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Results on Contractions of Reich Type in Graphical b-Metric Spaces with Applications

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Abstract. The main purpose of this article is to present some results concerning Reich type contractions in the graph structure in the framework of recently introduced graphical b-metric spaces. Our results are significant extensions and generalizations of some pioneer results in the existing theory. Innovative examples along with directed graphs are propounded to support the newfangled results, making the established theory more comprehensible. Final section is devoted to apply our results to the existence of solutions of some nonlinear problems along with some open problems which may be fruitful for the further scope of the study.

1. Introduction

The Banach contraction mapping principle is the opening and vital results in the direction of Fixed point theory. Subsequently several authors have devoted their concentration to expanding and improving this theory. For this, the authors consider to generalize some renowned results to different abstract spaces (see, *e.g.*, [1, 2, 8, 9, 12–14, 16–18, 20, 21, 25]).

Jachymski [15] introduced graphic version of Banach contraction with a new approach by replacing the order structure with a graph structure on a metric space. The idea behind their work was to club the concept of graph theory with metric fixed point theory. Utilizing this new concept Bojor [4] proved fixed point theorems for Reich type contractions on metric spaces endowed with a graph. For more sythesis on fixed point theory along with graph structure, reader can refer to [3, 4, 6] and the references mentioned therein.

In 2017, Shukla et al. [24] proposed graphical structure of metric spaces and introduced the notation of graphical metric spaces. Most recently, in 2018, Chuensupantharat et al. [7] extended the idea given in [24] and introduced the concept of graphical b-metric spaces along with suitable graphs.

On the other side, Reich [23] generalized Banach fixed point theorem for single valued as well as multivalued mappings. Since then Reich type mappings have been the center of intensive research for many authors.

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Bojor [4] proved a fixed point result for Reich type mappings in the framework of complete metric spaces along with a directed graph. For more details in this direction, we refer the reader to [22, 27]. In this paper, by extending Reich type mappings in graph structure, we inaugurate Reach-graph contraction in the context of graphical b-metric spaces. The article implements the idea of graphical structure of Reich mappings. Based on this structure, we show that every Reich contraction is Reich-graph contraction but inverse implication is not true in general. Moreover, some novel examples are furnished equipped with suitable graphs to validate the established concepts. Our main result in this article is an answer to the open problem (ii) posed in [26]. Last section is devoted to apply the established results, as application, to find the existence of solutions of some class of integral equations and differential equations. In the subsequent analysis, we assume the graphs to be studied are directed graphs encompassing nonempty set of edges.

2. Notations ans basic facts

Following Jachymski [15], let the diagonal of $Y \times Y$ be denoted by Δ for a nonempty set Y. Further suppose that G be a directed graph possessing no parallel edges and $\mathfrak{U}(G)$ be the set of all vertices such that $\mathfrak{U}(G)$ coincides with the set Y. Let $\mathfrak{E}(G)$ be the set of all edges of G containing all loops (i.e., $\mathfrak{E}(G) \supseteq \Delta$) and symbolically this is expressed as $G = (\mathfrak{U}(G), \mathfrak{E}(G))$. If we reverse the direction of edges of G, resultant graph is denoted by G^{-1}. Furthermore, the letter $\breve{G}$ denotes a directed graph with symmetric edges. More precisely, we define

$$\mathfrak{E}(\breve{G}) = \mathfrak{E}(G^{-1}) \cup \mathfrak{E}(G).$$

Let $v, w \in \mathfrak{U}(G)$, where the graph G is directed. A path (or directed path) of length m between v and w in G is defined to be a sequence $\{x_j\}_{j=0}^{m}$ of $(m+1)$ vertices with $v = x_0,\ w = x_m$ and $(x_{j-1}, x_j) \in \mathfrak{E}(G)$ for $j = 1, 2, \ldots, m$. If any two vertices of G contains a path between them, then G is called a connected graph. If $\exists$ a path between every two vertices in a undirected graph G, then G is said to be weakly connected. We call $G^* = (\mathfrak{U}(G^*), \mathfrak{E}(G^*))$ a subgraph of $G = (\mathfrak{U}(G), \mathfrak{E}(G))$ if $\mathfrak{U}(G) \supseteq \mathfrak{U}(G^*)$ and $\mathfrak{E}(G) \supseteq \mathfrak{E}(G^*)$.
Consistent with Shukla [24], we denote

$$[u]_G^l = \{v \in Y : \text{there exists a path directing from } u \text{ to } v \text{ having length } l\}.$$

