \mu_2 = \mu$, then f is called a μ-condensing operator.

Theorem 1.4. (see [9]) *Let Ω be a nonempty, closed, bounded and convex subset of a Banach space E and let $f : \Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in [0, 1)$ with the property given by*

$$\mu_2(f(\Omega)) < k\mu_1(\Omega).$$

Then f has a fixed point in Ω.

The contractive maps and the compact maps are condensing if we take as measures of noncompactness the diameter of a set and the indicator function of a family of non-relatively compact sets, respectively (see [4]). In the year 1955, Darbo [9] proved a fixed point theorem by using the concept of measures of noncompactness, which guarantees the existence of a fixed point for condensing operators. Subsequently, in the year 1969, Meir and Keeler [18] proved an interesting fixed point theorem (see Theorem 1.6 below), which is a generalization of the Banach contraction principle [7]. Darbo's theorem has provided an abundance of applications in the existence of solutions for differential and integral equations (see, for details [2] and [19]). It extends both the classical Schauder's fixed point theorem and the celebrated Banach's contraction principle.

Definition 1.5. (see [18]) Let (X, d) be a metric space. Then a mapping T on X is said to be a Meir-Keeler contraction if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leqq d(x, y) < \epsilon + \delta \implies d(Tx, Ty) < \epsilon \quad (\forall\, x, y \in X).$$

Theorem 1.6. (see [18]) *Let (X, d) be a complete metric space. If $T : X \to X$ is a Meir-Keeler contraction, then T has a unique fixed point.*

In [3], the following definitions and associated results are given, which will be needed in our study here.

Definition 1.7. (see [3]) Let C be a nonempty subset of a Banach space E and let μ be an arbitrary measure of noncompactness on E. We say that an operator $T : C \to C$ is a Meir-Keeler condensing operator if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leqq \mu(X) < \epsilon + \delta \implies \mu(T(X)) < \epsilon \tag{1}$$

for any bounded subset X of C.

Theorem 1.8. (see [3]) Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let μ be an arbitrary measure of noncompactness on E. If $T : C \to C$ is a continuous and Meir-Keeler condensing operator, then T has at least one fixed point and the set of all fixed points of T in C is compact.

The main object of this article is to establish the existence of solution of some infinite systems of integral equations in two variables in the sequence space $BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$ by using the Meir-Keeler condensing operators.

2. Measure of Noncompactness

In the Banach space $(c, \| \,.\, \|_c)$, the measure of noncompactness cannot be expressed as a simple rule. Nevertheless, we have an equivalent measure of noncompactness in c, which can be formulated as follows (see [8]):

$$\mu_c(\bar{Q}) = \lim_{n\to\infty} \left\{ \sup_{x\in\bar{Q}} \left(\sup_{k\geqq n} \left| x_k - \lim_{m\to\infty} x_m \right| \right) \right\}, \tag{2}$$

where $x = (x_i)_{i=1}^{\infty} \in c$ and $\bar{Q} \in \mathcal{M}_c$.

Let us denote by $BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$ the space of all bounded continuous functions on $\mathbb{R}_+ \times \mathbb{R}_+$ with values in c. Then $BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$ is also a Banach space with norm $\| x(t, s) \|_{BC(\mathbb{R}_+\times\mathbb{R}_+, c)} = \sup \{\| x(t, s) \|_c : t, s \in \mathbb{R}_+\}$ where $x(t, s) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$.

For any non-empty bounded subset $\hat{E}$ of $BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$ and $t, s \in \mathbb{R}_+$, let $\hat{E}(t, s) = \left\{ x(t, s) : x \in \hat{E} \right\}$. Now, using (2), we conclude that the measure of noncompactness for $\hat{E} \subset BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$ can be defined by

$$\mu_{BC(\mathbb{R}_+\times\mathbb{R}_+, c)}(\hat{E}) = \sup \left\{ \mu_c(\hat{E}(t, s)) : t, s \in \mathbb{R}_+ \right\}.$$

Let $(E, \|\,.\,\|)$ be some Banach sequence space.

Consider the following system:

$$x_n(t,s) = F_n\left(t, s, \int_0^{\alpha(s)} \int_0^{\beta(t)} G_n\,(t,s,v,w,x(v,w))\,dv\,dw, x(t,s)\right), \tag{3}$$

where

$$x(t,s) = \left(x_i(t,s)\right)_{i=1}^{\infty} \in E,\ (t,s) \in \mathbb{R}_+ \times \mathbb{R}_+,\ \ n \in \mathbb{N} \quad \text{and} \quad x_i(t,s) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$$

for all $i \in \mathbb{N}$.

3. Existence of Solution for an Infinite Systems of Nonlinear Integral Equations in Two Variables in the Space $BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$

Assume that

(i) $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous.

