Filomat 33:17 (2019), 5651–5659
https://doi.org/10.2298/FIL1917651L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Study of Global Stability of a Diffusive Michaelis-Menten and Tanner Predator-Prey Model

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Abstract. In this paper, we consider a parabolic predator-prey model of Michaelis-Menten and Tanner functional response with random diffusion:

$$u_t = d_1\Delta u + au - bu^2 - \frac{\delta uv}{\alpha u + v},$$
$$v_t = d_2\Delta v + rv - \gamma\frac{v^2}{u}$$

with $d_1, d_2, a, b, r, \alpha, \gamma, \delta > 0$ under the no-flux boundary condition in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 1, 2, 3)$. By applying a new method, we establish much improved global asymptotic stability of the unique positive equilibrium solution than works in literature. We also show the result can be extended to more general type of systems with heterogeneous environment.

1. Introduction

The main purpose of this article is to consider the following parabolic predator-prey model with Michaelis-Menten and Tanner functional response

$$(I)\begin{cases} u_t = d_1\Delta u + au - bu^2 - \dfrac{\delta uv}{\alpha u + v}, & x \in \Omega, \quad t \in (0,\infty), \\ v_t = d_2\Delta v + rv - \gamma\dfrac{v^2}{u}, & x \in \Omega, \quad t \in (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0,\infty), \\ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) \geq 0 (\not\equiv 0), & x \in \overline{\Omega}, \end{cases} \tag{1}$$

where $d_1, d_2, a, b, r, \alpha, \gamma, \delta > 0$, Ω is a bounded domain in $\mathbb{R}^n (n = 1, 2, 3)$ with smooth boundary $\partial\Omega$, $0 < T \leq +\infty$, and $u(x,t)$ and $v(x,t)$ are the density of prey and predator, respectively. Throughout this article, we suppose that the two diffusion coefficients d_1 and d_2 are equal, but not necessarily constants. We shall apply

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2010 *Mathematics Subject Classification*. Primary 35K57, 34C37; Secondary 92D25

Keywords. Global asymptotic stability, Positive equilibrium solution, Comparison principle, Diffusive predator-prey systems
Received: 26 February 2019; Accepted: 24 April 2019
Communicated by Maria Alessandra Ragusa
This work was supported in part by the Natural Science Foundation of Guangdong Province (No.2018A030313505).
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d to stand for the common value from now on. It may rely on both time variables and spatial but strictly positive in $\overline{\Omega} \times [0, \infty)$. The no-flux boundary condition is proposed to guarantee that ecological system is not disturbed by exterior factors which may influence population flow cross the boundary, and therefore internal forces are the sole reason to generate interesting dynamical behavior of the system.

The functional response $\frac{\delta uv}{\alpha u+v}$ was introduced firstly by Michaelis and Menten [1]. They established the following predator-prey model with Michaelis-Menten functional response

$$\begin{cases} \dot{x} = rx - \theta x^2 - \dfrac{\gamma xy}{ax+by}, \\ \dot{y} = -dy + \dfrac{\delta xy}{ax+by}, \end{cases} \tag{2}$$

Wang and Chen [2] established a three-dimensional pulsed input models with delayed Michaelis-Menten functional response. Hsu et al [3] studied the global analysis of a predator-prey system with Michaelis-Menten functional response, i.e. ratio-dependent.

Besides the Michaelis-Menten functional responses mentioned above, there are exist many other well-known functional responses, such as Monod-Haldane type, Holling type (I, II, III, IV) and Hassel-Verley type functional responses and so on. Several researchers investigated and raised many open problems for built predator-prey systems with different types of functional responses. Particularly, In 2005, Wang and Peng [4] considered the positive steady states of a Holling-Tanner prey-predator system with random diffusion

$$\begin{cases} u_t = d_1\Delta u + au - u^2 - \dfrac{uv}{m+u}, & x \in \Omega, \quad t \in (0,\infty), \\ v_t = d_2\Delta v + bv - \dfrac{v^2}{\gamma u}, & x \in \Omega, \quad t \in (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0,\infty), \\ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) \geq 0 (\not\equiv 0), & x \in \overline{\Omega}. \end{cases} \tag{3}$$

They obtained the existence and non-existence of positive non-constant steady states for the above system (3). In addition, they also obtained a certain condition which can guarantee that (3) possesses no positive non-constant steady solution. In the another paper [5], the authors studied the stability of diffusive predator-prey model of Holling-Tanner type (3) by the construction of a standard linearization procedure and a Lyapunov function. Chen and Shi [6] focused attention on the steady states of (3). They applied the defined iteration and comparison principle sequences to prove the global asymptotic stability. Their scientific research achievement improves the earlier one proposed by Wang and Peng [5] which used Lyapunov method. We also note here that the (non-spatial) kinetic equation of system (3) was first introduced by May [8] and Tanner [7], see also [9, 10] and references therein.