Further, a relation P on Y is such that

$$(uPv)_G \text{ if there exists a path directing from } u \text{ to } v \text{ in } G$$

and $w \in (uPv)_G$ if w is contained in the path $(uPv)_G$. For a sequence $\{x_m\} \in Y$ if $(x_m, Px_{m+1})_G$ for all $m \in \mathbb{N}$, we say $\{x_m\}$ to be a G-termwise connected (in short G-TWC) sequence.
Recently in a paper [7], authors amalgamated the concepts of graph theory and metric fixed point theory in a very interesting way and introduced graphical b-metric space as a generalization of b-metric space as follows:

Definition 2.1. *[7] A graphical b-metric on a nonempty set Y is a mapping $b_G : Y \times Y \to [0, \infty)$ with $s \geq 1$ satisfying the following conditions:*

(G_bM1) $b_G(x, y) = 0$ *if and only if* $x = y$;

(G_bM2) $b_G(x, y) = b_G(y, x)$ *for all* $x, y \in Y$;

(G_bM3) $(xPy)_G, z \in (xPy)_{G_b} \Longrightarrow b_G(x, y) \leq s[b_G(x, z) + b_G(z, y)]$.
The pair (Y, b_G) is called graphical b-metric space with coefficient s on Y.

Example 2.2. Let $Y = \{1, 2, 3, 4, 5\}$ be endowed with graphical b-metric b_G defined by:

$$b_G(u, v) = \begin{cases} 1 & ; \quad u \text{ or } v \notin \{1, 4\} \text{ and } u \neq v, \\ 3 & ; \quad u, v \in \{1, 4\} \text{ and } u \neq v, \\ 0 & ; \quad u = v. \end{cases}$$

It is easy to show that (Y, b_G) is a graphical b-metric space with coefficient $s = \frac{3}{2} > 1$ encompassing the graph $G = (\mathfrak{U}(G), \mathfrak{E}(G))$ equipped with $\mathfrak{U}(G) = Y$ and $\mathfrak{E}(G)$ as displayed in Figure 1.

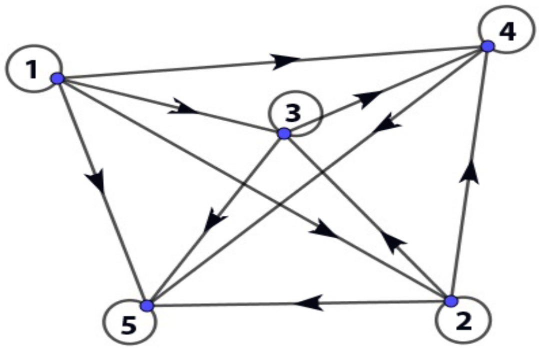

Figure 1: Graph depicting graphical b-metric space

Definition 2.3. *[7] Every open ball $B_{b_G}(y, \gamma)$ in a graphical b-metric space bearing center y and radius γ is open set. Moreover, the corresponding topological space (Y, b_G) is T_1 but not T_2.*

Definition 2.4. *[7]* A sequence $\{x_m\}$ in a graphical b-metric space (Y, b_G) is said to be:

(i) convergent sequence if there exists $y \in Y$ such that $b_G(y_m, y) \to 0$ as $m \to \infty$;

(ii) Cauchy sequence if $b_G(y_m, y_n) \to 0$ as $m, n \to \infty$.

For more details related to the topology of the underlying space one may refer to [7].

