



# On Some New Sharp Embedding Theorems for Multifunctional Herz-Type and Bergman-Type Spaces in Pseudoconvex Domains

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**Abstract.** We introduce new multifunctional mixed norm analytic Herz-type spaces in strongly pseudoconvex domains and provide new sharp embedding theorems for them. Some results are new even in case of onefunctional holomorphic spaces. Some new related sharp results for new multifunctional Bergman-type spaces will be also provided under one condition on Bergman kernel. Similar results with similar proofs in unbounded tubular domains over symmetric cones and bounded symmetric domains will be also shortly mentioned.

## 1. Introduction

Let  $D = \{z : \rho(z) < 0\}$  be a bounded strictly pseudoconvex domain of  $\mathbb{C}^n$  with  $C^\infty$  boundary. We assume, as usual, that the strictly plurisubharmonic function  $\rho$  is of class  $C^\infty$  in a neighbourhood of  $\bar{D}$ , that,  $-1 \leq \rho(z) < 0, z \in D, |\partial\rho| \geq c_0 > 0$  for  $|\rho| \leq r_0$ .

Denote by  $H(D)$  the space of all analytic functions on  $D$ . Let also (see [1])  $A_{\delta,k}^{p,q} = \{f \in H(D) : \|f\|_{p,q,\delta,k} < \infty\}$ , where

$$\|f\|_{p,q,\delta,k} = \left( \sum_{|\alpha| \leq k} \int_0^{r_0} \left( \int_{\partial D_r} |D^\alpha f|^p d\sigma_r \right)^{q/p} r^{\delta q - 1} dr \right)^{1/q}$$

be the mixed norm space in  $D$ . Here  $D_r = \{z \in \mathbb{C}^n : \rho(z) < -r\}$ ,  $\partial D_r$  is a boundary of the  $D_r$ ,  $d\sigma_r$  is the normalized surface measure on  $\partial D_r$  and by  $dr$  we denote the normalized volume element on  $(0, r)$ ,  $0 < p < \infty, 0 < q \leq \infty, \delta > 0, k = 0, 1, 2, \dots$  and

$$\|f\|_{p,\infty,\delta,k} = \sup \left\{ \left( \sum_{|\alpha| \leq k} r^\delta \int_{\partial D_r} |D^\alpha f|^p d\sigma_r \right)^{1/p} : 0 < r < r_0 \right\}$$

(for  $p, q < 1$  it is quasinorm), where  $D^\alpha$  is a derivative of  $f$  (see [21]).

2010 *Mathematics Subject Classification.* Primary 32A10; Secondary 46E15

*Keywords.* analytic functions, strongly pseudoconvex domains, embedding theorems, multifunctional spaces, Herz type spaces

Received: 25 March 2019; Revised: 23 September 2019; Accepted: 22 October 2019

Communicated by Miodrag Mateljević

Research supported by MNTR Serbia, Project 174017

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For  $p = q$  we denote

$$\|f\|_{A_{\delta,k}^p} = \|f\|_{p,\delta,k} = \left( \sum_{|\alpha| \leq k} \int_D |D^\alpha f|^p (-\rho)^\delta dv(\zeta) \right)^{1/p}, \quad \delta > -1, k \geq 0,$$

where  $dv$  is a Lebesgue measure on  $D$ . Some interesting properties of these classes can be seen in lemmas which we listed below.

We however consider in this paper analytic spaces with zero smoothness ( $\alpha = k = 0$ , see [21]). For  $p = q$ ,  $k = 0$  we get the usual Bergman spaces  $A_\delta^p(D)$  (see [21]).

The problem which we consider in this paper is classical (see, for example, [27], [29]), we wish to find sharp (or not) conditions on positive Borel measure  $\mu$  in  $D$  so that

$$\int_D |f(z)|^p d\mu(z) \leq c \|f\|_Y, \quad Y \subset H(D),$$

where  $Y$  is a quasinormed subspace of  $H(D)$ ,  $0 < p < \infty$ .

Throughout this paper constants are denoted by  $C$  and  $C_i$ ,  $i = 1, 2, \dots$ , or by  $C$  with other indices, they are positive and may not be the same at each occurrence. We mention that in [11] in the case of the Hardy  $H^p$  space in the unit disk and for  $H^p(B^k)$  in the ball such type result was obtained by L. Hörmander in [16] and L. Carleson. The case of weighted Bergman spaces investigated in [19].

For Bergman space  $Y = A_\alpha^p(D)$ ,  $0 < p < \infty$ ,  $\alpha > -1$ , (or Bergman type function spaces) this type of problem was considered before by various authors and solved for example in papers [28], [29], [35], [36]. For various other cases (spaces with more complicated norms or quasinorms) it is still open.

We mention a series of new sharp results of first author and authors of this paper (see [27] - [33]).

The plan of this paper is the following. We collect preliminaries and related assertions in our next section. In the third section we collect some known sharp results closely related with our work. The last section is devoted to some new embedding theorems. Note we use actively some machinery which was recently developed in [2], [3], [21].

Note also some assertions of this paper were taken from our previous paper [27], [28] where some results were proved in less general situation namely in case of unit ball in  $\mathbb{C}^n$  and for one function case  $m = 1$  (a simplest model of spaces and of pseudoconvex  $D$  domains which we consider in this paper).

One of the intentions of this paper is to generalize them to arbitrary bounded pseudoconvex domains.

Various related assertions (sharp embedding theorems in analytic function spaces in pseudoconvex domains) can be seen in [7], [18], [21], [29].

The theory of analytic spaces on pseudoconvex domains is well-developed by various authors during last decades (see [1], [2], [21], [29] and various references there). One of the goals of this paper among other things is to define for the first time in literature new mixed norm analytic spaces in strictly pseudoconvex domains and to establish some basic properties of these spaces. We believe that this new interesting objects can serve as a base for further generalizations and investigations in this active research area.

