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Almost Polar Elements in a Ring

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Abstract. In this paper we define and study a generalized polar elements in a ring. Various characterizations of such elements have explored. The relations between quasipolar and almost polar elements in rings are obtained. Finally, we characterize almost polar elements in a ring by their Drazin-like property.

1. Introduction

Suppose A is an associative ring with identity 1, and $U(A)$ denote the set of invertible elements of A. The commutant of $a \in A$ is defined by $comm(a) = \{x \in A \mid xa = ax\}$. The double commutant of $a \in A$ is defined by $comm^2(a) = \{x \in A \mid xy = yx \text{ for all } y \in comm(a)\}$. Then an element $a \in A$ has been defined (Koliha [4]) to be quasinilpotent provided that

$$a \in A^{qnil} = \{a \in A : 1 - comm(a)a \subseteq U(A)\}.$$

When A is a complex Banach algebra, then (Harte [3])

$$a \in A^{qnil} \Leftrightarrow \| a^n \|^{\frac{1}{n}} \mapsto 0 \ (n \mapsto \infty).$$

We define $a \in A$ to be quasipolar provided that there is a spectral idempotent, $e^2 = e \in comm^2(a)$ such that

$$a + e \in U(A), \text{ and } ae \in A^{qnil}. \qquad (*)$$

Equivalently, $a \in A$ has a Drazin inverse

$$a^{\times} = (a + e)^{-1}(1 - e),$$

where $e = e^2$ is the spectral idempotent of $(*)$. We shall write $QP(A)$ for the set of $a \in A$ for which $(*)$ is valid; following Koliha, we say that the ring A is quasipolar provided that

$$A = QP(A).$$

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In the present paper we extend these ideals to almost polar ring elements, where have more general version of spectral idempotent, $e \in comm^2(a)$ for which

$$e - e^2 \in J(A), a + e \in U(A), \text{ and } ae \in A^{qnil}.$$

Here $J(A)$ is the Jacobson radical of the ring A:

$$J(A) = \{a \in A : 1 - Aa \subseteq U(A)\}. \qquad (**)$$

We shall call the ring A almost polar when every element $a \in A$ is almost polar in the sense $(**)$. While we make the definition for general rings, we of course prove the theorems for complex Banach algebras.

In Section 2, we shall investigate various characterizations of such elements in a ring. For a Banach algebra, we prove that an element $a \in A$ is almost polar if and only if there exists $r > 0$ such that $\lambda - a \in U(A)$ for all $0 <\mid \lambda \mid< r$ and

$$a + \frac{1}{2\pi i}\int_\gamma (\lambda - a)^{-1} d\lambda \in U(A),$$
$$\lim_{n\to\infty} \|(\frac{a}{2\pi i}\int_\gamma (\lambda - a)^{-1} d\lambda)^n\|^{\frac{1}{n}} = 0,$$
$$\frac{1}{2\pi i}\int_\gamma (\lambda - a)^{-1} d\lambda - \Big(\frac{1}{2\pi i}\int_\gamma (\lambda - a)^{-1} d\lambda\Big)^2 \in J(A),$$

where γ is a $(\{\lambda \mid \quad \mid \lambda \mid< r\}, \{0\})$-cycle.

In Section 3, we are concern with the relations between quasipolar and almost polar elements in a ring. Let A be a ring with nil Jacobson radical $J(A) \subseteq N(A)$, where $N(A)$ is the set of nilpotent elements of A. We prove that A is almost polar if and only if A is quasipolar.

Recall that an element $a \in A$ has a generalized Drazin inverse $b \in A$ if, $b \in comm^2(a), b = b^2a, a - a^2b \in A^{qnil}$. Finally, in the last section, we prove that a ring A is almost polar if and only if for any $a \in A$ there exists $b \in comm^2(a)$ such that

$$b - b^2a \in J(A), a - a^2b \in A^{qnil}.$$

A new characterization of generalized Drazin inverses in a ring is thereby obtained.

Throughout the paper, all rings are associative and Banach algebras are complex with an identity. $\mathbb{N}$ stands for the set of all natural numbers.

2. Equivalent characterizations

The aim of this section is to explore characterizations of almost polar elements in a ring. As a consequence, we give a new characterization of quasipolar element in Banach algebras by using the function calculus.