3. Main results

Consider $\mathcal{H}_{\mathcal{G}}$ to be a subgraph of G with $\mathfrak{E}(\mathcal{H}_{\mathcal{G}}) \supseteq \Delta$ and further assume $\mathcal{H}_{\mathcal{G}}$ to be a weighted graph. Let $y_0 \in Y$ be the initial value of a sequence $\{y_m\}$, we say $\{y_m\}$ to be a g-Picard sequence (g-PS) for a mapping $g : Y \to Y$ if $y_m = gy_{m-1}$ for all $m \in \mathbb{N}$.
Furthermore, we say a graph $\mathcal{H}_{\mathcal{G}} = (\mathfrak{U}(\mathcal{H}_{\mathcal{G}}), \mathfrak{E}(\mathcal{H}_{\mathcal{G}}))$ satisfies the property $(\mathcal{P})$ [24], if a $\mathcal{H}_{\mathcal{G}}$-$TWC$ g-PS $\{y_m\}$ converging in Y ensures that there is a limit $w \in Y$ of $\{y_m\}$ and $m_0 \in \mathbb{N}$ such that $(y_m, w) \in \mathfrak{E}(\mathcal{H}_{\mathcal{G}})$ or $(w, y_m) \in \mathfrak{E}(\mathcal{H}_{\mathcal{G}})$ for all $m > m_0$.
The main Definition of this article runs as follows.

Definition 3.1. Let (Y, b_G) be a graphical b-metric space. A self mapping $g : Y \to Y$ is said to be Reich-graph contraction for the subgraph $\mathcal{H}_{\mathcal{G}}$ on (Y, b_G) if

$(\mathbb{R}_1)$ for all $y_1, y_2 \in Y$ if $(y_1, y_2) \in \mathfrak{E}(\mathcal{H}_{\mathcal{G}})$ implies $(gx_1, gx_2) \in \mathfrak{E}(\mathcal{H}_{\mathcal{G}})$, i.e., $\mathcal{H}_{\mathcal{G}}$ is graph preserving;

$(\mathbb{R}_2)$ there exist non-negative constants c_1, c_2, c_3 such that $c_1 + c_2 + c_3 < \frac{1}{s}$ and for every $y_1, y_2 \in Y$ with $(y_1, y_2) \in \mathfrak{E}(\mathcal{H}_{\mathcal{G}})$, we have

$$b_G(gy_1, gy_2) \leq c_1 b_G(y_1, y_2) + c_2 b_G(y_1, gy_1) + c_3 b_G(y_2, gy_2) \tag{1}$$

From now onwards, "Reich-graph contraction" stands for "Reich-graph contraction for the subgraph $\mathcal{H}_{\mathcal{G}}$".

Example 3.2. *Any Reich type contraction is a Reich-graph contraction along with the graph $\mathcal{H}_{\mathcal{G}} = G$ defined by $\mathfrak{U}(\mathcal{H}_{\mathcal{G}}) = Y$ and $\mathfrak{E}(\mathcal{H}_{\mathcal{G}}) = Y \times Y$.*
For instance, let $Y = [0, 4]$ via the grahical b-metric b_G defined by

$$b_G(y_1, y_2) = \begin{cases} (y_1 - y_2)^2 & \text{if} \quad y_1 \neq y_2, \\ 0 & \text{if} \quad y_1 = y_2. \end{cases}$$

Obviously, (Y, b_G) is a graphical b-metric space with $s = 2$. The mapping defined by

$$gy = \frac{\sin y}{1 + \sin y}$$

is a Reich type contraction for $c_1 = 0.36, c_2 = 0.04$ and $c_3 = 0.9$.
Now examine the graph $\mathcal{H}_{\mathcal{G}}$ along with $Y = \mathfrak{U}(\mathcal{H}_{\mathcal{G}})$ and $\mathfrak{E}(\mathcal{H}_{\mathcal{G}}) = \{(y_1, y_2) \in Y \times Y : x \leq y\} \cup \Delta$. One can clearly see that g is a Reich-graph contraction for the graph $\mathcal{H}_{\mathcal{G}}$. Figure 2 illustrates the directed graph for the set of points $\{0, 1, 2, 3, 4\}$ contained in $\mathfrak{U}(\mathcal{H}_{\mathcal{G}})$.