In main part of paper we will turn to study of certain embedding theorems for some new mixed norm analytic classes in strictly pseudoconvex domains in  $\mathbb{C}^n$ . Proving estimates and embedding theorems in pseudoconvex domains we heavily use the technique which was developed recently in [1], [2]. In our embedding theorem and inequalities for analytic function spaces in pseudoconvex domains with smooth boundary the so-called Carleson-type measures constantly appear. We are ending with some historical remarks on this important topic now. Carleson measures were introduced by Carleson [11] in his solution of the corona problem in the unit disk of the complex plane, and, since then, have become an important tool in analysis, and an interesting object of study *per se*. Let  $A$  be a Banach space of holomorphic functions a domain  $D \subset \mathbb{C}^n$ ; given  $p \geq 1$ , a finite positive Borel measure  $\mu$  on  $D$  is a *Carleson measure* of  $A$  (for  $p$ ) if there is a continuous inclusion  $A \rightarrow L^p(\mu)$ , that is there exists a constant  $C > 0$  such that

$$\forall f \in A \quad \int_D |f|^p d\mu \leq C \|f\|_A^p;$$

we shall furthermore say that  $\mu$  is a *vanishing Carleson measure* of  $A$  if the inclusion  $A \rightarrow L^p(\mu)$  is compact. Carleson studied this property [11] taking as Banach space  $A$  the Hardy spaces in unit disk  $H^p(\Delta)$ , and proved that a finite positive Borel measure  $\mu$  is a Carleson measure of  $H^p(\Delta)$  for  $p$  if and only if there exists a constant  $C > 0$  such that  $\mu(S_{\theta_0,h}) \leq Ch$  for all sets

$$S_{\theta_0,h} = \{re^{i\theta} \in \Delta : 1 - h \leq r < 1, |\theta - \theta_0| < h\}$$

(see also [12], [19]); in particular the set of Carleson measures of  $H^p(\Delta)$  does not depend on  $p$ .

In 1975, W. Hastings [14] (see also V. Oleinik and B. Pavlov [19] and L. Oleinik [20]) proved a similar characterization for the Carleson measure of the Bergman spaces  $A^p(\Delta)$ , still expressed in terms of the sets  $S_{\theta_0,h}$ . Later J. Cima and W. Wogen [35] characterized Carleson measures for Bergman spaces in the unit ball  $B^n \subset \mathbb{C}^k$ , and J. Cima and P. Mercer [18] characterized Carleson measures of Bergman spaces in strongly pseudoconvex domains, showing in particular that the set of Carleson measures of  $A^p(D)$  is independent of  $p \geq 1$ .

J. Cima and P. Mercer’s characterization of Carleson measures of Bergman spaces is expressed using interesting generalizations of  $S_{\theta_0,h}$ . Given  $z_0 \in D$  and  $0 < r < 1$ , let  $B_D(z_0, r)$  denote the ball of center  $z_0$  and radius  $\frac{1}{2} \log \frac{1+r}{1-r}$  for the Kobayashi distance  $k_D$  of  $D$  (that is, of radius  $r$  with respect to the pseudohyperbolic distance  $\rho = \tanh(k_D)$ ). Then it is possible to prove (see D. Luecking [17] for  $D = \Delta$ , P. Duren and R. Weir [34] for  $D = B$ , and [2], [25] for  $D$  strongly pseudoconvex) that a finite positive measure  $\mu$  is a Carleson measure of  $A^p(D)$  for  $p$  if and only if for some (and hence all)  $0 < r < 1$  there is a constant  $C_r > 0$  such that

$$\mu(B_D(z_0, r)) \leq C_r \nu(B_D(z_0, r))$$

for all  $z_0 \in D$ . (The proof of this equivalence in [25] relied on J. Cima and P. Mercer’s characterization [18]).

## 2. Preliminaries on geometry of strongly pseudoconvex domains with smooth boundary

In this section we provide a chain of facts, properties on the geometry of strongly convex domains which we will use heavily in all our proofs below. In this section we also introduce in detail all basic lemmas in context of pseudoconvex domains which are needed for formulations and proofs of our results see [2], [25]. In particular, we following these papers provide several results on the boundary behavior of Kobayashi balls, and we formulate a vital submean property for nonnegative plurisubharmonic function in Kobayashi balls.

We now recall first the standard definition and the main properties of the Kobayashi distance which can be seen in various books and papers; we refer for example to [1] and [22] for details. Let  $k_\Delta$  denote the Poincare distance on the unit disk  $\Delta \subset \mathbb{C}^n$ . If  $X$  is a complex manifold, the Lempert function  $\delta_X : X \times X \rightarrow \mathbb{R}^+$  of  $X$  is defined by

$$\delta_X(z, \omega) = \inf\{k_\Delta(\zeta, \eta) \mid \text{there exists a holomorphic } \phi : \Delta \rightarrow X \\ \text{with } \phi(\zeta) = z \text{ and } \phi(\eta) = \omega\}$$

for all  $z, \omega \in X$ . The Kobayashi pseudodistance  $k_X : X \times X \rightarrow \mathbb{R}^+$  of  $X$  is the smallest pseudodistance on  $X$  bounded below by  $\delta_X$ . We say that  $X$  is (Kobayashi) hyperbolic if  $k_X$  is a true distance, and in that case it is known that the metric topology induced by  $k_X$  coincides with the manifold topology of  $X$  (see, e.g., Theorem 2.3.10 in [1]). For instance, all bounded domains are hyperbolic (see, e.g., Theorem 2.3.14 in [1]). The following properties are well known in literature. The Kobayashi (pseudo) distance is contracted by holomorphic maps: if  $f : X \rightarrow Y$  is a holomorphic map then

$$\forall z, \omega \in X \quad k_Y(f(z), f(\omega)) \leq k_X(z, \omega).$$

If  $X$  is a hyperbolic manifold,  $z_0 \in X$  and  $r \in (0, 1)$  we shall denote by  $B_X(z_0, r)$  the Kobayashi ball of center  $z_0$  and radius  $\frac{1}{2} \log \frac{1+r}{1-r}$ :

$$B_X(z_0, r) = \{z \in X \mid \tanh k_X(z_0, z) < r\}.$$

We can see that  $\rho_X = \tanh k_X$  is still a distance on  $X$ , because  $\tanh$  is a strictly convex function on  $\mathbb{R}^+$ . In particular,  $\rho_{B^n}$  is the pseudohyperbolic distance of  $B^n$ .