Recently, Qi and Zhu [11] studied the global stability of a reaction-diffusion system of predator-prey model (3). Indeed, they established improved global asymptotic stability of the unique positive equilibrium solution in [11]. Besides the papers mentioned above, one can see [12–18, 20, 21] for more detailed information and biological significances of the studied system.

In the present paper by incorporating the ratio-dependent Michaelis-Menten functional response and diffusion term into system (3), motivated by the previous works [6], we will study the global stability of the positive equilibrium solution by applying a new comparison argument, which is more complicated and different from the method applied in paper such as [6]. Therefore, we argue that it is interesting, beneficial and significant to study the global asymptotic stability of (1) since it possesses biological implications and extends the former researches.

Definition 1.1 (Global stability). *Let (u^*, v^*) be a positive solution of model (1). We say that it is global asymptotically stable if any other positive solution $(u(x,t), v(x,t))$ of model (1) has the property*

$$\lim_{t\to\infty} (u(x,t), v(x,t)) = (u^*, v^*).$$

Our main theorem is as follows.

Theorem 1.2. *Suppose $d = d(x,t)$ is strictly positive, bounded and continuous in $\Omega \times [0,+\infty)$, a, b, r, α, γ, and δ are positive constants, $r < a$, then the positive equilibrium solution (u^*, v^*) is globally asymptotically stable in the sense that every solution $u(x,t)$ of (1) satisfies*

$$\lim_{t\to\infty}(u(x,t), v(x,t)) = (u^*, v^*)$$

uniformly in $x \in \Omega$.

Remark 1.3. *The approach we apply here is more powerful than that applied in [6] and more flexible than the Lyapunov function and linear analysis approaches, and the results cover more general settings such as when the Laplace operator is replaced by a uniform elliptic operator. It means we can cover cases with heterogeneous environment.*

Let us denote by

$$\mathcal{L}u = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}$$

a uniform elliptic operator in Ω with continuous coefficients $a_{ij}(x)$, $i, j = 1, 2, \cdots, N$. Therefore, we can easily reveal a outcome similar to Theorem 1.2 for the following initial-boundary value problem:

$$(II)\begin{cases} u_t = \mathcal{L}u + au - bu^2 - \dfrac{\delta uv}{\alpha u + v}, & x \in \Omega, \quad t \in (0,\infty), \\ v_t = \mathcal{L}v + rv - \gamma\dfrac{v^2}{u}, & x \in \Omega, \quad t \in (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0,\infty), \\ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) \geq 0 (\not\equiv 0), & x \in \overline{\Omega}. \end{cases}$$

Theorem 1.4. *Suppose $r < a$ and $a, b, r, \alpha, \gamma, \delta$ are positive constants satisfying the assumption in Theorem 1.2 and $\mathcal{L}$ is a uniform elliptic operator in Ω with continuous coefficients. Then, the unique positive equilibrium (u^*, v^*) of (II) is globally asymptotically stable.*

It is known via a direct and simple computation that (1) possesses a unique positive equilibrium (u^*, v^*), where

$$u^* = \frac{a\left(\alpha + \frac{r}{\gamma}\right) - \delta\frac{r}{\gamma} + \sqrt{\left(a\left(\alpha + \frac{r}{\gamma}\right) - \delta\frac{r}{\gamma}\right)^2}}{2b\left(\alpha + \frac{r}{\gamma}\right)},$$

$$v^* = \frac{r}{\gamma}u^*.$$

Remark 1.5. *To guarantee that the ecosystem (1) has a unique positive equilibrium (u^*, v^*), the condition, $a\left(\alpha + \frac{r}{\gamma}\right) - \delta\frac{r}{\gamma} > 0$, is imposed according to the expressions of u^* and v^*.*

The rest of the paper is organized as follow. In Sect. 2, we prove our main result. We shall argue how to generalize our results to more general setting in Sect. 3, such as different functional responses and time delay.