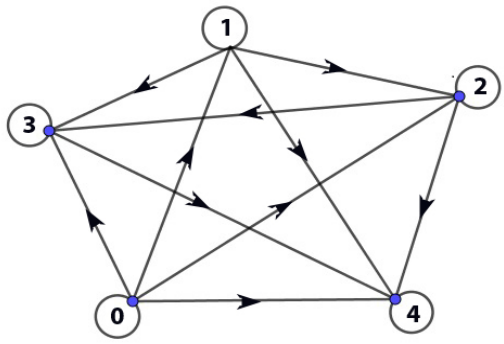

Figure 2: Graph associated with Reich-graph contraction

Following example endores that the class of Reich-graph contraction is different from that of Reich contractions.

Example 3.3. Let $Y = \{0, 1, 2, 3, 4, 5\}$ be endowed with the graphical b-metric defined as follows

$$r_{G_b}(y_1, y_2) = \begin{cases} |y_1 - y_2|^2 & \text{if} \quad y_1 \neq y_2, \\ 0 & \text{if} \quad y_1 = y_2. \end{cases}$$

Then (Y, b_G) is a graphical b-metric space with the coefficient $s = 2$. Define the mapping $g : Y \to Y$ by

$$gy = \begin{cases} 1 & \text{if} \quad y \in \{0, 1\}, \\ 2 & \text{if} \quad y \in \{2, 3, 4, 5\}. \end{cases}$$

Now consider the graph $\mathcal{H}_{\mathcal{G}}$ for which $Y = \mathfrak{U}(\mathcal{H}_{\mathcal{G}})$ and
$\mathfrak{E}(\mathcal{H}_{\mathcal{G}}) = \{(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5),$
$(3, 4), (3, 5), (4, 5)\} \cup \Delta$. Then, g is Reich-graph contraction for $c_1 = 0.33, c_2 = 0.11$ and $c_3 = 0.05$. Figure 3 illustrates the directed graph associated with graphical b-metric space.

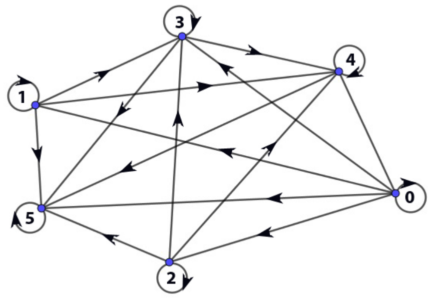

Figure 3: Graph depicting graphical b-metric space

Notice that, g is not a Reich type contraction, since

$$b_G(g1, g2) = 1 > 0.33 = c_1\, b_G(1, 2) + c_2\, b_G(1, g1) + c_3\, b_G(2, g2)$$

Remark 3.4.

(i) If g is a Reich-graph contraction with paraemeters c_1, c_2, c_3 and $c_2 = c_3 = 0$, then g is a Banach-graph contraction in the framework of graphical b-metric spaces.

(ii) If g is a Reich-graph contraction with paraemeters c_1, c_2, c_3 and $c_1 = 0$, then g is a Kannan-graph contraction and hence our results generalize the results of Younis et al. [26].

(iii) Since a graphical b-metric space is a graphical metric space and every metric space is graphical metric space, hence our results in this article are sharp generalizations of a number of results concerning Reich type contractions (see eg., [4, 5, 15]).

Now we present our main result concerning Reich-graph contraction as follows.

Theorem 3.5. *Let $g : Y \to Y$ be Reich-graph contraction on a $\mathcal{H}_\mathcal{G}$-complete graphical b-metric space (Y, b_G). If following conditions hold:*

(a) $\mathcal{G}$ satisfies the property $(\mathcal{P})$;

(b) there exists $y_0 \in Y$ with $gy_0 \in [y_0]^r_{\mathcal{H}_\mathcal{G}}$ for some $r \in \mathbb{N}$.