Theorem 2.1. *Let A be a ring, and $a \in A$. Then the following are equivalent:*

(1) *a is almost polar.*
(2) *There exists $e \in comm^2(a)$ such that*

$$1 - e \in aAa, e - e^2 \in J(A), ae \in A^{qnil}.$$

Proof. (1) ⇒ (2) Let $a \in A$. Then there exists $e \in comm^2(a)$ such that $u := a + e \in U(A)$ and $ae \in A^{qnil}$. Hence, $au^{-1} + eu^{-1} = 1$, and so $au^{-1}(1 - e) = 1 - e - eu^{-1}(1 - e)$. Likewise, $(1 - e)u^{-1}a = 1 - e - (1 - e)u^{-1}e$. Clearly, $u^{-1} \in comm^2(a)$. Set $g = e + (1 - e)u^{-1}e$. Then $g \in comm^2(a), g^2 - g \in J(A)$ and $1 - g = au^{-1}(1 - e)$. Choose $f = g(2 - g)$. Then $f \in comm^2(a), af \in A^{qnil}, f - f^2 \in J(A)$. Moreover, $1 - f = (1 - g)^2 \in aAa$, as desired.

(2) $\Rightarrow$ (1) Let $a \in A$. Then there exists $e \in comm^2(a)$ such that $e - e^2 \in J(A), 1 - e \in aAa$ and $ae \in A^{qnil}$. Write $1 - e = ara$. Then

$$\begin{aligned}(a+e)(arara+e) &= (a+e)((1-e)ra+e)\\ &= (1-e)^2 + ae + (e-e^2)ra + e^2\\ &= (1+ae) + (e-e^2)(ra-2)\\ &\in U(A).\end{aligned}$$

Similarly, $(arara + e)(a + e) \in U(A)$. Accordingly, $a + e \in U(A)$, as required. □

Corollary 2.2. *Let A be a ring, and $a \in A$. Then the following are equivalent:*

(1) *a is almost polar.*
(2) *There exists $e \in comm^2(a)$ such that $e - e^2 \in J(A), ae \in A^{qnil}$ and $(1 - ax)(1 - e) \in J(A)$ for some $x \in comm^2(a)$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in A$. Then there exists $e \in comm^2(a)$ such that $a+e = u \in U(A), ae \in A^{qnil}$ and $e-e^2 \in J(A)$. Hence, $au^{-1}+eu^{-1} = 1$, and so $au^{-1}(1-e) = 1-e-eu^{-1}(1-e)$. This shows that $(1-au^{-1})(1-e) = u^{-1}(e-e^2) \in J(A)$, as desired.

(2) $\Rightarrow$ (1) Let $a \in A$. There exists $e \in comm^2(a)$ such that $e-e^2 \in J(A), ae \in A^{qnil}$ and $w := (1-ax)(1-e) \in J(A)$ for some $x \in comm^2(a)$. Hence, $ax(1 - e) = 1 - e - w$. Let $f = e + w$. Then $f \in comm^2(a), f - f^2 \in J(A)$ and $1-f \in aA \bigcap Aa$. Set $g = f(2-f)$. Then $g \in comm^2(a), g-g^2 = (1-f)(2-f)(f-f^2) \in J(A)$ and $1-g = (1-f)^2 \in aAa$.

Moreover, $af = ae + aw \in A^{qnil} + J(A) \subseteq A^{qnil}$, and so $ag \in A^{qnil}$ by [2, Theorem 7.4.3]. This completes the proof, by Theorem 2.1. □

Proposition 2.3. *Let A be a ring, and $a \in A$. Then the following are equivalent:*

(1) *a is almost polar.*
(2) *There exists $e \in comm^2(a)$ such that $e - e^2 \in J(A), ae \in A^{qnil}$ and $ua - e \in U(A)$ for any $u \in U(A) \cap comm(a)$.*
(3) *There exists $e \in comm^2(a)$ such that $e - e^2 \in J(A), ae \in A^{qnil}$ and $ua - e \in U(A)$ for some $u \in U(A) \cap comm(a)$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in A$. Then there exists $e \in comm^2(a)$ such that $a + e \in U(A)$, $e - e^2 \in J(A), ae \in A^{qnil}$. Let $u \in U(A) \cap comm(a)$. Since $ae \in A^{qnil}$, $1 - (ae)u \in U(A)$ and so $ae - u^{-1} \in U(A)$. Consider $b = (ae - u^{-1})^{-1}u^{-1}e + (a + e)^{-1}u^{-1}(1 - e)$. Then,