2. Proof of the main result

We define $w = \frac{v}{u}$, then we obtain

$$\begin{aligned}
w_t &= \frac{v_t u - v u_t}{u^2} = \frac{v_t}{u} - \frac{u_t v}{u^2},\\
\nabla w &= \frac{\nabla v \cdot u - \nabla u \cdot v}{u^2} = \frac{\nabla v}{u} - \frac{\nabla u \cdot v}{u^2},\\
\Delta w &= \frac{\Delta v u^3 + u^2 \nabla u \cdot \nabla v - u^2 v \Delta u - u^2 \nabla u \cdot \nabla v - (\nabla v \cdot u - \nabla u \cdot v)\, 2u \nabla u}{u^4}\\
&= \frac{\Delta v}{u} - \frac{v \Delta u}{u^2} - \frac{2\nabla u \cdot \nabla v}{u^2} + \frac{2v|\nabla u|^2}{u^3}.
\end{aligned}$$

Therefore the equation satisfied by $w(x,t)$ is

$$\begin{aligned}
w_t - d\Delta w =& \left(\frac{v_t}{u} - \frac{u_t v}{u^2}\right) - d\left(\frac{\Delta v}{u} - \frac{v\Delta u}{u^2} - \frac{2\nabla u \cdot \nabla v}{u^2} + \frac{2v|\nabla u|^2}{u^3}\right)\\
=& \frac{v_t - d\Delta v}{u} - \frac{v(u_t - d\Delta u)}{u^2} + 2d\frac{\nabla u}{u}\left(\frac{\nabla v}{u} - \frac{v \nabla u}{u^2}\right)\\
=& \frac{v(r - \gamma\frac{v}{u})}{u} - \frac{v\left(au - bu^2 - \frac{\delta u v}{\alpha u + v}\right)}{u^2} + 2d\frac{\nabla u}{u} \cdot \nabla w\\
=& w\left(r - a + bu + w\left(\frac{\delta u}{\alpha u + v} - \gamma\right)\right) + 2d\frac{\nabla u}{u} \cdot \nabla w.
\end{aligned} \tag{4}$$

Proposition 2.1. *Suppose $r < a$ and $\varepsilon_1 > 0$ small. There exists a sufficiently large constant $T > 0$ such that the solution u of (1) satisfies*

$$u \le \overline{u}_2(\varepsilon_1) \equiv \frac{a\alpha - \frac{r}{\gamma}\underline{u}_1 b + \sqrt{\left(a\alpha - \frac{r}{\gamma}\underline{u}_1 b\right)^2 + 4ab\alpha\frac{r}{\gamma}\underline{u}_1}}{2b\alpha} + O(\varepsilon_1),$$

for $x \in \Omega$ and $t \ge T$, where

$$\begin{aligned}
\underline{u}_1 =& \frac{a(\alpha + \underline{w}_1(\varepsilon_1)) - \delta\underline{w}_1(\varepsilon_1) + \sqrt{\left[a(\alpha + \underline{w}_1(\varepsilon_1)) - \delta\underline{w}_1(\varepsilon_1)\right]^2}}{2b(\alpha + \overline{w}_1(\varepsilon_1))},\\
\overline{w}_1 =& \frac{(r+\delta)\overline{u}_1 + b\overline{u}_1^2 - (a + \alpha\gamma)\overline{u}_1}{2\gamma\overline{u}_1}\\
&+ \frac{\sqrt{\left[(a+\alpha\gamma)\overline{u}_1 - (r+\delta)\overline{u}_1 - b\overline{u}_1^2\right]^2 + 4\gamma\overline{u}_1\left[(r\alpha - a\alpha + b\alpha\overline{u}_1)\overline{u}_1\right]}}{2\gamma\overline{u}_1}.
\end{aligned}$$

and $\overline{u}_1 \equiv \frac{a}{b}$.