Then, there exists $y' \in Y$ such that the g-PS $\{y_p\}$ with initial value $y_0 \in Y$ is $\mathcal{H}_\mathcal{G}$-TWC and converges to y'.

Proof. Let $y_0 \in Y$ be such that $gy_0 \in [y_0]^r_{\mathcal{H}_\mathcal{G}}$, for some $r \in \mathbb{N}$. Since $\{y_p\}$ is a g-*PS* starting from y_0, therefore there exists a path $\{x_i\}_{i=0}^r$ such that $y_0 = x_0$, $gy_0 = x_r$ and $(x_{i-1}, x_i) \in \mathfrak{E}(\mathcal{H}_\mathcal{G})$ for $i = 1, 2, \ldots, r$. By hypothesis g being Reich-graph contraction, therefore from assertion $(\mathbb{R}_1)$ we have $(gx_{i-1}, gx_i) \in \mathfrak{E}(\mathcal{H}_\mathcal{G})$ for $i = 1, 2, \ldots, r$. This implies that $\{gx_i\}_{i=0}^r$ is a path from $gx_0 = gy_0 = y_1$ to $gx_r = g^2 y_0 = y_2$ having length r, and hence $y_2 \in [y_1]^r_{\mathcal{H}_\mathcal{G}}$. Pursuing this process, we acquire that $\{g^p x_i\}_{i=0}^r$ is a path from $g^p x_0 = g^p y_0 = y_p$ to $g^p x_r = g^p g y_0 = y_{p+1}$ of length r, i.e., $\{g^p x_i\}_{i=0}^r$ is a path possessing length r from y_p to y_{p+1} and hence, $y_{p+1} \in [y_p]^r_{\mathcal{H}_\mathcal{G}}$, for all $p \in \mathbb{N}$. Thus we attain that $\{y_p\}$ is a $\mathcal{H}_\mathcal{G}$-*TWC* sequence.
Now $(g^p x_{i-1}, g^p x_i) \in \mathfrak{E}(\mathcal{H}_\mathcal{G})$ for $i = 1, 2, \ldots, r$ and $p \in \mathbb{N}$, thanks to $(\mathbb{R}_2)$, we obtain

$$\begin{aligned} b_G(g^p x_{i-1}, g^p x_i) &= b_G\Big(g(g^{p-1} x_{i-1}), g(g^{p-1} x_i)\Big) \\ &\le c_1\, b_G\Big(g^{p-1} x_{i-1}, g^{p-1} x_i\Big) + c_2\, b_G\Big(g^{p-1} x_{i-1}, g(g^{p-1} x_{i-1})\Big) + c_3\, b_G\Big(g^{p-1} x_i, g(g^{p-1} x_i)\Big) \\ &= c_1\, b_G\Big(g^{p-1} x_{i-1}, g^{p-1} x_i\Big) + c_2\, b_G\Big(g^{p-1} x_{i-1}, g^{p-1} x_i\Big) + c_3\, b_G\Big(g^p x_{i-1}, g^p x_i\Big). \end{aligned}$$

This implies that

$$b_G(g^p x_{i-1}, g^p x_i) \le \left(\frac{c_1 + c_2}{1 - c_3}\right) b_G\Big(g^{p-1} x_{i-1}, g^{p-1} x_i\Big).$$

Since, $c_1 + c_2 + c_3 < \frac{1}{s}$, taking $\frac{c_1+c_2}{1-c_3} = \xi \in [0, \frac{1}{s})$, it follows from the above inequality that

$$b_G(g^p x_{i-1}, g^p x_i) \leq \xi\, b_G\Big(g^{p-1} x_{i-1}, g^{p-1} x_i\Big); \quad for\ all \quad \xi \in \Big[0, 1\Big).$$