$$\begin{aligned}&b(ua-e)\\ =\ &((ae-u^{-1})^{-1}u^{-1}e + (a+e)^{-1}u^{-1}(1-e))(ua-e)\\ =\ &(ae-u^{-1})^{-1}u^{-1}e(ua-e) + (a+e)^{-1}u^{-1}(1-e)(ua-e)\\ =\ &(ae-u^{-1})^{-1}(ae-u^{-1}e^2) + (a+e)^{-1}(1-e)(a-u^{-1}e)\\ =\ &1 + j \text{ for some } j \in J(A),\end{aligned}$$

as $e - e^2 \in J(A)$. Hence $b(ua - e) \in U(A)$. Therefore $ua - e$ is left invertible. Similarly, it can be shown that $(ua - e)b \in U(A)$. Consequently, $ua - e \in U(A)$.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Let $a \in A$. Then there exists $e \in comm^2(a)$ such that

$$e - e^2 \in J(A), ae \in A^{qnil} \text{ and } ua - e \in U(A)$$

for some $u \in U(A) \cap comm(a)$. Hence $a - u^{-1}e \in U(A)$. Since $(a - u^{-1}e)(1 - e) = a(1 - e) = (1 - e)a$, we easily check that

$$(1-e)^2 = a(1-e)(a-u^{-1}e)^{-2}(1-e)a.$$

Set $g = e(2 - e)$. Then $g \in comm^2(a), g - g^2 = 2e - e^2 - 4e^2 + 4e^3 - e^4 \in J(A)$ and $1 - g = (1 - e)^2 = aAa$. In view of [2, Theorem 7.4.3], we see that $ag = (2 - e)(ae) \in A^{qnil}$. Therefore a is almost polar, by Theorem 2.1. □

Corollary 2.4. *A Banach algebra A is almost polar if and only if for any $a \in A$ there exists $e \in comm^2(a)$ such that*

$$e - e^2 \in J(A), \lim_{n\to\infty} \|(ae)^n\|^{\frac{1}{n}} = 0 \text{ and } a + \alpha e \in U(A)$$

for some $\alpha \in \mathbb{C}$.

Proof. $\Longrightarrow$ This is obvious by choosing $\alpha = 1$.

$\Longleftarrow$ Let $a \in A$. Then there exists $e \in comm^2(a)$ such that

$$e - e^2 \in J(A), \lim_{n\to\infty} \|(ae)^n\|^{\frac{1}{n}} = 0 \text{ and } a + \alpha e \in U(A)$$

for some $\alpha \in \mathbb{C}$. If $\alpha = 0$, we choose $u = -1$. If $\alpha \neq 0$, we choose $u = -\frac{1}{\alpha}$. By using Proposition 2.3, A is almost polar, as asserted. $\square$

Now we come to the main result of this section.

Theorem 2.5. *Let A be a Banach algebra, and $a \in A$. Then the following are equivalent:*

(1) *a is almost polar.*
(2) *There exists $r > 0$ such that $\lambda - a \in U(A)$ for all $0 <| \lambda |< r$ and*

$$a + \frac{1}{2\pi i}\int_\gamma (\lambda - a)^{-1} d\lambda \in U(A),$$
$$\lim_{n\to\infty} \|(\frac{a}{2\pi i}\int_\gamma (\lambda - a)^{-1} d\lambda)^n\|^{\frac{1}{n}} = 0,$$
$$\frac{1}{2\pi i}\int_\gamma (\lambda - a)^{-1} d\lambda - \Big(\frac{1}{2\pi i}\int_\gamma (\lambda - a)^{-1} d\lambda\Big)^2 \in J(A),$$

where γ is a $(\{\lambda \mid \ | \lambda |< r\}, \{0\})$-cycle.

Proof. (1) $\Rightarrow$ (2) Let $a \in A$. Then there exists $e \in comm^2(a)$ such that

$$a + e \in U(A), e - e^2 \in J(A) \text{ and } ae \in A^{qnil}.$$

For any λ we have

$$\lambda - a = (\lambda - ae)e + (\lambda - (e + a))(1 - e) + w,$$

where $w = (1 - a)(e^2 - e) \in J(A)$. Taking $r = \frac{1}{\|(e+a)^{-1}\|}$. Then $\lambda - (e + a) \in U(A)$ if $|\lambda| < r$. As $ae \in A^{qnil}$, we see that $\lambda - ae \in U(A)$ for all $\lambda \neq 0$.