(ii) $F_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times BC(\mathbb{R}_+ \times \mathbb{R}_+, c) \to \mathbb{R}\,(n \in \mathbb{N})$ are continuous and there exists continuous functions $\hat{a}_n : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R},\ b_n : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+\,(n \in \mathbb{N})$ such that

$$F_n\left(t, s, \phi(t,s), x(t,s)\right) = \hat{a}_n(t,s)x_n(t,s) + b_n(t,s)\phi(t,s),$$

where $x(t,s) = (x_i(t,s))_{i=1}^{\infty} \in BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$ and $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$. Also, there exists a non-negative sequence $(a_n)_{i=1}^{\infty}$ such that $|\hat{a}_n(t,s)| \leq a_n$ for all $n \in \mathbb{N},\ t, s \in \mathbb{R}_+$ with $\lim_{n\to\infty} a_n = 0$.

(iii) $G_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times BC(\mathbb{R}_+ \times \mathbb{R}_+, c) \to \mathbb{R}\ (n \in \mathbb{N})$ are continuous and there exists a constant

$$H_n = \sup\left\{ b_n(t,s)\left|\int_0^{\alpha(s)}\int_0^{\beta(t)} G_n\,(t,s,v,w,x(v,w))\,dvdw\right| : t,s,v,w \in \mathbb{R}_+, x(v,w) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, c)\right\}.$$

Also

$$\lim_{t,s\to\infty}\left| b_n(t,s)\int_0^{\alpha(s)}\int_0^{\beta(t)} \left[G_n\,(t,s,v,w,x(v,w)) - G_n\,(t,s,v,w,y(v,w))\right] dvdw \right| = 0.$$

(iv) Define an operator $\mathcal{F}$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$ to $BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$ as follows

$$(t, s, x(t,s)) \to (\mathcal{F}x)\,(t,s), \ \text{where}$$

$$(\mathcal{F}x)\,(t,s) = \left(F_n(t, s, \gamma_n(x(t,s)), x(t,s))\right)_{n=1}^{\infty},$$

and $\gamma_n(x(t,s)) = \int_0^{\alpha(s)}\int_0^{\beta(t)} G_n\,(t,s,v,w,x(v,w))\,dvdw$.

(v) As $n \to \infty,\ H_n \to 0$. Also we assume

$$\sup_n H_n = H \text{ and}$$

$$\sup\{a_n : n \in \mathbb{N}\} = A < \infty \text{ such that } 0 < A < 1.$$

Theorem 3.1. Under the hypothesis $(i) - (v)$, infinite system (3) has at least one solution $x(t,s) = (x_i(t,s))_{i=1}^{\infty} \in BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$ for all $t, s \in \mathbb{R}_+$. Also, $x_i(t,s) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ for all $i \in \mathbb{N}$.

Proof. By using (3) and (i)-(v), we have for arbitrarily fixed $t, s \in \mathbb{R}_+$,

$$\begin{aligned}
&\| x(t,s) \|_c \\
&= \sup_{n\geq 1} \left| F_n \left(t, s, \int_0^{\alpha(s)} \int_0^{\beta(t)} G_n\left(t,s,v,w,x(v,w)\right) dvdw, x(t,s) \right) \right| \\
&= \sup_{n\geq 1} \left| \hat{a}_n(t,s) x_n(t,s) + b_n(t,s) \int_0^{\alpha(s)} \int_0^{\beta(t)} G_n\left(t,s,v,w,x(v,w)\right) dvdw \right| \\
&\leq \sup_{n\geq 1} \left[|\hat{a}_n(t,s)| \, |x_n(t,s)| + b_n(t,s) \left| \int_0^{\alpha(s)} \int_0^{\beta(t)} G_n\left(t,s,v,w,x(v,w)\right) dvdw \right| \right] \\
&\leq A \| x(t,s) \|_c + H.
\end{aligned}$$

i.e. $(1 - A) \| x(t,s) \|_c \leq H$ implies $\| x(t,s) \|_c \leq \frac{H}{1-A} = r$(say).
Therefore $\| x(t,s) \|_{BC(\mathbb{R}_+\times\mathbb{R}_+,c)} \leq r$ which gives $x(t,s) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$.

Let $\bar{E} = \bar{E}\left(x^0(t,s), r\right)$ be the closed ball with center at $x^0(t,s) = (x_i^0(t,s))_{i=0}^{\infty}$ where $x_i^0(t,s) = 0$ for all $i \in \mathbb{N}, t,s \in \mathbb{R}_+$ and radius r, thus $\bar{E}$ is an non-empty, bounded, closed and convex subset of $BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$. Assume $\mathcal{F} = (\mathcal{F}_n)$ be an operator defined as follows for all $t, s \in \mathbb{R}_+$,

$$(\mathcal{F}x)(t,s) = \{(\mathcal{F}_n x)(t,s)\}_{n=1}^{\infty} = \{F_n\left(t,s,\gamma_n(x(t,s)), x(t,s)\right)\}_{n=1}^{\infty},$$

where $x(t,s) = (x_i(t,s))_{i=1}^{\infty} \in \bar{E}$ and $x_i(t,s) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$, $\forall i \in \mathbb{N}$.