Proof. Since $u > 0$, $v \ge 0$, it is easily to verify by a direct calculation that u satisfies

$$u_t - d\Delta u \le u(a - bu), \quad \text{in} \quad \Omega \times (0, \infty).$$

By the well established fact and a simple comparison that any positive solution of

$$\begin{cases}
u_t - d\Delta u \le u(a - bu), & x \in \Omega, \quad t \in (0,\infty),\\
\dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0,\infty),
\end{cases}$$

converges to the asymptotic stable equilibrium $\frac{a}{b}$ uniformly as $t \to \infty$, i.e. $\lim_{t\to\infty} u = \frac{a}{b}$, we can obtain that $\forall \varepsilon_1 > 0$, $\exists t_1 > 0$ such that

$$u(x,t) < \overline{u}_1(\varepsilon_1) \equiv \frac{a}{b} + \frac{\varepsilon_1}{5} \tag{5}$$

for $x \in \Omega$ and $t \geq t_1$. Therefore, for $t \geq t_1$,

$$w_t - d\Delta w \leq w\left(r - a + b\overline{u}_1(\varepsilon_1) + w\left(\frac{\delta \overline{u}_1(\varepsilon_1)}{\alpha \overline{u}_1(\varepsilon_1) + \overline{u}_1(\varepsilon_1) w} - \gamma\right)\right) + 2d\frac{\nabla u}{u} \cdot \nabla w.$$

It is obvious that the following ordinary differential equation (ODE) about $W(t)$

$$W_t = W\left(r - a + b\overline{u}_1(\varepsilon_1) + W\left(\frac{\delta \overline{u}_1(\varepsilon_1)}{\alpha \overline{u}_1(\varepsilon_1) + \overline{u}_1(\varepsilon_1) W} - \gamma\right)\right) \tag{6}$$

possesses three solutions:

$$\begin{aligned} W_0 &= 0, \\ W_{1,2} &= \frac{(r+\delta)\overline{u}_1(\varepsilon_1) + b\overline{u}_1^2(\varepsilon_1) - (a + \alpha\gamma)\overline{u}_1(\varepsilon_1)}{2\gamma \overline{u}_1(\varepsilon_1)} \\ &\pm \frac{\sqrt{\left[(a+\alpha\gamma)\overline{u}_1(\varepsilon_1) - (r+\delta)\overline{u}_1(\varepsilon_1) - b\overline{u}_1^2(\varepsilon_1)\right]^2 + 4\gamma \overline{u}_1(\varepsilon_1)(r - a + b\overline{u}_1(\varepsilon_1))\beta \overline{u}_1(\varepsilon_1)}}{2\gamma \overline{u}_1(\varepsilon_1)}. \end{aligned} \tag{7}$$

It is obvious that $W_1(t)$ is the unique asymptotically stable positive equilibrium point of (6), and $W_0(t) = 0$ is unstable. Since the trajectories of (6) cannot cross the x-axis, then all positive solutions $W(t)$ of (6) will converge to the unique positive asymptotically stable equilibrium point $W_1(t)$. By a simple comparison argument, we obtain that there possesses a positive constant $t_2 \geq t_1$ satisfies

$$0 < w(x,t) = \frac{v(x,t)}{u(x,t)} \leq \overline{w}_1(\varepsilon_1) \equiv W_1 + \frac{\varepsilon_1}{5} \tag{8}$$

for all $x \in \Omega$ and $t \geq t_2$. Therefore, $v \leq \overline{w}_1(\varepsilon_1)u$, and

$$u_t - d\Delta u \geq u(a - bu) - \frac{\delta \overline{w}_1(\varepsilon_1) u}{\alpha + \overline{w}_1(\varepsilon_1)} = \frac{u\left[(a - bu)(\alpha + \overline{w}_1(\varepsilon_1)) - \delta \overline{w}_1(\varepsilon_1)\right]}{\alpha + \overline{w}_1(\varepsilon_1)}$$

for all $x \in \Omega$ and $t \geq t_2$. Let

$$(a - bu)(\alpha + \overline{w}_1(\varepsilon_1)) - \delta \overline{w}_1(\varepsilon_1) = 0,$$

then we obtain only one positive solution

$$u_R = \frac{a\alpha + (a - \delta)\overline{w}_1(\varepsilon_1)}{b(\alpha + \overline{w}_1(\varepsilon_1))} \tag{9}$$

which is a stable equilibrium point of the corresponding ordinary differential equation

$$u_t = \frac{u\left[(a - bu)(\alpha + \overline{w}_1(\varepsilon_1)) - \delta \overline{w}_1(\varepsilon_1)\right]}{\alpha + \overline{w}_1(\varepsilon_1)}. \tag{10}$$