Clearly, $\lambda - a \in U(A)$. Suppose that $(\lambda - a)v = 1$ for some $v \in A$. Then $((\lambda - ae)e + (\lambda - (e+a))(1 - e))v + wv = 1$; hence, $((\lambda - ae)e + (\lambda - (e+a))(1 - e))v(1 - wv)^{-1} = 1$. As $(1 - wv)(1 - wv)^{-1} = 1$, we see that $(1 - wv)^{-1} - (1 - wv)^{-1}vw = 1$; and so $(1 - wv)^{-1} = 1 + (1 - wv)^{-1}vw$. We deduce that $((\lambda - ae)e + (\lambda - (e + a))(1 - e))v(1 + (1 - wv)^{-1}vw) = 1$. This implies that

$$\begin{aligned} & ((\lambda - ae)e + (\lambda - (e + a))(1 - e))v \\ = \ & 1 - ((\lambda - ae)e + (\lambda - (e + a))(1 - e))v(1 + (1 - wv)^{-1}vw) \\ = \ & 1. \end{aligned}$$

Thus

$$\begin{aligned} & ((\lambda - ae)e + (\lambda - (e + a))(1 - e))v \\ = \ & 1 - ((\lambda - ae)e + (\lambda - (e + a))(1 - e))v \\ = \ & 1 - ((\lambda - ae)e + (\lambda - (e + a))(1 - e))v \\ = \ & 1 - ((\lambda - ae)e + (\lambda - (e + a))(1 - e))v(1 - wv)^{-1}vw. \end{aligned}$$

Hence, $v = ((\lambda - ae)^{-1}e + (\lambda - (e + a))^{-1}(1 - e))(1 - ((\lambda - ae)e + (\lambda - (e + a))(1 - e))v(1 - wv)^{-1}vw)$. Therefore

$$(\lambda - a)^{-1} = (\lambda - ae)^{-1}e + (\lambda - (e + a))^{-1}(1 - e) + s,$$

where $s \in A(e^2 - e) \subseteq J(A)$ whenever $0 < |\lambda| < r$. Furthermore, we have

$$\begin{aligned} & \frac{1}{2\pi i}\int_\gamma (\lambda - a)^{-1} d\lambda \\ = \ & \frac{1}{2\pi i}\int_\gamma (\lambda - ae)^{-1} e d\lambda + \frac{1}{2\pi i}\int_\gamma (\lambda - (e + a))^{-1}(1 - e) d\lambda + t \\ = \ & \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} \lambda^{-n-1} a^n e d\lambda + 0 + t \\ = \ & p + t \end{aligned}$$

where $t \in A(e^2 - e) \subseteq J(A)$ and γ is a $(\{\lambda \mid \ \mid \lambda \mid < r\}, \{0\})$-cycle. Thus, $e = \frac{1}{2\pi i} \int_{\gamma} (\lambda - a)^{-1} d\lambda - t$ for some $t \in J(A)$. We easily check the preceding conditions in (2) are satisfied.

(2) $\Rightarrow$ (1) Let $x \in comm(a)$. Then $ax = xa$, and so $x(\lambda - a)^{-1} = (\lambda - a)^{-1}x$. Hence,

$$x \cdot \frac{1}{2\pi i} \int_{\gamma} (\lambda - a)^{-1} d\lambda = \frac{1}{2\pi i} \int_{\gamma} (\lambda - a)^{-1} d\lambda \cdot x.$$

Hence, $\frac{1}{2\pi i} \int_{\gamma} (\lambda - a)^{-1} d\lambda \in comm^2(a)$. Therefore a is almost polar. $\square$

3. Strong Lifting

As is well known, idempotents in Banach algebras always lift modulo the Jacobson radical, but idempotents in an almost polar ring might not be liftable modulo its Jacobson radical, we now give the relations between quasipolar and almost polar rings by means of the lifting property. We say that every idempotent strongly lifts modulo $J(A)$ if for any $a \in A, e \in comm^2(a)$, $e - e^2 \in J(A)$ implies that there exists $f^2 = f \in comm^2(a)$ such that $e - f \in J(A)$. We have