Now, we have to show that for arbitrary fixed $t, s \in \mathbb{R}_+$, $(\mathcal{F}x)(t,s) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$.

Let us consider fixed $x(t,s) \in \bar{E}$ and $t, s \in \mathbb{R}_+$. For arbitrary $m, n \in \mathbb{N}$ we have

$$\begin{aligned}
&\left| F_n\left(t,s,\gamma_n(x(t,s)), x(t,s)\right) - F_m\left(t,s,\gamma_m(x(t,s)), x(t,s)\right) \right| \\
&= \left| \hat{a}_n(t,s) x_n(t,s) b_n(t,s) \gamma_n(x(t,s)) - \hat{a}_m(t,s) x_m(t,s) - b_m(t,s) \gamma_m(x(t,s)) \right| \\
&\leq |\hat{a}_n(t,s)| \, |x_n(t,s)| + b_n(t,s) \left| \gamma_n(x(t,s)) \right| + |\hat{a}_m(t,s)| \, |x_m(t,s)| + b_m(t,s) \left| \gamma_m(x(t,s)) \right| \\
&\leq a_n |x_n(t,s)| + H_n + a_m |x_m(t,s)| + H_m
\end{aligned}$$

As $m, n \to \infty$ we have $\left| F_n\left(t,s,\gamma_n(x(t,s)), x(t,s)\right) - F_m\left(t,s,\gamma_m(x(t,s)), x(t,s)\right) \right| \to 0$ which gives $|(\mathcal{F}_n x)(t,s) - (\mathcal{F}_m x)(t,s)| \to 0$. Thus $(\mathcal{F}x)(t,s)$ is a real Cauchy sequence hence it is convergent. Since $t, s \in \mathbb{R}_+$ are arbitrary therefore $(\mathcal{F}x)(t,s) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$.

Again, $\| (\mathcal{F}x)(t,s) - x^0(t,s) \|_{BC(\mathbb{R}_+\times\mathbb{R}_+,c)} \leq r$ thus $\mathcal{F}$ is self mapping on $\bar{E}$.

We have to show that $\mathcal{F}$ is continuous on $\bar{E}$.

Let $x(t,s) = (x_i(t,s))_{i=1}^{\infty}$, $y(t,s) = (y_i(t,s))_{i=1}^{\infty} \in \bar{E}$ and $\epsilon > 0$ be such that $\| x(t,s) - y(t,s) \|_{BC(\mathbb{R}_+\times\mathbb{R}_+,c)} < \frac{\epsilon}{2A}$.

For $t, s \in \mathbb{R}_+$ we have

$$\begin{aligned}
&\left| (\mathcal{F}_n x)(t,s) - (\mathcal{F}_n y)(t,s) \right| \\
&= \left| F_n\left(t,s,\gamma_n(x(t,s)), x(t,s)\right) - F_n\left(t,s,\gamma_n(y(t,s)), y(t,s)\right) \right| \\
&\leq a_n \left| x_n(t,s) - y_n(t,s) \right| + b_n(t,s) \left| \gamma_n(x(t,s)) - \gamma_n(y(t,s)) \right| \\
&< \frac{\epsilon}{2} + b_n(t,s) \left| \int_0^{\alpha(s)} \int_0^{\beta(t)} \left[G_n\left(t,s,v,w,x(v,w)\right) - G_n\left(t,s,v,w,y(v,w)\right) \right] dvdw \right|.
\end{aligned}$$

By using assumption (iii), we can choose $t_1 > 0$ such that $\max(t,s) > t_1$,

$$\left| b_n(t,s) \int_0^{\alpha(s)} \int_0^{\beta(t)} \left[G_n(t,s,v,w,x(v,w)) - G_n(t,s,v,w,y(v,w)) \right] dvdw \right| < \frac{\epsilon}{2}.$$

Hence $\left|(\mathcal{F}_n x)(t,s) - (\mathcal{F}_n y)(t,s)\right| < \epsilon$.

For $t,s \in [0,t_1]$, let

$$\begin{aligned}
&\bar{\alpha} = \sup\{\alpha(t) : t \in [0,t_1]\},\\
&\bar{\beta} = \sup\{\beta(t) : t \in [0,t_1]\},\\
&\bar{b} = \sup\{b_n(t,s) : t,s \in [0,t_1], n \in \mathbb{N}\}\\
&\textit{and}\\
&G = \sup_n \left\{ \left| G_n(t,s,v,w,x(v,w)) - G_n(t,s,v,w,y(v,w)) \right| : t,s \in [0,t_1],\ v \in [0,\bar{\beta}]\ \ w \in [0,\bar{\alpha}], x,y \in \bar{E} \right\}.
\end{aligned}$$

Then $\left|(\mathcal{F}_n x)(t,s) - (\mathcal{F}_n y)(t,s)\right| < \frac{\epsilon}{2} + \bar{b}G\bar{\alpha}\bar{\beta}$.