The Kobayashi distance of bounded strongly pseudoconvex domains with smooth boundary has several important properties. First of all, it is complete (see, e.g., Corollary 2.3.53 in [1]), and hence closed Kobayashi balls are compact. It is vital that we can describe the boundary behavior of the Kobayashi distance: if  $D \subset \mathbb{C}^n$  is a strongly pseudoconvex bounded domain and  $z_0 \in D$ , there exist  $c_0, C_0 > 0$  such that

$$\forall z \in D \quad c_0 - \frac{1}{2} \log d(z, \partial D) \leq k_D(z_0, z) \leq C_0 - \frac{1}{2} \log d(z, \partial D),$$

where  $d(\cdot, \partial D)$  denotes the Euclidean distance from the boundary of  $D$  (see Theorems 2.3.51 and 2.3.52 in [1]). We provide some facts on Kobayashi balls of  $B^n$ ; for proofs see Section 2.2.2 in [1], Section 2.2.7 in [24] and [34]. The ball  $B_{B^n}(z_0, r)$  is given by

$$B_{B^n}(z_0, r) = \left\{ z \in B^n \mid \frac{(1 - \|z_0\|^2)(1 - \|z\|^2)}{|1 - \langle z, z_0 \rangle|^2} > 1 - r^2 \right\}.$$

Geometrically, it is an ellipsoid of (Euclidean) center

$$c = \frac{1 - r^2}{1 - r^2\|z_0\|^2} z_0,$$

its intersection with the complex line  $C_{z_0}$  is an Euclidean disk of radius

$$r \frac{1 - \|z_0\|^2}{1 - r^2\|z_0\|^2},$$

and its intersection with the affine subspace through  $z_0$  orthogonal to  $z_0$  is an Euclidean ball of the larger radius

$$r \sqrt{\frac{1 - \|z_0\|^2}{1 - r^2\|z_0\|^2}}.$$

Let  $\nu$  denote the Lebesgue volume measure of  $\mathbb{R}^{2n}$ , normalized so that  $\nu(B^n) = 1$ . Then the volume of a Kobayashi ball  $B_{B^n}(z_0, r)$  is given by (see [34])

$$\nu(B_{B^n}(z_0, r)) = r^{2n} \left( \frac{1 - \|z_0\|^2}{1 - r^2\|z_0\|^2} \right)^{n+1}.$$

A similar estimate is valid for the volume of Kobayashi balls in strongly pseudoconvex bounded domains:

**Lemma 2.1.** (see [2], [25]) *Let  $D \subset \mathbb{C}^n$  be a strongly pseudoconvex bounded domain. Then there exists  $c_1 > 0$  and, for each  $r \in (0, 1)$ , a  $C_{1,r} > 0$  depending on  $r$  such that*

$$c_1 r^{2n} d(z_0, \partial D)^{n+1} \leq \nu(B_D(z_0, r)) \leq C_{1,r} d(z_0, \partial D)^{n+1}$$

for every  $z_0 \in D$  and  $r \in (0, 1)$ .

Let  $D \subset \mathbb{C}^n$  be a bounded strongly pseudoconvex domain in  $\mathbb{C}^n$ . We shall use the following notations:

- $\delta : D \rightarrow \mathbb{R}^+$  will denote the Euclidean distance from the boundary, that is  $\delta(z) = d(z, \partial D)$ . Let  $dv_t(z) = (\delta(z))^t dv(z)$ ,  $t > -1$ ;
- $\nu$  will be the Lebesgue measure on  $D$ ;

- $H(D)$  will denote the space of holomorphic on  $D$ , endowed with the topology of uniform convergence on compact subsets;
- $K : D \times D \rightarrow \mathbb{C}$  will be the Bergman kernel of  $D$ . The  $K_t$  is a kernel of type  $t$  defined with the help of Henkin-Ramirez function  $\Phi$ . Note that if  $K$  is kernel of type  $t$ ,  $t \in \mathbb{N}$ , then  $K^s$  is kernel of type  $st$ ,  $s \in \mathbb{N}, t \in \mathbb{N}$ . This follows directly from definition (see [7]). Note  $K = K_{n+1}$  (see [2], [7]);
- given  $r \in (0, 1)$  and  $z_0 \in D$ , we shall denote by  $B_D(z_0, r)$  the Kobayashi ball of center  $k_{z_0} : D \rightarrow \mathbb{C}$  and radius  $\frac{1}{2} \log \frac{1+r}{1-r}$ .

See, for example, [1], [22] for definitions, basic properties and applications to geometric function theory of the Kobayashi distance and [15], [23] for definitions and basic properties of the Bergman kernel. Let us now recall a number of vital results proved in  $D$ . The first two give information about the shape of Kobayashi balls:

**Lemma 2.2. (Lemma 2.1 in [25])** Let  $D \subset \mathbb{C}^n$  be a bounded strongly pseudoconvex domain, and  $r \in (0, 1)$ . Then

$$v(B_D(\cdot, r)) \approx \delta^{n+1},$$

where the constant depends on  $r$ .

**Lemma 2.3. (Lemma 2.2 in [25])** Let  $D \subset \mathbb{C}^n$  be a bounded strongly pseudoconvex domain. Then there is  $C > 0$  such that

$$\frac{C}{1-r} \delta(z_0) \leq \delta(z) \leq \frac{1-r}{C} \delta(z_0)$$

for all  $r \in (0, 1)$ ,  $z_0 \in D$  and  $z \in B_D(z_0, r)$ .

**Definition 2.4.** Let  $D \subset \mathbb{C}^n$  be a bounded domain, and  $r > 0$ . An  $r$ -lattice in  $D$  is a sequence  $\{a_k\} \subset D$  such that  $D = \bigcup_k B_D(a_k, r)$  and there exists  $m > 0$  such that any point in  $D$  belongs to at most  $m$  balls of the form  $B_D(a_k, R)$ , where  $R = \frac{1}{2}(1+r)$ .

Note by Lemma 2.3,  $v_\alpha(B_D(a_k, R)) = \int_{B_D(a_k, R)} \delta^\alpha(z) dv(z) = (\delta^\alpha(a_k))v(B_D(a_k, R))$ ,  $\alpha > -1$ .

The existence of  $r$ -lattice in bounded strongly pseudoconvex domains is ensured by the following

**Lemma 2.5. (Lemma 2.5 in [25])** Let  $D \subset \mathbb{C}^n$  be a bounded strongly pseudoconvex domain. Then for every  $r \in (0, 1)$  there exists an  $r$ -lattice in  $D$ , that is there exists  $m \in \mathbb{N}$  and a sequence  $\{a_k\} \subset D$  of points such that  $D = \bigcup_{k=0}^\infty B_D(a_k, r)$  and no point of  $D$  belongs to more than  $m$  of the balls  $B_D(a_k, R)$ , where  $R = \frac{1}{2}(1+r)$ .

We will call  $r$ -lattice sometimes the family  $B_D(a_k, r)$ . Dealing with  $K$  Bergman kernel we always assume  $|K(z, a_k)| \asymp |K(a_k, a_k)|$  for any  $z \in B_D(a_k, r)$ ,  $r \in (0, 1)$  (see [2], [25]). Let  $m = (n+1)l, l \in \mathbb{N}$ . Then  $|K_m(z, a_k)| \asymp |K_m(a_k, a_k)|$ ,  $z \in B_D(a_k, r), r \in (0, 1)$ . This fact is crucial for embedding theorems in pseudoconvex domains (see also [29], [33]).