Lemma 3.1. *A ring A is quasipolar if and only if*

(1) *A is almost polar;*
(2) *Every idempotent strongly lifts modulo $J(A)$.*

Proof. $\Longrightarrow$ Clearly, A is almost polar. Suppose that $e - e^2 \in J(A)$ with $e \in comm^2(a)$. Since A is quasipolar, there exists $f^2 = f \in comm^2(e)$ such that $e - f \in U(A)$ and $ef \in A^{qnil}$. As $e \in comm^2(a)$, we see that $ea = ae$. Thus $ef = fe$. Clearly, $\overline{(e-f)^3} = \overline{e-f}$; and so $\overline{e-f}^2 = \overline{1}$. Hence, $\overline{e - 2ef + f} = \overline{1}$; hence, $\overline{e} = \overline{(1-2f)(1-f)} = \overline{1-f}$. Therefore $e - (1 - f) \in J(A)$. If $x \in comm(a)$, then $xa = ax$; and so $ex = xe$. This shows that $fx = xf$, and then $(1 - f)x = x(1 - f)$, i.e., $1 - f \in comm^2(a)$, as desired.

$\Longleftarrow$ Let $a \in A$. Then there exists $e \in comm^2(a)$ such that

$$(-a) + e \in U(A), e - e^2 \in J(A) \text{ and } (-a)e \in A^{qnil}.$$

By hypothesis, we can find $f^2 = f \in comm^2(a)$ such that $e - f \in J(A)$. Hence $a - f = (a - e) + (e - f) \in U(A) + J(A) \subseteq U(A)$. Moreover, $af = ae + a(f - e) \in A^{qnil} + J(A) \subseteq A^{qnil}$, as asserted. $\square$

Lemma 3.2. *Every idempotent in a ring strongly lifts modulo any nil ideal.*

Proof. Let $I \subseteq N(A)$ be a nil ideal of a ring A. Let $a \in A$ and $e \in comm^2(a)$ such that $e^2 - e \in I$. As I is a nil ideal, there exists $n \in \mathbb{N}$ such that $(e - e^2)^n = e^n(1 - e)^n = 0$. Since $(e + (1 - e))^{2n} = 1$,

$$1 = \sum_{i=0}^{n} \begin{pmatrix} 2n \\ i \end{pmatrix} e^i (1-e)^{2n-i} + \sum_{i=n+1}^{2n} \begin{pmatrix} 2n \\ i \end{pmatrix} e^i (1-e)^{2n-i}.$$

Take $f = \sum_{i=0}^{n} \begin{pmatrix} 2n \\ i \end{pmatrix} e^i (1-e)^{2n-i}$ and $g = 1 - f$. Hence, $f + g = 1$ and $f^2 + fg = f$. Since $fg = 0$, $f^2 = f$ is an idempotent such that $f \in comm^2(a)$. Also $g^2 = g \in comm^2(a)$. It is easy to show that $e - g \in I$. Therefore, every idempotent strongly lifts modulo any nil ideal. $\square$

We now have at our disposal all the information necessary to prove the following.

Theorem 3.3. *Let A be a ring with nil Jacobson radical. Then the following are equivalent for $a \in A$:*

(1) *a is almost polar.*
(2) *a is quasipolar.*

Proof. In view of Lemma 3.2, every idempotent strongly lifts modulo $J(A)$. Therefore we complete the proof by Lemma 3.1. □

Recall that a ring A is polar if and only if for any $a \in A$ there exists $e^2 = e \in comm^2(a)$ such that $a - e \in U(A)$ and $ae \in N(A)$ (see [5]).

Corollary 3.4. *A ring A is polar if and only if*

(1) *A is almost polar;*
(2) $A^{qnil} = N(A)$.

Proof. This is obvious, by [1, Theorem 2.6]. □

4. Drazin-like Characterizations

As is well known, an element $a \in A$ is quasipolar if and only if there exists $b \in comm^2(a)$ such that $b = b^2a, a - a^2b \in A^{qnil}$. We now characterize almost polar element in a ring by a kind of Drazin-like property.