Since G_n is continuous on $[0,t_1] \times [0,t_1] \times [0,\bar{\beta}] \times [0,\bar{\alpha}] \times \bar{E}$ for all $n \in \mathbb{N}$ therefore we have $G \to 0$ as $\epsilon \to 0$ which gives $\| (\mathcal{F}_n x)(t,s) - (\mathcal{F}_n y)(t,s) \|_{BC(\mathbb{R}_+ \times \mathbb{R}_+, c)} \to 0$ as $\| x(t,s) - y(t,s) \|_{BC(\mathbb{R}_+ \times \mathbb{R}_+, c)} \to 0$. Thus $\mathcal{F}$ is continuous on $\bar{E} \subset BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$.

Again for arbitrarily fixed $t,s \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\mu_c(\mathcal{F}(\bar{E}))\\
&= \lim_{n\to\infty} \left[\sup_{x(t,s)\in\bar{E}} \left\{ \sup_{k\geq n} \left| F_k(t,s,\gamma_k(x(t,s)), x(t,s)) - \lim_{m\to\infty} F_m(t,s,\gamma_m(x(t,s)), x(t,s)) \right| \right\} \right]\\
&= \lim_{n\to\infty} \left[\sup_{x(t,s)\in\bar{E}} \left\{ \sup_{k\geq n} \left| \hat{a}_k(t,s)x_k(t,s) + b_k(t,s)\gamma_k(x(t,s)) - \lim_{m\to\infty} \left(\hat{a}_m(t,s)x_m(t,s) + b_m(t,s)\gamma_m(x(t,s)) \right) \right| \right\} \right]\\
&\leq \lim_{n\to\infty} \left[\sup_{x(t,s)\in\bar{E}} \left\{ \sup_{k\geq n} \left(A\left| x_k(t,s) - \lim_{m\to\infty} x_m(t,s) \right| + \left| \lim_{m\to\infty} (\hat{a}_k(t,s) - \hat{a}_m(t,s))\, x_m(t,s) \right| + H_k + \lim_{m\to\infty} H_m \right) \right\} \right]\\
&= A\mu_c(\bar{E}).
\end{aligned}$$

Thus $\mu_{BC(\mathbb{R}_+\times\mathbb{R}_+,c)}(\mathcal{F}(\bar{E})) \leq A\mu_{BC(\mathbb{R}_+\times\mathbb{R}_+,c)}(\bar{E})$.

Observe that $\mu_{BC(\mathbb{R}_+\times\mathbb{R}_+,c)}(\mathcal{F}(\bar{E})) \leq A\mu_{BC(\mathbb{R}_+\times\mathbb{R}_+,c)}(\bar{E}) < \epsilon$ implies $\mu_{BC(\mathbb{R}_+\times\mathbb{R}_+,c)}(\bar{E}) < \frac{\epsilon}{A}$.

Taking $\delta = \frac{\epsilon(1-A)}{A}$ we get $\epsilon \leq \mu_{BC(\mathbb{R}_+\times\mathbb{R}_+,c)}(\bar{E}) < \epsilon + \delta$. Therefore $\mathcal{F}$ is a Meir-Keeler condensing operator defined on the set $\bar{E} \subset BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$. Since t,s are arbitrarily fixed so $\mathcal{F}$ satisfies all the conditions of Theorem 1.8 which implies $\mathcal{F}$ has a fixed point in $\bar{B}$ for all $t,s \in \mathbb{R}_+$. The system (3) has a solution in $BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$. $\square$

4. An Illustrative Example

In this section, we give the following illustrative example.

Example. Consider the following infinite system of integral equations:

$$x_n(t,s) = \frac{1}{t^2s^2 + n^2} \sum_{i=n}^{3n} \left(\frac{x_n(t,s)}{4i^2} \right) + \frac{1}{n^4 e^{t^2s^2}} \int_0^{s^2} \int_0^{t^2} \frac{\cos\left(\sum_{i=1}^{n} x_i(v,w) \right)}{4 + \sin\left(1 + \sum_{i=1}^{2n} x_i(v,w) \right)} dvdw, \tag{4}$$

where $n \in \mathbb{N}$. Here $\alpha(t) = \beta(t) = t^2$,

$$F_n(t,s,\gamma_n(x(t,s)),x(t,s)) = \frac{1}{t^2s^2+n^2}\sum_{i=n}^{3n}\left(\frac{x_n(t,s)}{4i^2}\right) + \frac{1}{n^4e^{t^2s^2}}\gamma_n(x(t,s)),$$

where

$$\gamma_n(x(t,s)) = \int_0^{s^2}\int_0^{t^2} G_n\left(t,s,v,w,x(v,w)\right)dvdw,$$

and

$$G_n\left(t,s,v,w,x(v,w)\right) = \frac{\cos\left(\sum\limits_{i=1}^{n} x_i(v,w)\right)}{4+\sin\left(\sum\limits_{i=1}^{2n} x_i(v,w)\right)}.$$