We shall use a submean estimate for non-negative plurisubharmonic functions on Kobayashi balls:

**Lemma 2.6. (Corollary 2.8 in [25])** Let  $D \subset \mathbb{C}^n$  be a bounded strongly pseudoconvex domain. Given  $r \in (0, 1)$ , set  $R = \frac{1}{2}(1+r) \in (0, 1)$ . Then there exists a  $C_r > 0$  depending on  $r$  such that

$$\forall z_0 \in D, \forall z \in B_D(z_0, r), \chi(z) \leq \frac{C_r}{v(B_D(z_0, r))} \int_{B_D(z_0, R)} \chi dv$$

for every nonnegative plurisubharmonic function  $\chi : D \rightarrow \mathbb{R}^+$ .

We will use this lemma for  $\chi = |f(z)|^q, f \in H(D), q \in (0, +\infty)$ .

We now collect a few facts on the (possibly weighted)  $L^p$ -norms of the Bergman kernel and the normalized Bergman kernel. The first result is classical (see, for example, [2], [25]):

**Lemma 2.7.** (see [2]) Let  $D \subset \mathbb{C}^n$  be a bounded strongly pseudoconvex domain. Then

$$\|K(\cdot, z_0)\|_2 = \sqrt{K(z_0, z_0)} \approx \delta^{-\frac{n+1}{2}}(z_0) \text{ and } \|k_{z_0}\|_2 \equiv 1$$

for all  $z_0 \in D$ .

The next result is the main result of this section, and contains the weighted  $L^p$ -estimates we shall need:

**Proposition 2.8.** (see [2]) Let  $D \subset \mathbb{C}^n$  be a bounded strongly pseudoconvex domain, and let  $z_0 \in D$  and  $1 \leq p < \infty$ . Then

$$\int_D |K(\zeta, z_0)|^p \delta^\beta(\zeta) d\nu(\zeta) \leq \begin{cases} \delta^{\beta-(n+1)(p-1)}(z_0), & -1 < \beta < (n+1)(p-1); \\ |\log \delta(z_0)|, & \beta = (n+1)(p-1); \\ 1, & \beta > (n+1)(p-1). \end{cases}$$

**Definition 2.9.** Let  $D \subset \subset \mathbb{C}^n$  be a bounded domain, and  $r > 0$ . An  $r$ -lattice in  $D$  is a sequence  $\{a_k\} \subset D$  such that  $D = \bigcup_k B_D(a_k, r)$  and there exists  $m > 0$  such that any point in  $D$  belongs to at most  $m$  balls of the form  $B_D(a_k, R)$ , where  $R = \frac{1}{2}(1+r)$ .

The existence of  $r$ -lattices in bounded strongly pseudoconvex domains is ensured by the following

**Lemma 2.10.** (see [7], Lemma 2.5) Let  $D \subset \subset \mathbb{C}^n$  be a bounded strongly pseudoconvex domain. Then for every  $r \in (0, 1)$  there exists an  $r$ -lattice in  $D$ , that is there exists  $m \in \mathbb{N}$  and a sequence  $\{a_k\} \subset D$  of points such that  $D = \bigcup_{k=0}^\infty B_D(a_k, r)$  and no point of  $D$  belongs to more than  $m$  of the balls  $B_D(a_k, R)$ , where  $R = \frac{1}{2}(1+r)$ .

We provide another sharper known lemma a decomposition theorem via Koranyi (or Kobayashi) ball.

**Lemma 2.11.** (see [21]). Whitney type decomposition for pseudoconvex domains (existence of  $\eta_0 - \eta_1$  lattice  $\{a_{m,j}\}$ ): Let  $D$  be a strictly pseudoconvex domain of  $\mathbb{C}^n$  with smooth boundary,

$$d(\zeta, z) = \left| \sum_{i=1}^n \frac{\partial \rho(\zeta)}{\partial \zeta_i}(\zeta_i, z_i) \right| + \left| \sum_{i=1}^n \frac{\partial \rho(z)}{\partial z_i}(z_i, \zeta_i) \right| + |\zeta z|^2$$

be the Koranyi pseudodistance,  $\tilde{B}(\zeta, \varepsilon) = \tilde{B}_\varepsilon(\zeta)$  the Koranyi ball centered at  $\zeta$  of radius  $\varepsilon$ , and  $d(z) = \text{dist}(z, \partial D) = d(z, \partial D)$ . Then for each  $\eta > 0, \varepsilon \in (0, 1)$  there exist  $0 < \eta_0 < \eta_1 < \eta$  and a sequence  $\{a_{m,j}\}$  of points of  $D$  which satisfies  $\bigcup_{m,j} \tilde{B}(a_{m,j}, \eta_1 d(a_{m,j})) = D, \tilde{B}(a_{m,j}, \eta_0 d(a_{m,j})) \cap \tilde{B}(a_{m',j'}, \eta_0 d(a_{m',j'})) \neq \emptyset$  iff  $m = m', j = j', (-\rho)(a_{m,j}) = \varepsilon^m, m \geq 0, j = 0, \dots, j_m$ .

We fix  $\{a_k\}$   $r$ -lattice till the end of paper.

### 3. Preliminary theorems

In this section we first review some known sharp embedding theorem in bounded strictly pseudoconvex domain with smooth boundary .

Let first  $D \subset \subset \mathbb{C}^n$  be a bounded domain,  $\beta, \theta \in \mathbb{R}, p > 0$ . A (analytic) Carleson measure of  $A^p(D, \beta) = A^p_\beta$  is a finite positive Borel measure on  $D$  such that there is a positive continuous inclusion  $A^p(D, \beta) \subset L^p(\mu)$  that is there is a constant  $c > 0$  such that  $\int_D |f|^p d\mu \leq c \|f\|_{A^p_\beta}^p, \forall f \in A^p(D, \beta), \theta$  Carleson measure is a finite positive Borel measure on  $D$  such that  $\mu(B_D(\cdot, r)) \leq c v(B_D(\cdot, r))^\theta$  for all  $r \in (0, 1)$  where the constant  $c$  might depend on  $r$ . If  $\theta = 1$  we have usual Carleson measure.

We first list two known sharp embeddings in this direction.

**Theorem A.** (see [7], [8], [10]) Let  $D \subset C^n$  be a bounded strictly pseudoconvex domain. Let  $\mu$  be a positive Borel measure on  $D$ ,  $f \in H(D)$ . Let  $1 \leq p < \infty$ . We have  $\int_D |f|^p d\mu \leq c \|f\|_{A_0^p}^p$  iff  $\mu(B_D(a_k, r)) \leq v(B_D(a_k, r))$ ,  $r \in (0, 1)$  or iff  $\mu(B_D(\cdot, r)) \leq cv(B_D(\cdot, r))$  or iff  $\mu(B_D(a_k, r)) \leq c(\delta^{n+1}(a_k))$  for certain sequence  $\{a_k\}$  which is  $r$ -lattice for  $D$ .