Hence, all positive solution of (10) will converge to u_R, which means that there exists $t_3 > t_2$ such that

$$u \geq \underline{u}_1(\varepsilon_1) \equiv u_R - \frac{\varepsilon_1}{5} = \frac{a\alpha + (a - \delta)\overline{w}_1(\varepsilon_1)}{b(\alpha + \overline{w}_1(\varepsilon_1))} - \frac{\varepsilon_1}{5}, \tag{11}$$

for all $x \in \Omega$ and $t \geq t_3$. On the other hand, by applying the second equation of (1), we have

$$v_t - d\Delta v \geq rv - \gamma \frac{v^2}{\underline{u}_1(\varepsilon_1)}$$

for all $x \in \Omega$ and $t \geq t_3$. Therefore, there exists a constant $t_4 > t_3$ such that

$$v \geq \underline{v}_1(\varepsilon_1) = \frac{r\underline{u}_1(\varepsilon_1)}{\gamma} - \frac{\varepsilon_1}{5} \tag{12}$$

for all $x \in \Omega$ and $t \geq t_4$. Setting the estimate $v \geq \underline{v}_1(\varepsilon_1)$ into the first equation of (1), we have

$$u_t - d\Delta u \leq au - bu^2 - \frac{\delta u \underline{v}_1(\varepsilon_1)}{\alpha u + \underline{v}_1(\varepsilon_1)} = \frac{u\left[(a-bu)(\alpha u + \underline{v}_1(\varepsilon_1)) - \delta \underline{v}_1(\varepsilon_1)\right]}{\alpha u + \underline{v}_1(\varepsilon_1)}$$

The quadratic equation of one variable

$$(a-bu)(\alpha u + \underline{v}_1(\varepsilon_1)) - \delta \underline{v}_1(\varepsilon_1) = 0$$

possesses only one positive solution

$$u^R = \frac{a\alpha - \underline{v}_1(\varepsilon_1)b + \sqrt{\left(a\alpha - \underline{v}_1(\varepsilon_1)b\right)^2 + 4\left(a\underline{v}_1(\varepsilon_1) - \delta\underline{v}_1(\varepsilon_1)\right)b\alpha}}{2b\alpha} \tag{13}$$

By comparison principle, we can draw a conclusion that there exists $t_5 > t_4$ such that if $t \geq t_5$,

$$\begin{aligned} u &\leq \overline{u}_2(\varepsilon_1) \equiv u^R + \frac{\varepsilon_1}{5} \\ &= \frac{a\alpha - \underline{v}_1(\varepsilon_1)b + \sqrt{\left(a\alpha - \underline{v}_1(\varepsilon_1)b\right)^2 + 4\left(a\underline{v}_1(\varepsilon_1) - \delta\underline{v}_1(\varepsilon_1)\right)b\alpha}}{2b\alpha} + \frac{\varepsilon_1}{5}. \end{aligned} \tag{14}$$

The expression of $\overline{u}_2(\varepsilon_1)$ and that of $\underline{u}_1(\varepsilon_1)$ and $\overline{w}_1(\varepsilon_1)$ are valid by a simple computation using (5), (7), (8) and (11)-(14). The proof is complete. □

By repeating the above step, there exists a sufficiently large T such that when $t \geq T$,

$$\begin{aligned} u &\leq \overline{u}_{n+1}(\varepsilon_1) \equiv \frac{a\alpha - \underline{v}_n(\varepsilon_1)b + \sqrt{\left(a\alpha - \underline{v}_n(\varepsilon_1)b\right)^2 + 4\left(a\underline{v}_n(\varepsilon_1) - \delta\underline{v}_n(\varepsilon_1)\right)b\alpha}}{2b\alpha} + \frac{\varepsilon_1}{5}, \\ u &\geq \underline{u}_n(\varepsilon_1) \equiv \frac{a(\alpha + \overline{w}_n(\varepsilon_1)) - \delta\overline{w}_n(\varepsilon_1) + \sqrt{[a(\alpha + \overline{w}_n(\varepsilon_1)) - \delta\overline{w}_n(\varepsilon_1)]^2}}{2b(\alpha + \overline{w}_n(\varepsilon_1))} - \frac{\varepsilon_1}{5} \end{aligned}$$