If $x(t,s) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$, then we have

$$F_n(t,s,\gamma_n(x(t,s)),x(t,s)) = \hat{a}_n(t,s)x_n(t,s) + b_n(t,s)\gamma_n(x(t,s))$$

where $\hat{a}_n(t,s) = \frac{1}{t^2s^2+n^2}\sum_{i=n}^{3n}\frac{1}{4i^2},\ b_n(t,s) = \frac{1}{n^4e^{t^2s^2}}$ are continuous functions on $\mathbb{R}_+ \times \mathbb{R}_+$. Also, we have $|\hat{a}_n(t,s)| \leq a_n$ for all $t,s \in \mathbb{R}_+$ where $a_n = \frac{\pi^2}{24n^2}$ It is obvious that $(a_n)_{n=1}^{\infty}$ is a sequence converging to zero and for all $n \in \mathbb{N}, b_n$ is a continuous function on $\mathbb{R}_+ \times \mathbb{R}_+$. Also $0 < A < 1$. Since

$$\int_0^{s^2}\int_0^{t^2} \frac{\cos\left(\sum\limits_{i=1}^{n} x_i(v,w)\right)}{4+\sin\left(1+\sum\limits_{i=1}^{2n} x_i(v,w)\right)}dvdw \leq \frac{t^2s^2}{3}.$$

therefore

$$\begin{aligned} &H_n \\ &= \sup\left\{\frac{1}{n^4e^{t^2s^2}}\left|\int_0^{s^2}\int_0^{t^2} \frac{\cos\left(\sum\limits_{i=1}^{n} x_i(v,w)\right)}{4+\sin\left(1+\sum\limits_{i=1}^{2n} x_i(v,w)\right)}dvdw\right| : t,s,v,w \in \mathbb{R}_+, x(v,w) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, c)\right\} \\ &= \frac{1}{3en^4}. \end{aligned}$$

Thus $H_n \to 0$ as $n \to \infty$ and $H = \frac{1}{3e}$. Moreover we get as $t,s \to \infty$

$$\left|\frac{1}{n^4e^{t^2s^2}}\int_0^{s^2}\int_0^{t^2}\left\{G_n\left(t,s,v,w,x(t,s)\right) - G_n\left(t,s,v,w,y(t,s)\right)\right\}dvdw\right| \to 0.$$

It is obvious that F_n and G_n are continuous functions for all $n \in \mathbb{N}$.Also for fixed $t,s \in \mathbb{R}_+$ and $x(t,s) \in BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$ we have $(F_n(t,s,\gamma(x(t,s)),x(t,s))_{n=1}^{\infty} \in BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$ So all the assumptions from (i)-(v) are satisfied. Hence by Theorem 3.1 we conclude that the system 4 has a solution in $BC(\mathbb{R}_+ \times \mathbb{R}_+, c)$.

5. Semi-Analytic Method to Solve Infinite System (4)

In Section 4, as an application of Theorem 3.1, we have proved the existence of solution of the problem considered in the above example. Here, in this section, we find an approximation of solution for the above problem by a coupled semi-analytic method in the case of two variables. In fact, this method is a combination of the modified homotopy perturbation method with infinite functions of two variables and the Adomian decomposition method. Applications of the modified homotopy perturbation method to solve nonlinear Fredholm integral equations and nonlinear differential equations can be found in [13, 23, 24], respectively. Adomian [1] introduced a decomposition method for solving some frontier problems of Physics and this technique was used in [25] to solve a system of Fredholm integro-differential equations. Hazarika et al. [16] applied a modified homotopy perturbation method and the Adomian decomposition method to solve nonlinear quadratic integral equations in the case of one variable. But, in this article, we have generalized the modified homotopy perturbation method to infinite functions of two variables and also, for simplification of nonlinear terms, we use the Adomian decomposition method in a suitable form. For this purpose, we consider a nonlinear problem with infinite functions of two variables in the following general form:

$$\begin{cases} A(x_1(t,s), x_2(t,s), \cdots, x_n(t,s), \cdots) - f(t,s,n) = 0 & \Big((t,s) \in \Omega;\ n \in \mathbb{N}\Big) \\ B\left(x_i, \frac{\partial x_i}{\partial \eta}\right) = 0 & (\eta \in \Gamma), \end{cases} \tag{5}$$

where A is a general nonlinear operator, B is a boundary operator and f is a known analytic function. Similar to the works in [16] and [23], we divide the general operator A into two nonlinear operators denoted by N_1 and N_2. Of course, N_1 or N_2 can be a linear operator in special cases. We also convert f to the functions f_1 and f_2. So, clearly, we can write

$$\begin{cases} N_1(x_1(t,s), \cdots, x_n(t,s), \cdots) - f_1(t,s,n) \\ \qquad +N_2(x_1(t,s), \cdots, x_n(t,s), \cdots) - f_2(t,s,n) = 0, (n \in \mathbb{N}). \end{cases}$$