This vital theorem was extended recently.

**Theorem B.** (see [7], [8], [10]) Let  $D \subset C^n$  be a bounded strictly pseudoconvex domain. Let  $\mu$  be a positive Borel measure on  $D$ ,  $f \in H(D)$ . Let  $1 - \frac{1}{n+1} < \theta < 2$ ,  $1 \leq p < \infty$ . Then the following assertions are equivalent

1.  $\int_D |f(z)|^p d\mu(z) \leq c \int_D |f(z)|^p \delta^{(n+1)(\theta-1)}(z) dv(z)$ ,
2.  $\mu$  is  $\theta$ -Carleson measure
3. for every  $r \in (0, 1)$  and every  $r$ -lattice  $\{a_k\}$  in  $D$  one has  $\mu(B_D(a_k, r)) \leq [v(B_D(a_k, r))]^\theta$ ,  $r \in (0, 1)$ .
4. there exists  $r_0 \in (0, 1)$  so that for every  $r_0$ -lattice  $\{a_k\}$  in  $D$ ,  $\mu(B_D(a_k, r_0)) \leq c[v(B_D(a_k, r_0))]^\theta$ .

The following theorem is another sharp embedding theorem for mixed norm spaces. In the unit ball case Theorem B can be seen in [3].

**Theorem C.** (see [30]) Let  $\mu$  be positive Borel measure on  $D$ ,  $f \in H(D)$ . Let  $\{a_k\}$  be  $r$ -lattice. Assume  $q < p$  or  $q = p$ ,  $r \leq p$ . Then

$$\left( \int_D |f(z)|^p d\mu(z) \right)^{1/p} \leq c_0 \|f\|_{A_{v,0}^{q,r}}$$

iff  $\mu(B_D(a, r)) \leq c_1 \delta^{\frac{mp}{q} + \frac{\delta}{q} p}(a)$ ,  $a \in D$  or iff  $\mu(B_D(a_k, r)) \leq c_2 \delta^{\frac{mp}{q} + \frac{\delta}{q} p}(a_k)$  for  $k = 1, 2, \dots$  and for some constants  $c_1, c_2$ .

We also refer the reader to [4], [5], [30], [31], [32] where new sharp interesting results (embedding theorems) in this direction were recently provided.

#### 4. Main results

Namely, we show first our two main theorems in this section. Here we show new two sharp embedding theorems and our arguments are mostly standard and sketchy. Results from our point are interesting enough and enlarge the list of previously known sharp results in this direction.

We define new analytic general multifunctional Herz (for  $p = q$  Bergman) spaces in bounded pseudoconvex domains with smooth boundary as spaces of  $(f_k)_{k=1}^m$  functions analytic in  $D$ , so that

$$\|(f_1, \dots, f_m)\|_{A(p,q,m,d\mu)}^q = \sum_{k=1}^{\infty} \left( \int_{\Delta_k} \prod_{i=1}^m |f_i(z)|^p d\mu(z) \right)^{q/p} < \infty, \quad 0 < p, q < \infty,$$

$\mu$  is a positive Borel measure on  $D$ .

Note for  $m = 1$ ,  $q = p$  in the unit ball, polydisk, pseudoconvex domains these spaces are well-known (see [2], [3] and references there). For  $m = 1$  these are known Herz spaces.

For  $d\mu = \delta^s(z) dv(z) = dv_s(z)$  we write  $A(p, q, m, s)$  and if  $m = 1$  we write  $A(p, q, d\mu)$  or  $A(p, q, s)$  if  $d\mu = dv_s$ . Also, we put, for convenience, below  $\Delta_k = B(a_k, r)$  and we put  $\Delta_k^* = B(a_k, R)$ ,  $R = \frac{1+r}{2}$ ,  $k = 1, 2, \dots$

In this paper we, for  $\alpha q > \beta + n + 1$ , always assume (this is true in the ball, see [36] for  $\beta > 0$ )

$$\int_{B(\tilde{z}, r)} |K_\alpha(z, w)|^q dv_\beta(w) \leq \tilde{C} |K_{\alpha q - \beta - n - 1}(z, \tilde{z})|,$$

for all  $\beta > -1$ ,  $q > 0$ ,  $\alpha > 0$ , namely in Theorems 4.3 and 4.6,  $z, \tilde{z} \in D$ .

The proof of the necessity of condition on measure in all theorems uses standard arguments with standard test function (see, for example, [2], [29], [30]). We put for some  $\tau > 0$

$$f_i(z) = f_w(z) = K_\tau(w, z), \quad i = 1, \dots, m, \quad z, w, a_k \in D$$

and reduce the problem to onefunctional case noting that

$$\|(f_1, \dots, f_m)\|_{A(p,q,m,d\mu)} = \|(f_w, \dots, f_w)\|_{A(p,q,m,d\mu)} \geq c \left( \int_{\Delta_k} |f_w(z)|^{mp} d\mu(z) \right)^{\frac{1}{p}} \geq c (\mu(\Delta_k))^{v_1} (\delta(a_k))^v,$$

for some  $v, v_1 > 0$  and for some fixed  $\{a_k\}$   $r$ -lattice or

$$\int_D \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \geq c \int_{\Delta_k} \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \geq \tilde{c} (\mu(\Delta_k))^{\tilde{v}_1} (\delta(a_k))^{\tilde{v}},$$

for some  $\tilde{v}_1, \tilde{v} > 0$ . On the other hand, the condition on kernel we put (see below) and then Forrelly-Rudin estimates must be used. We will partial omit these standard details.

**Theorem 4.1.** *Let  $0 < p, q < \infty, 0 < s \leq q, \beta_j > -1$  for  $j = 1, \dots, m$  and let  $\mu$  be a positive Borel measure on  $D$ . Then the following conditions are equivalent:*

1. *If  $f_i \in A_{\beta_i}^s, i = 1, \dots, m$ , then*

$$\|(f_1, \dots, f_m)\|_{A(p,q,m,d\mu)} \leq C \prod_{i=1}^m \|f_i\|_{A_{\beta_i}^s}. \tag{1}$$

2. *The measure  $\mu$  satisfies a Carleson type condition:*

$$\mu(\Delta_k) \leq C \delta(a_k)^{\sum_{i=1}^m \frac{p(n+1+\beta_i)}{s}}, \quad k \geq 1,$$

for any  $\{a_k\}$   $r$ -lattice in  $D$ .