Theorem 4.1. *An element a in a ring A is almost polar if and only if there exists $b \in comm^2(a)$ such that*

$$b - b^2a \in J(A), a - a^2b \in A^{qnil}.$$

Proof. $\Longrightarrow$ By hypothesis, there exists some $p \in comm^2(a)$ such that $a + p \in U(A), p - p^2 \in J(A)$ and $ap \in A^{qnil}$. Set $b = (1 - p)(a + p)^{-1}$. Then $b \in comm^2(a)$ and $b^2a - b \in J(A)$. Further, we have

$$\begin{array}{rcl} a - a^2b & = & a - a^2(1 - p)(a + p)^{-1} \\ & = & (a + p)^{-1}(a(a + p) - a^2(1 - p)) \\ & = & (a + p)^{-1}(a^2p + ap) \\ & = & (a + p)^{-1}ap(a + 1) \in A^{qnil}, \end{array}$$

as desired.

$\Longleftarrow$ Let $p = 1-ab$. Then $p \in comm^2(a), p-p^2 \in J(A)$ and $a+p = a+1-ab$. We check that $(a+1-ab)(b+1-ab) = 1 + a(1 - ab) - ((-b + b^2a)(1 - a)) = 1 - (a - a^2b) + ((-b + b^2a)(1 - a)) \in U(A)$, and so $a + p \in U(A)$. Additionally, $ap = a - a^2b \in A^{qnil}$, as desired. □

Corollary 4.2. *Let A be a Banach algebra and $a \in A$. Then the following are equivalent:*

(1) *a is almost polar.*
(2) *There exists $b \in comm^2(a)$ such that $ba - (ba)^2 \in J(A)$, $a - a^2b \in A^{qnil}$.*

Proof. (1) ⇒ (2) Let $a \in A$. In view of Theorem 4.1, there exists $b \in comm^2(a)$ such that

$$b - b^2a \in J(A), a - a^2b \in A^{qnil}.$$

Then $ba - (ba)^2 = a(b - b^2a) \in J(A)$, as required.

(2) ⇒ (1) Let $a \in A$. Then there exists $b \in comm^2(a)$ such that

$$ba - (ba)^2 \in J(A), a - a^2b \in A^{qnil}.$$

Set $c = bab$. Then $c \in comm^2(a)$. One easily checks that

$$c - c^2a = bab - (bab)^2a = (ba - (ba)^2)b + (ba - (ba)^2)bab \in J(A).$$

Moreover,

$$\lim_{n\to\infty} \|(a - a^2c)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|((1 + ab)(a - a^2b))^n\|^{\frac{1}{n}} = 0.$$

In light of [2, Definition 7.4.1], $a - a^2c \in A^{qnil}$. This completes the proof by Theorem 4.1. □

Corollary 4.3. *Let A be a Banach algebra. Then the following are equivalent:*

(1) *A is polar.*
(2) *For any $a \in A$, there exists $b \in comm^2(a)$ such that $b - b^2a, a - a^2b$ are nilpotent.*

Proof. (1) $\Rightarrow$ (2) This is obvious by [5, Proposition 4.9].

(2) $\Rightarrow$ (1) In light of Theorem 4.1, A is almost polar. Let $a \in A^{qnil}$. Then $a - a^2b \in N(A)$ for some $b \in comm^2(a)$. Hence, $a(1 - ab) \in N(A)$, and so $a \in N(A)$, i.e., $A^{qnil} = N(A)$. Therefore A is polar, by Corollary 3.4. □

We are now ready to prove:

Theorem 4.4. *Let A be a Banach algebra, and $a \in A$. Then a is almost polar if and only if there exists $b \in comm^2(a)$ and $k, m \in \mathbb{N}$ such that*

$$b^k - b^{k+1}a \in J(A), a^m - a^{m+1}b \in A^{qnil}.$$

Proof. $\Longrightarrow$ This is obvious by choosing $k = m = 1$ in Theorem 4.1.

$\Longleftarrow$ Step 1. $m = 1$. Let $a \in A$. Then there exists $b \in comm^2(a)$ and $k \in \mathbb{N}$ such that

$$b^k - b^{k+1}a \in J(A), \lim_{n\to\infty} \|(a - a^2b)^n\|^{\frac{1}{n}} = 0.$$

Then $b^{k-1}(b - b^2a) \in J(A)$, and so

$$\begin{array}{rcl} (ba - b^2a^2)^k & = & (b - b^2a)^k a^k \\ & = & (b - b^2a)^{k-1}(b - b^2a)a^k \\ & = & (1 - ba)^{k-1}b^{k-1}(b - b^2a)a^k \\ & = & (1 - ba)^{k-1}(b^k - b^{k+1}a)a^k \\ & \in & J(A). \end{array}$$