We now introduce a modified homotopy perturbation for infinite functions of two variables:

$$\begin{cases} H(u_1(t,s), u_2(t,s), \cdots, p) & = N_1(u_1(t,s), \cdots, u_n(t,s), \cdots) - f_1(t,s,n) \\ & \quad +p(N_2(u_1(t,s), \cdots, u_n(t,s), \cdots) - f_2(t,s,n)) = 0 \\ (p \in [0,1]), & \end{cases} \tag{6}$$

where p is an embedding parameter and u_i $(i = 1, \cdots, n)$ are approximation of x_i for $i \in \mathbb{N}$. According to the variations of $p = 0$ to $p = 1$, we can get

$$N_1(u_1(t,s), \cdots, u_n(t,s), \cdots) = f_1(t,s,n)$$

$$\vdots$$

$$A(u_1(t,s), u_2(t,s), \cdots, u_n(t,s), \cdots) - f(t,s,n) = 0.$$

So, in (6), the solution of (5) is given for $p = 1$ and we also have

$$\begin{cases} x_n(t,s) \approx u_n(t,s) = \sum_{j=0}^{\infty} p^j u_{j,n}(t,s) & (n \in \mathbb{N}) \\ x_n(t,s) = \lim_{p \to 1} u_n(t,s). \end{cases} \tag{7}$$

In order to solve the infinite system of nonlinear integral equations in (4), we choose the operators N_1 and N_2 and the function f as follows:

$$\begin{aligned}
&N_1(x_1(t,s),\cdots,x_n(t,s),\cdots) = x_n(t,s),\\
&N_2(x_1(t,s),\cdots,x_n(t,s),\cdots) = -\frac{1}{t^2s^2+n^2}\sum_{i=n}^{3n}\left(\frac{x_n(t,s)}{4i^2}\right)\\
&\qquad\qquad - \frac{1}{n^4e^{t^2s^2}}\int_0^{s^2}\int_0^{t^2}\frac{\cos\left(\sum\limits_{i=1}^{n}x_i(v,w)\right)}{4+\sin\left(1+\sum\limits_{i=1}^{2n}x_i(v,w)\right)}dvdw,\\
&f(t,s,n) = 0.
\end{aligned} \tag{8}$$

To simplify the computations, we can choose $f_1(t,s)$ as a simple function. By substituting (8) and (7) into the homotopy perturbation (6), we have

$$\begin{aligned}
&\Big(\sum_{j=0}^{\infty}p^ju_{j,n}(t,s) - f_1(t,s,n)\Big) + p\Bigg(-\frac{1}{t^2s^2+n^2}\sum_{i=n}^{3n}\left(\frac{x_n(t,s)}{4i^2}\right)\\
&-\frac{1}{n^4e^{t^2s^2}}\int_0^{s^2}\int_0^{t^2}\frac{\cos\left(\sum\limits_{i=1}^{n}\sum\limits_{j=0}^{\infty}p^ju_{j,i}(v,w)\right)}{4+\sin\left(1+\sum\limits_{i=1}^{2n}\sum\limits_{j=0}^{\infty}p^ju_{j,i}(v,w)\right)}dv\,dw - f_2(t,s,n)\Bigg) = 0.
\end{aligned} \tag{9}$$

In (9), for decomposing the nonlinear terms to smaller separable nonlinear terms, we apply the Adomian decomposition method in the following form:

$$\begin{aligned}
&\sum_{i=n}^{3n}\frac{\sum\limits_{j=0}^{\infty}p^ju_{j,n}(t,s)}{4i^2} = \sum_{j=0}^{\infty}p^jA_{j,n}(t,s)\\
&\frac{\cos\left(\sum\limits_{i=1}^{n}\sum\limits_{j=0}^{\infty}p^ju_{j,i}(t,s)\right)}{4+\sin\left(1+\sum\limits_{i=1}^{2n}\sum\limits_{j=0}^{\infty}p^ju_{j,i}(t,s)\right)} = \sum_{j=0}^{\infty}p^j\widehat{A}_{j,n}(t,s),
\end{aligned} \tag{10}$$

where the Adomian polynomials are given by

$$\begin{aligned}
&A_{k,n}(t,s) = \frac{1}{k!}\left(\frac{d^k}{dp^k}\sum_{i=n}^{3n}\frac{\sum\limits_{j=0}^{\infty}p^ju_{j,n}(t,s)}{4i^2}\right)_{p=0}\\
&\widehat{A}_{k,n}(t,s) = \frac{1}{k!}\left(\frac{d^k}{dp^k}\frac{\cos\left(\sum\limits_{i=1}^{n}\sum\limits_{j=0}^{\infty}p^ju_{j,i}(t,s)\right)}{4+\sin\left(1+\sum\limits_{i=1}^{2n}\sum\limits_{j=0}^{\infty}p^ju_{j,i}(t,s)\right)}\right)_{p=0}.
\end{aligned} \tag{11}$$