*Proof.* Assume 2 holds and choose  $f_i \in A_{\beta_i}^s, i = 1, \dots, m$ . Since  $0 < s/q \leq 1$  we have, using Lemma 5 and properties of  $r$ -lattices we listed above:

$$\begin{aligned} \|(f_1, \dots, f_m)\|_{A(p,q,m,d\mu)}^s &= \left( \sum_{k=1}^{\infty} \left( \int_{\Delta_k} \prod_{i=1}^m |f_i(z)|^p d\mu(z) \right)^{q/p} \right)^{s/q} \\ &\leq \left( \sum_{k=1}^{\infty} \mu(\Delta_k)^{q/p} \max_{\Delta_k} \prod_{i=1}^m |f_i|^q \right)^{s/q} \leq C \sum_{k=1}^{\infty} \prod_{i=1}^m \delta(a_k)^{n+1+\beta_i} \max_{\Delta_k} |f_i|^s \\ &\leq C \prod_{i=1}^m \left( \sum_{k=1}^{\infty} \delta(a_k)^{n+1+\beta_i} \max_{\Delta_k} |f_i|^s \right) \leq C \prod_{i=1}^m \|f_i\|_{A_{\beta_i}^s}^s, \end{aligned}$$

and we proved implication  $2 \Rightarrow 1$ .

Conversely, assume 1 holds and choose  $l \in \mathbb{N}$  such that  $s(n+l-1) > n + \beta_i + 1$ . We use, as test functions, functions  $f_{a_k}, k \in \mathbb{N}$  where

$$f_{a_k}(z) = K_{\tau}(a_k, z), \quad z \in D, \tag{2}$$

see [1] for norm and pointwise estimates related to these functions. In particular we have based on lemmas above

$$\|f_{a_k}\|_{A_{\beta_i}^s} \leq C \delta(a_k)^{-n+1-l+\frac{n+\beta_i+1}{s}}, \quad k \geq 1, \quad 1 \leq i \leq m \tag{3}$$

and

$$\|(f_{a_k}, \dots, f_{a_k})\|_{A(p,q,m,d\mu)} \geq \left( \int_{\Delta_k} |f_{a_k}(z)|^{mp} d\mu(z) \right)^{\frac{1}{p}} \geq C\mu(\Delta_k)^{1/p} \delta(a_k)^{-m(n+l-1)}. \tag{4}$$

The last two estimates give 2.  $\square$

The corresponding more general result for mixed norm spaces is the following theorem.

**Theorem 4.2.** *Let  $0 < p, q < \infty$ ,  $0 < s \leq q$ ,  $\beta_j > -1$  for  $j = 1, \dots, m$ ,  $0 < t_i \leq s$  for  $i = 1, \dots, m$  and let  $\mu$  be a positive Borel measure on  $D$ . Then the following conditions are equivalent:*

1. *If  $f_i \in A_{(\beta_i+1)/s}^{s,t_i}$ ,  $i = 1, \dots, m$ , then*

$$\|(f_1, \dots, f_m)\|_{A(p,q,m,d\mu)} \leq C \prod_{i=1}^m \|f_i\|_{A_{\frac{\beta_i+1}{s}}^{s,t_i}}. \tag{5}$$

2. *The measure  $\mu$  satisfies a Carleson type condition:*

$$\mu(\Delta_k) \leq C\delta(a_k)^{\sum_{i=1}^m \frac{p(n+1+\beta_i)}{s}}, \quad k \geq 1,$$

for any  $\{a_k\}$   $r$ -lattice in  $D$ .

*Proof.* Since  $A(s, \beta) = A_{(\beta+1)/s}^{s,s}$ , the previous theorem, combined with embeddings  $A_{(\beta+1)/s}^{s,t} \hookrightarrow A_{(\beta+1)/s}^{s,s}$  (see [21]) valid for  $0 < t \leq s$ , gives implications  $2 \Rightarrow 1$ .

We fix a test function as in previous theorem

$$f_{a_k}(w) = K_\alpha(a_k, w), \quad \alpha > \alpha_0, \quad a_k, w \in D, \quad k = 1, 2, \dots$$

where  $\{a_k\}$  is  $r$ -lattice and  $\alpha_0$  is large enough. Let us fix a cube  $\Delta_k$ . Using pointwise estimates from below for functions  $f_{a_k}$ , see [1], we obtain:

$$\begin{aligned} \mu(\Delta_k)^{1/p} \delta(a_k)^{-m(n-1+l)} &\leq C \left( \int_{\Delta_k} |f_{a_k}(z)|^{mp} d\mu(z) \right)^{1/p} \\ &\leq C \|(f_1, \dots, f_m)\|_{A(p,q,m,d\mu)}, \quad k \geq 1. \end{aligned} \tag{6}$$

We then must also estimate  $\|f_{a_k}\|_{A_{\frac{\beta_i+1}{s}}^{s,t_i}}$  from above.

Note here, from [13] for estimate of test  $K_\alpha(z, w)$  function, the following estimate further must be used

$$\int_{x:\rho(x)=t} |K_\alpha(x, y)|^s d\sigma(x) \leq C(\rho(y) + t)^{n-q},$$

where  $n < q$ ,  $\alpha > -1$ ,  $s > 0$ ,  $q = \alpha s$ ,  $y \in \Omega$ ,  $t \in (0, t_0)$ , and that

$$\prod_{i=1}^m \|K_\tau\|_{A_{\frac{\beta_i+1}{s}}^{s,t_i}} \leq C\delta(a_k)^{v-m(n-1+l)}, \quad \tau = n - 1 + l, \quad v = \sum_{i=1}^m \frac{\beta_i + n + 1}{s}.$$

We must now repeat these arguments for  $f_{a_k}(w) = \delta^s(a_k)K_\alpha(a_k, w)$  with certain fixed  $s$ ,  $s = s(\alpha)$  and use that  $\rho(y) \asymp \delta(y)$ . The rest is clear.  $\square$

The following is another new sharp theorem for multifunctional analytic spaces in pseudoconvex domains.

**Theorem 4.3.** Let  $0 < p, q < \infty$ ,  $0 < \sigma_i \leq q$ ,  $-1 < \alpha_i < \infty$  for  $i = 1, \dots, m$ . Let  $\mu$  be a positive Borel measure on  $D$ . Then the following two conditions are equivalent:

1.

$$\|(f_1, \dots, f_m)\|_{A(p,q,m,d,\mu)}^q \leq C \prod_{i=1}^m \int_D \left( \int_{B(w,r)} |f_i(z)|^{\sigma_i} \delta^{\alpha_i}(z) dv(z) \right)^{q/\sigma_i} dw. \tag{7}$$

2. The measure  $\mu$  satisfies a Carleson type condition:

$$\mu(\Delta_k) \leq C \delta(a_k)^{m(n+1)\frac{p}{q} + \sum_{i=1}^m \frac{p(n+1+\alpha_i)}{\sigma_i}}, \quad k \geq 1,$$

for any  $\{a_k\}$   $r$ -lattice in  $D$ .