Set $p = 1 - ba$. Then $p^k(1 - p)^k \in J(A)$. One easily checks that

$$\begin{array}{rcl} 1 & = & (p + (1 - p))^{2k} \\ & = & \sum\limits_{i=0}^{k} \binom{2k}{i} p^{2k-i}(1 - p)^i + \sum\limits_{i=k+1}^{2k} \binom{2k}{i} p^{2k-i}(1 - p)^i. \end{array}$$

Take $e = \sum\limits_{i=0}^{k} \binom{2k}{i} p^{2k-i}(1 - p)^i$ and $f = 1 - e$. Clearly, $e + f = 1$ and $ef = fe \in J(A)$. Thus $e \in comm^2(p)$ and $e - e^2 = ef \in J(A)$. As $p \in comm^2(a)$, we have $e \in comm^2(a)$. In view of [2, Theorem 7.4.3], we see that

$$ae = \Big(\sum_{i=0}^{k} \binom{2k}{i} p^{2k-i-1}(1 - p)^i\Big)(ap) \in A^{qnil}.$$

Obviously, we have $(a - 1 + ab)(b - 1 + ab) = 1 - (1 + b)(a - a^2b) - (b - ab^2)$. Since $a - a^2b \in A^{qnil}$, we see that $1-(1+b)(a-a^2b) \in U(A)$. As $(b-b^2a)^k = (1-ba)^{k-1}(b^{k-1}(b-b^2a)) \in J(A)$, we see that $\big(1-(1+b)(a-a^2b)\big)^k - (b-b^2a)^k \in U(A)$, and then $(a - 1 + ab)(b - 1 + ab) \in U(A)$. That is, $a - p \in U(A)$. On the other hand,

$$\begin{array}{rcl} p - e & = & p - \sum\limits_{i=0}^{k} \binom{2k}{i} p^{2k-i}(1 - p)^i \\ & = & \sum\limits_{i=0}^{2k-2} p^i(p - p^2) + \sum\limits_{i=1}^{k} \binom{2k}{i} p^{2k-i}(1 - p)^i \\ & = & z(p - p^2) \end{array}$$

for some $z \in comm^2(p)$. As $(p-p^2)^k \in J(A)$, we have $(p-e)^k \in J(A)$. Thus $(a-p)^k - (e-p)^k \in U(A)$, and therefore $a-e = (a-p)-(e-p) \in U(A)$. Accordingly, A is almost polar.

Step 2. Let $a \in A$. There exists $b \in comm^2(a)$ and $k, m \in \mathbb{N}$ such that $b^k - b^{k+1}a \in J(A), a^m - a^{m+1}b \in A^{qnil}$. Hence $a^{m-1}(a-a^2b) \in A^{qnil}$, and so $(a-a^2b)^m \in A^{qnil}$. Let $x \in comm(a-a^2b)$. Then $x \in comm(a-a^2b)^m$; hence, $1-(a-a^2b)^m x^m \in U(A)$. This implies that $1-(a-a^2b)x \in U(A)$. Therefore $a - a^2b \in A^{qnil}$. By using Step 1, we see that a is almost polar, as asserted. $\square$

We now derive a new characterization of quasipolar elements in a Banach algebra.

Corollary 4.5. *Let A be a Banach algebra, and $a \in A$. Then a is qusipolar if and only if there exists $b \in comm^2(a)$ and $k, m \in \mathbb{N}$ such that*

$$b^k = b^{k+1}a, a^m - a^{m+1}b \in A^{qnil}.$$

Proof. $\Longrightarrow$ This is clear by [5, Theorem 4.2].

$\Longleftarrow$ In view of Theorem 4.4, A is almost polar. Suppose that $p - p^2 \in J(A)$ with $p \in comm^2(a), a \in A$. As in the proof of Theorem 4.4, there exists an idempotent $e \in comm^2(p)$ such that $p - e \in U(A)$. Hence, $(p-e)(1-(p+e)) = (p-e)-(p^2-e^2) = p-p^2 \in J(A)$. It follows that $p-(1-e) \in J(A)$. Obviously, $(1-e)^2 = 1-e \in comm^2(a)$. That is, every idempotent strongly lifts modulo $J(A)$. This completes the proof, by Theorem 3.1. $\square$

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