Upon substituting from (10) into (9), we obtain

$$\begin{aligned}&\left(\sum_{j=0}^{\infty} p^j u_{j,n}(t,s) - f_1(t,s,n)\right) + p\left(-\frac{1}{t^2s^2+n^2}\sum_{j=0}^{\infty} p^j A_{j,n}(t,s)\right.\\ &\qquad\left. -\frac{1}{n^4 e^{t^2s^2}}\int_0^{s^2}\int_0^{t^2}\sum_{j=0}^{\infty} p^j \widehat{A}_{j,n}(v,w)dv\,dw - f_2(t,s,n)\right) = 0.\end{aligned} \tag{12}$$

By rearranging the equation (12) in terms of p powers, we can write

$$\begin{aligned}&p^0 : \left(u_{0,n}(t,s) - f_1(t,s,n)\right)\\ &p^1 : \left(u_{1,n}(t,s) - \frac{1}{t^2s^2+n^2}A_{0,n}(t,s) - \frac{1}{n^4e^{t^2s^2}}\int_0^{s^2}\int_0^{t^2}\widehat{A}_{0,n}(v,w)dv\,dw - f_2(t,s,n)\right)\\ &p^k : \left(u_{k,n}(t,s) - \frac{1}{t^2s^2+n^2}A_{k-1,n}(t,s) - \frac{1}{n^4e^{t^2s^2}}\int_0^{s^2}\int_0^{t^2}\widehat{A}_{k-1,n}(v,w)dv\,dw\right) \quad (k \geqq 2).\end{aligned}$$

According to the definition of the modified homotopy perturbation (6), the coefficients of p powers are equal to zero, so we can get an iterative algorithm to solve (4).

Algorithm:

$$\begin{aligned}&u_{0,n}(t,s) = f_1(t,s,n)\\ &u_{1,n}(t,s) = f_2(t,s,n) + \frac{1}{t^2s^2+n^2}A_{0,n}(t,s) + \frac{1}{n^4e^{t^2s^2}}\int_0^{s^2}\int_0^{t^2}\widehat{A}_{0,n}(v,w)dv\,dw\\ &u_{k,n}(t,s)) = \frac{1}{t^2s^2+n^2}A_{k-1,n}(t,s) + \frac{1}{n^4e^{t^2s^2}}\int_0^{s^2}\int_0^{t^2}\widehat{A}_{k-1,n}(v,w)dv\,dw \quad (k \geq 2).\end{aligned} \tag{13}$$

For sample, we compute some elements of $\{x_1(t,s), x_2(t,s), \cdots\}$ by using the above algorithm, in which the Adomian polynomials are given by

$$A_{0,n}(t,s) = \sum_{i=n}^{3n}\frac{u_{0,n}(t,s)}{4i^2}; \tag{14}$$

$$\widehat{A}_{0,n}(t,s) = \frac{\cos\left(\sum_{i=1}^{n} u_{0,i}(t,s)\right)}{4+\sin\left(1+\sum_{i=1}^{2n} u_{0,i}(t,s)\right)}. \tag{15}$$

Since in (4) $f(t,s) = 0$ then we can choose $f_1(t,s) = f_2(t,s) = 0$ or $f_1(t,s) = -f_2(t,s) = \alpha \in \mathbb{R}_+$. For sample in the algorithm (13), we set $u_{0,n}(t,s) = f_1(t,s,n) = f_2(t,s,n) = 0$ and using Adomian polynomials (14)-(15)) for $n = 1$, we conclude that,

$$\begin{aligned}&u_{0,1}(t,s) = f_1(t,s,1) = 0,\\ &u_{1,1}(t,s) = \frac{2}{t^2s^2+1^2}A_{0,1}(t,s) + \frac{1}{1^4e^{t^2s^2}}\int_0^{s^2}\int_0^{t^2}\widehat{A}_{0,1}(v,w)dv\,dw = 0.206549\, e^{-s^2t^2}s^2t^2.\end{aligned}$$