*Proof.* The implication  $1 \Rightarrow 2$  is very similar to the same implication in Theorem 4.1 and we omit details. The only new ingredient is condition on kernel we put before Theorem 4.1.

Let us prove  $2 \Rightarrow 1$  assuming  $m = 1$  (general  $m > 1$  case needs small modification). Let  $f = f_1 \in H(D)$ , then we have, using Lemma 1, Lemma 2 and Lemma 3:

$$\begin{aligned} \|(f_1, \dots, f_m)\|_{A(p,q,m,d,\mu)}^p &= \sum_{k=1}^{\infty} \left( \int_{\Delta_k} |f(z)|^p d\mu(z) \right)^{q/p} \\ &= \sum_{k=1}^{\infty} \max_{\Delta_k} |f|^q \delta(a_k)^{q(\frac{n+1+\alpha}{\sigma} + \frac{n+1}{q})} \\ &\leq \sum_{k=1}^{\infty} \left( \int_{\Delta_k^*} |f(z)|^\sigma \delta(z)^{-n-1} dv(z) \right)^{q/\sigma} \delta(a_k)^{q(\frac{n+1+\alpha}{\sigma} + \frac{n+1}{q})} \\ &\leq \sum_{k=1}^{\infty} \left( \int_{\Delta_k^*} |f(z)|^\sigma dv(z) \right)^{q/\sigma} \delta(a_k)^{n+1+q\alpha/\sigma}. \end{aligned}$$

We continue this estimation, using Lemma 2 and Lemma 3, and obtain

$$\begin{aligned} &\|(f_1, \dots, f_m)\|_{A(p,q,m,d,\mu)}^p \\ &\leq C \sum_{k=1}^{\infty} \left( \int_{\Delta_k^*} \left( \int_{B(z,r)} |f(w)|^\sigma \delta(w)^\alpha dv(w) \right) \delta(z)^{-n-1-\alpha} dv(z) \right)^{q/\sigma} \delta(a_k)^{n+1+q\alpha/\sigma} \\ &\leq C \sum_{k=1}^{\infty} \left( \int_{\Delta_k^*} \left( \int_{B(z,r)} |f(w)|^\sigma \delta(w)^\alpha dv(w) \right) \delta(z)^{-n-1} dv(z) \right)^{q/\sigma} \delta(a_k)^{n+1} \\ &\leq C \sum_{k=1}^{\infty} \int_{\Delta_k^*} \left( \int_{B(z,r)} |f(w)|^\sigma \delta(w)^\alpha dv(w) \right)^{q/\sigma} \delta(z)^{-n-1} dv(z) \delta(a_k)^{n+1}. \end{aligned} \tag{8}$$

arriving at (7) for  $m = 1$ . Note that at (8) we used Hölder’s inequality and at the last step we used finite overlapping property of the cubes  $\Delta_k^*$ , (see [2], [3], [21]).

The case of  $m$  functions use Hölders inequality for  $m$  functions, only multiple sums appears as in proof of Theorem 4.1. Since  $m > 1$  case uses same ideas, we omit easy details.  $\square$

In our Theorems 4.4 and 4.6 below we consider another two new scales of Herz-type spaces (multifunctional) defined with the help of the following expressions:

$$\prod_{i=1}^m \left( \sum_{i=1}^{\infty} \left( \int_{\Delta_k^*} |f_i(z)|^{p_i} \delta^\alpha(z) dv(z) \right)^{q_i} \right)^{\frac{1}{q_i}},$$

$$\prod_{i=1}^m \int_D \left( \int_{B(w,r)} |f_i(z)|^{\sigma_i} dv_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}},$$

and prove some sharp assertions also for them. Proofs here will be sketchy since ideas we already provided will be repeated partially by us below.

**Theorem 4.4.** *Let  $\mu$  be a positive Borel measure on  $D$ . Let  $0 < p_i, q_i < \infty, i = 1, \dots, m$ , let  $\sum_{i=1}^m \frac{1}{q_i} = 1, \alpha > -1$ . Then the following conditions are equivalent:*

1) *If  $f_i \in H(D), i = 1 \dots m$  then*

$$\int_D \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \leq c \prod_{i=1}^m \left( \sum_{i=1}^{\infty} \left( \int_{\Delta_k^*} |f_i(z)|^{p_i} \delta^\alpha(z) dv(z) \right)^{q_i} \right)^{\frac{1}{q_i}}.$$

2) *The measure  $\mu$  satisfies a Carleson type condition  $\mu(\Delta_k) \leq C|\Delta_k|^{m(1+\frac{\alpha}{n+1})}$  for  $k \geq 1$ .*

*Proof.* Assume that positive Borel measure satisfies  $\mu(\Delta_k) \leq c\Delta_k^{m(1+\frac{\alpha}{n+1})}, k \geq 1$ .

Then we have using Lemmas 1 and 2

$$\int_D \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \leq \sum_{k=1}^{\infty} \mu(\Delta_k) \prod_{i=1}^m \sup_{\Delta_k} |f_i|^{p_i}$$

$$\leq c \sum_{k=1}^{\infty} \mu(\Delta_k) \delta(a_k)^{-m(1+n+\alpha)} \prod_{i=1}^m \int_{\Delta_k^*} |f_i(w)|^{p_i} \delta^\alpha(w) dv(w)$$

$$\leq \bar{c} \sum_{k=1}^{\infty} \prod_{i=1}^m \int_{\Delta_k^*} |f_i(w)|^{p_i} \delta^\alpha(w) dv(w),$$

so we get what we need now.

Let us show the reverse of assertion in theorem. We fix  $\tilde{n} \in \mathbb{N}$ . Let

$$f_i(z) = f_i(\tilde{n}, w, z) = K_{\tilde{n}-1}(z, w), \quad w \in D.$$

Clearly  $\prod_{i=1}^m |f_i(z)|^{p_i} = |K_{\sum_{i=1}^m p_i(\tilde{n}-1)}(z, w)|$ .