uniformly in Ω for any positive integer n, where

$$\underline{v}_n(\varepsilon_1) = \frac{r\underline{u}_n(\varepsilon_1)}{\gamma} - \frac{\varepsilon_1}{5},$$

$$\begin{aligned} \overline{w}_n =& \frac{(r+\alpha)\overline{u}_n + b\overline{u}_n^2 - (a+\alpha\gamma)\overline{u}_n}{2\gamma\overline{u}_n} \\ &+ \frac{\sqrt{\left[(a+\alpha\gamma)\overline{u}_n - (r+\delta)\overline{u}_n - b\overline{u}_n^2\right]^2 + 4\gamma\overline{u}_n\left[(r\alpha - a\alpha + b\alpha\overline{u}_n)\overline{u}_n\right]}}{2\gamma\overline{u}_n}. \end{aligned}$$

When setting $\varepsilon_1 = 0$, we obtain

$$\begin{aligned}
\overline{u}_{n+1} &= \frac{a\alpha - \frac{r}{\gamma}\underline{u}_n b + \sqrt{\left(a\alpha - \frac{r}{\gamma}\underline{u}_n b\right)^2 + 4\left(a\frac{r}{\gamma}\underline{u}_n - \delta\frac{r}{\gamma}\underline{u}_n\right)b\alpha}}{2b\alpha}, \\
\underline{u}_n &= \frac{a(\alpha + \overline{w}_n) - \delta\overline{w}_n + \sqrt{[a(\alpha + \overline{w}_n) - \delta\overline{w}_n]^2}}{2b(\alpha + \overline{w}_n)}, \\
\underline{v}_n &= \frac{r}{\gamma}\underline{u}_n
\end{aligned}$$

and $\overline{u}_1 = \frac{a}{b}$, $\overline{u}_1 > u^*$, $\underline{u}_1 < u^*$. It is known by a direct calculation with the first equality proposed above that

$$\begin{aligned}
&\left(a\alpha - \frac{r}{\gamma}\underline{u}_1 b\right)^2 + 4\left(a\frac{r}{\gamma}\underline{u}_1 - \delta\frac{r}{\gamma}\underline{u}_1\right)b\alpha \\
=&(a\alpha)^2 + \frac{r^2b^2\underline{u}_1^2}{\gamma^2} + 2(a\alpha)\frac{rb}{\gamma}\underline{u}_1 - \frac{4\delta b\alpha r\underline{u}_1}{\gamma} \\
<&\left(a\alpha + b\frac{r}{\gamma}\underline{u}_1\right)^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{u}_2 &= \frac{a\alpha - b\frac{r}{\gamma}\underline{u}_1 + \sqrt{\left(a\alpha - b\frac{r}{\gamma}\underline{u}_1\right)^2 + 4\left(a\frac{r}{\gamma}\underline{u}_1 - \delta\frac{r}{\gamma}\underline{u}_1\right)b\alpha}}{2b\alpha} \\
&< \frac{2a\alpha}{2b\alpha} \\
&= \overline{u}_1.
\end{aligned}$$

Then, by induction, we can obtain that the sequence $\{\overline{u}_n\}$ is decreasing as $n \to \infty$. Similarly, since

$$\begin{aligned}
\overline{w}_n =&\frac{r+\delta}{2\gamma} + \frac{b\overline{u}_n}{2\gamma} - \frac{a+\alpha\gamma}{2\gamma} \\
&+ \sqrt{\left(\frac{r+\delta}{2\gamma} + \frac{b\overline{u}_n}{2\gamma} - \frac{a+\alpha\gamma}{2\gamma}\right)^2 + \frac{1}{\gamma}(r\alpha - a\alpha + b\alpha\overline{u}_n)},
\end{aligned}$$

and

$$\underline{u}_n = \frac{1}{2}\left[\left(\frac{a}{b} - \frac{\delta\overline{w}_n}{b(\alpha + \overline{w}_n)}\right) + \sqrt{\left(\frac{a}{b} + \frac{\delta\overline{w}_n}{b(\alpha + \overline{w}_n)}\right)^2 - \frac{4a\delta\overline{w}_n}{b^2(\alpha + \overline{w}_n)}}\right],$$