Since the solution of (4) is equal to $\lim_{n\to\infty} x_n(t,s)$ and, according to (7),

$$x_n(t,s) \simeq \lim_{p\to 1} u_n(t,s) = \lim_{p\to 1}\sum_{j=0}^{\infty} p^j u_{j,n}(t,s),$$

we approximate the first few elements of the sequence $(x_n(t,s))_{n=1}^{\infty}$ in terms of the above approximations. So we have

$$x_1(t,s) \simeq \sum_{j=0}^{1} u_{j,1}(t,s) = 0.206549\ e^{-s^2t^2}s^2t^2. \tag{16}$$

Similarly, we compute some elements of the above sequence by using *Mathematica* (Version 10) as follows:

$$\begin{aligned} x_5(t,s) &= 0.000330478\ e^{-s^2t^2}s^2t^2; \\ x_{50}(t,s) &= 3.30478 \times 10^{-8} e^{-s^2t^2}s^2t^2; \\ x_{100}(t,s) &= 2.06549 \times 10^{-9} e^{-s^2t^2}s^2t^2; \\ x_{200}(t,s) &= 1.29093 \times 10^{-10} e^{-s^2t^2}s^2t^2; \\ x_{500}(t,s) &= 3.30478 \times 10^{-12} e^{-s^2t^2}s^2t^2. \end{aligned} \tag{17}$$

In light of (16)-(17), we see that $x_1(t,s), x_5(t,s), x_{50}(t,s), x_{100}(t,s), x_{200}(t,s), x_{500}(t,s)$ are decreasing and convergent to zero function (see values on the third axis in Figure 1) in the space c. Also, by plotting the elements of the above sequence, the convergence is verified.

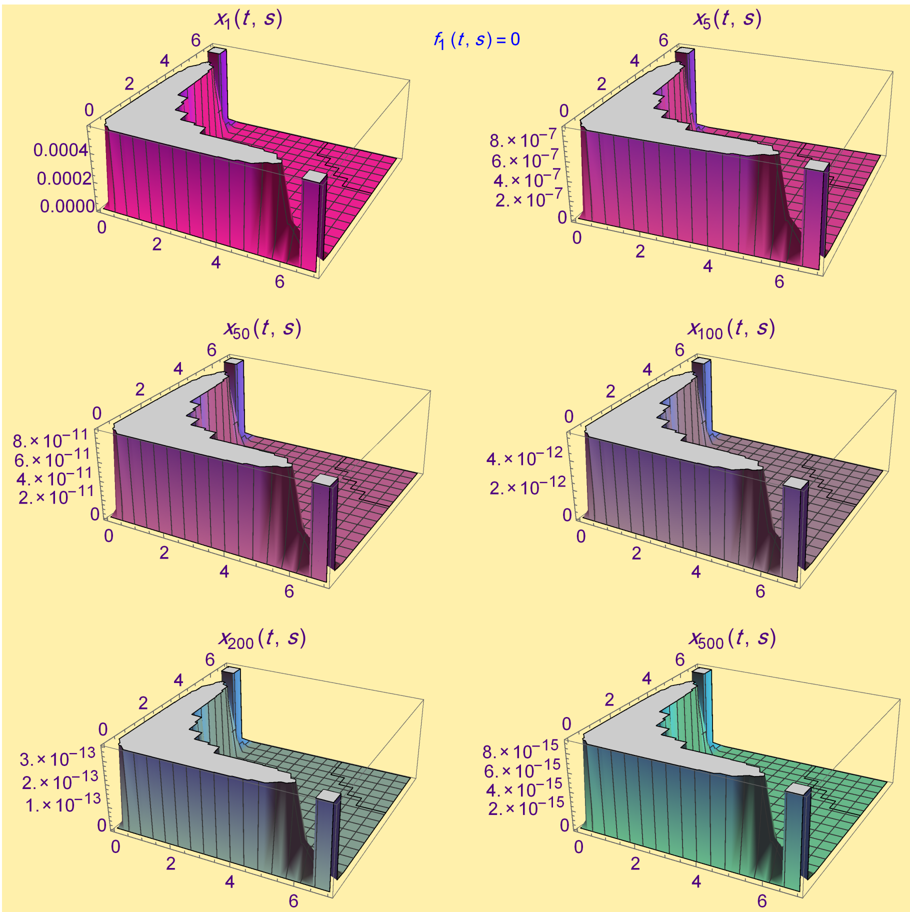

Figure 1: $x_1(t,s), x_5(t,s), x_{50}(t,s), x_{100}(t,s), x_{200}(t,s), x_{500}(t,s)$

6. Concluding Remarks and Observations

In this article, we proved existence results for the solution of an infinite systems of nonlinear integral equations in two variables. We presented an illustrative example to illustrate the efficiency of our results. Moreover, we introduced a coupled semi-analytic method in the case of two variables in order to construct an iteration algorithm to get the solution of the above-mentioned infinite system of nonlinear integral equations in two variables. The numerical results, which we presented in this article, show that the produced sequence for approximating the solution in the examples is in the sequence space c itself.

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