Hence using lemmas we have (see remark after Lemma 2.5 and Proposition 2.8)

$$\frac{\mu(\Delta_k)}{\delta(a_k)^{(\tilde{n}-1)\sum_{i=1}^m p_i}} \leq c \int_{\Delta_k} |K_\tau(z, w)| d\mu(z) \leq c_1 \int_D \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) = M.$$

Estimating from other side we have using  $(\sum_{k \geq 0} a_k)^q \leq \sum_{k \geq 0} a_k^q, q \leq 1, a_k \geq 0$ ,

$$M \leq c_2 \prod_{i=1}^m \int_{\Delta_k^*} \delta^\alpha(z) K_{p_i(\tilde{n}-1)}(z, w) dv(z) \leq c_3 \prod_{i=1}^m \delta(a_k)^{\alpha - p_i(\tilde{n}-1) + \tilde{n} + 1},$$

and we have what we need.  $\square$

**Remark 4.5.** Theorem 4.4 can be seen in [33]. We added it here for completeness of exposition. The same proof can be given in tube and bounded symmetric domains based on properties of  $r$ -lattice (in tube without any condition on kernel). Additional condition on kernel used in proofs of Theorems 4.3 and 4.6 can be probably removed.

The following additional condition on analytic  $(f_i)_{i=1}^m$  functions (which probably can be removed) is needed for our last theorem with condition on kernel (see before Theorem 4.1). We assume that for any Kobayashi ball  $B(w, r)$

$$(\delta(a_k))^{n+1} \left( \int_{B(a_k, r)} |f_i(z)|^{\sigma_i} \right)^{\frac{p_i}{\sigma_i}} dv_{\alpha_i}(z) \leq \tilde{c} \int_{B(a_k, r)} \left( \int_{B(w, r)} |f_i(z)|^{\sigma_i} dv_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}} dv(w),$$

where  $\{a_k\}$  is any  $r$ -lattice,  $0 < p_i, \sigma_i < \infty, \alpha_j > -1, j = 1, \dots, m, dv_\alpha = \delta^\alpha(z)dv(z)$ .

**Theorem 4.6.** Let  $0 < p_i, \sigma_i < \infty, -1 < \alpha_i < \infty, i = 1, \dots, m$ . Let  $\mu$  be a positive Borel measure on  $D$ . Then the following conditions are equivalent:

1) If  $f_i \in H(D), i = 1, \dots, m$  then

$$\int_D \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \leq c \prod_{i=1}^m \int_D \left( \int_{B(w, r)} |f_i(z)|^{\sigma_i} dv_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}} dv(w).$$

2) The positive Borel measure  $\mu$  satisfies the following Carleson type condition:  $\mu(\Delta_k) \leq c_1 \delta(a_k)^\tau$ , where  $\tau = m(n+1) + \sum_{i=1}^m \frac{p_i(n+1+\alpha_i)}{\sigma_i}, k \geq 1$  for some constants  $c$  and  $c_1$ , and for any  $\{a_k\}$   $r$ -lattice in  $D$ .

*Proof.* Let us assume  $\mu(\Delta_k) \leq c \delta(a_k)^{m(n+1) + \sum_{i=1}^m p_i \frac{n+1+\alpha_i}{\sigma_i}}, k \geq 1$  holds. Let farther  $p_i(n-1+l) > (n+1) + \frac{p_i}{\sigma_i}(n+1+\alpha_i), i = 1, \dots, m$ .

We have using same test function as in proof of our previous theorem, using condition on kernel and Proposition 2.8

$$\begin{aligned} \delta(a_k)^{-(n-1+l) \sum_{i=1}^m p_i} \mu(\Delta_k) &\leq c \int_D \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \\ &\leq \tilde{c} \prod_{i=1}^m \int_D \left( \int_{B(w, r)} |f_i(z)|^{\sigma_i} dv_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}} dv(w) \\ &\leq \tilde{c} \prod_{i=1}^m \int_D (|K_\tau(z, w)|)^{\frac{p_i}{\sigma_i}} dv(w) \leq c_3 \delta(a_k)^{\tilde{\tau}}; \end{aligned}$$

for some fixed  $\tau, \tilde{\tau}, \tilde{\tau} = -(n-1+l) \sum_{i=1}^m p_i + m(n+1) + \sum_{i=1}^m \frac{p_i}{\sigma_i}(n+1+\alpha_i), \tau > \tau_0$  which can be calculated easily.

Let us show the reverse, using finite overlapping property of  $B(a_k, r)$  balls. We have

$$\begin{aligned} \prod_{i=1}^m \int_D \left( \int_{B(w, r)} |f_i(z)|^{\sigma_i} dv_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}} dv(w) &\geq \\ &\geq c \sum_{k=1}^\infty \prod_{i=1}^m \int_{B(a_k, r)} \left( \int_{B(w, r)} |f_i(z)|^{\sigma_i} dv_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}} dv(w), \end{aligned}$$

and we have also that

$$\int_{B(a_k, r)} \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \leq \mu(\Delta_k) \prod_{i=1}^m \max_{z \in \Delta_k} |f_i(z)|^{p_i} \leq c \delta(a_k)^{\tilde{\tau}_1} \left( \prod_{i=1}^m \max_{z \in \Delta_k} |f_i(z)|^{p_i} \right),$$

for  $\tilde{\tau}_1 = m(n+1) + \sum_{i=1}^m \frac{p_i(n+1+\alpha_i)}{\sigma_i}$ .

Therefore it suffices to show that for  $k \geq 1, 1 \leq i \leq m$  and some  $\tau_2$  fixed number

$$\delta(a_k)^{\tau_2} \max_{z \in \Delta_k} |f_i(z)|^{p_i} \leq c \int_{\Delta_k^*} \left( \int_{B(w, r)} |f_i(z)|^{\sigma_i} dv_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}} dv(w),$$

where  $\tau_2 = (n+1) + \frac{p_i(n+1+\alpha_i)}{\sigma_i}$ .

But, using lemmas and additional condition above we have for some fixed  $\tau_2$

$$\begin{aligned} \delta(a_k)^{\tau_2} \max_{z \in \Delta_k} |f_i(z)|^{p_i} &\leq c_1 \delta(a_k)^{n+1} \left( \int_{\Delta_k^*} |f_i(z)|^{\sigma_i} dv_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}} \\ &\leq \tilde{c} \int_{\Delta_k^*} \left( \int_{B(w, r)} |f(z)|^{\sigma_i} dv_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}} dv(w), \end{aligned}$$

and the proof of our theorem is now completed.  $\square$

**Remark 4.7.** *Very similar results with similar proofs are valid for similar type spaces in unbounded tubular domains over symmetric cones (see [31], [32] for one functional case for similar results), based on known properties of  $r$ -lattices on these domains (see [31], [32]).*

**Remark 4.8.** *Very similar sharp results with similar proofs can be provided (under some condition on kernel) in bounded symmetric domains in  $\mathbb{C}^n$  based on known properties of  $r$ -lattices on these domains (see [8]).*

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