where $r < a$, we obtain that the sequence $\{\overline{w}_n\}$ is decreasing and the sequence $\{\underline{u}_n\}$ is increasing. Hence, we obtain

$$\lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} \underline{u}_n = u^*$$

under the assumption of Theorem 1.2. Thus, we obtain

$$\lim_{n\to\infty} \overline{v}_n = \lim_{n\to\infty} \underline{v}_n = v^*.$$

Now, we prove $\lim_{t\to\infty}(u(x,t), v(x,t)) = (u^*, v^*)$, uniformly in $x \in \Omega$.

Proof. [*Proof of Theorem 1.2*] For any $\varepsilon > 0$, there exists $N_1 \in \mathbb{Z}^+$ such that when $n > N_1$,

$$\left|\overline{u}_n - u^*\right| + \left|\underline{u}_n - u^*\right| < \frac{\varepsilon}{4}. \tag{15}$$

We can choose a sufficiently small positive number $\varepsilon_1 > 0$ such that

$$\left|\overline{u}_{N_1}(\varepsilon_1) - \overline{u}_{N_1}\right| + \left|\underline{u}_{N_1}(\varepsilon_1) - \underline{u}_{N_1}\right| < \frac{\varepsilon}{4}. \tag{16}$$

For any $\varepsilon > 0$, there exists $N_2 \in \mathbb{Z}^+$ such that when $n > N_2$,

$$\left|\overline{v}_n - v^*\right| + \left|\underline{v}_n - v^*\right| < \frac{\varepsilon}{4}. \tag{17}$$

We can choose a sufficiently small positive number $\varepsilon_2 > 0$ such that

$$\left|\overline{v}_{N_2}(\varepsilon_2) - \overline{v}_{N_2}\right| + \left|\underline{v}_{N_2}(\varepsilon_2) - \underline{v}_{N_2}\right| < \frac{\varepsilon}{4}. \tag{18}$$

Furthermore, there exists $t_{M_1}, t_{M_2} \gg a$ such that when $t \geq t_{M_1}$ and $t \geq t_{M_2}$, we have

$$\underline{u}_{N_1}(\varepsilon_1) \leq u(x,t) \leq \overline{u}_{N_1}(\varepsilon_1) \quad \text{in } \Omega$$
$$\underline{v}_{N_2}(\varepsilon_2) \leq u(x,t) \leq \overline{v}_{N_2}(\varepsilon_2) \quad \text{in } \Omega$$

respectively.

Let $N = \max\{N_1, N_2\}$, $\epsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $t_M = \max\{t_{M_1}, t_{M_2}\}$. Hence, by (15)- (18), when $t \geq t_M$, we obtain

$$|u(x,t) - u^*| < \varepsilon \quad \text{in } \Omega$$

and

$$|v(x,t) - v^*| < \varepsilon \quad \text{in } \Omega$$

This proves $\lim_{t\to\infty} u(x,t) = u^*$ and $\lim_{t\to\infty} v(x,t) = v^*$ uniformly in $x \in \Omega$. This completes the proof of Theorem 1.2. □

3. Generalization and future works

It is effortless to verify that the proof of Theorem 1.4 follows exactly the same way of argument as in Theorem 1.2. Hence, it is omitted.

The approach we propose in this paper is novel and can be used to many interesting reaction-diffusion type models where the stability of a unique positive equilibrium solution is an essential problem to be considered. For instance, the famous Gierer-Meinhardt model [19],

$$\begin{cases} u_t = \varepsilon^2 \Delta u - u + \dfrac{u^p}{v^q}, & x \in \Omega, t \in (0,\infty), \\ \tau v_t = \Delta v - v + \dfrac{u^m}{v^s}, & x \in \Omega, t \in (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,\infty), \\ u(x,0) = u_0(x) > 0, \quad v(x,0)) = v_0(x) \geq 0, & x \in \overline{\Omega}, \end{cases} \tag{19}$$

is an interesting system worth of looking into.

It will be interesting to see how can we corporate other interesting features such as time delay into our model.

Acknowledgment

The author would like to thank the anonymous referee when the author submitted previously. Referee's constructive comments led to improvements of this paper.

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