Repeating this process, we get

$$b_G(g^p x_{i-1}, g^p x_i) \leq \xi^p\, b_G(x_{i-1}, x_i). \tag{2}$$

Since the sequence $\{y_m\}$ being $\mathcal{G}$-TWC and $\mathcal{G}$ being a subgraph of G, using (2) and triangular inequality to obtain

$$\begin{aligned} b_G(x_p, x_{p+1}) &= b_G(g^p y_0, g^{p+1} y_0) \\ &= b_G(g^p x_0, g^p x_r) \\ &\leq s[b_G(g^p x_0, g^p x_1) + b_G(g^p x_1, g^p x_r)] \\ &\leq s[b_G(g^p x_0, g^p x_1)] + s^2[b_G(g^p x_1, g^p x_2)] + \ldots s^r[b_G(g^p x_{r-1}, g^p x_r)] \\ &\leq s\xi^p[b_G(x_0, x_1)] + s^2\xi^p[b_G(x_1, x_2)] + s^3\xi^p[b_G(x_2, x_3)] + \ldots + s^r\xi^p[b_G(x_{r-1}, x_l)] \\ &= s\xi^p \sum_{k=1}^{r} s^{k-1} b_G(x_{k-1}, x_k). \end{aligned} \tag{3}$$

Set $\mathcal{S}_b^r = \sum_{k=1}^{r} s^{k-1} b_G(x_{k-1}, x_k)$, inequality 3 reduces to

$$b_G(x_p, x_{p+1}) \leq s\xi^p \left(\mathcal{S}_b^r\right).$$

Again, $\{y_p\}$ is $\mathcal{H}_{\mathcal{G}}$-$TWC$, for $p, q \in \mathbb{N},\ q > p$, we get

$$\begin{aligned} b_G(y_p, y_q) &\leq s[b_G(y_p, y_{p+1}) + b_G(y_{p+1}, y_q)] \\ &\leq s[b_G(y_p, y_{p+1})] + s^2[b_G(y_{p+1}, y_{p+2})] + s^2[b_G(y_{p+2}, y_q)] \\ &\leq s[b_G(y_p, y_{p+1})] + s^2[b_G(y_{p+1}, y_{p+2})] + \ldots + s^{q-p}[b_G(y_{q-1}, y_q)] \\ &= \sum_{k=m}^{q-1} \Big[s^{k-p+1} b_G(y_k, y_{k+1})\Big] \\ &\leq s \sum_{k=p}^{q-1} [s^{k-p+1} \xi^k \mathcal{S}_b^r] \\ &= s^2 \xi^p \left[\sum_{k=p}^{q-1} (s\xi)^{k-p}\right] \mathcal{S}_b^r \\ &\leq s^2 \xi^p \left[\sum_{k=1}^{\infty} (s\xi)^{k-1}\right] \mathcal{S}_b^r \\ &= s^2 \xi^p \Big(\frac{1}{1 - s\xi}\Big) \mathcal{S}_b^r. \end{aligned} \tag{4}$$

Since $\xi \in [0, \frac{1}{s})$, we infer that $\lim_{p,q\to\infty} b_G(y_p, y_q) = 0$. Hence $\{y_p\}$ is a Cauchy sequence in Y. Also since Y is $\mathcal{H}_{\mathcal{G}}$-complete, therefore $\{y_p\}$ converges in Y and by hypothesis, there exists some $y' \in Y$, $p_0 \in \mathbb{N}$ such that $(y_p, y') \in \mathfrak{E}(\mathcal{G})$ or $(y', y_p) \in \mathfrak{E}(\mathcal{H}_{\mathcal{G}})$ for every $p > p_0$ and

$$\lim_{p\to\infty} b_G(y_p, y') = 0,$$

which shows that $\{y_p\}$ converges to y'. $\square$

For the existence of a fixed point of the underlying mapping, Shukla [24, Theorem 3.10] and Chuensupantharat [7, Theorem 3.4] used condition (S) i.e., if a *G-TWC g-PS* $\{y_m\}$ has two limits u and v; $u \in Y, v \in g(Y)$ then $u = v$. However, we drop this condition and assume that the subgraph $\mathcal{H}_{\mathcal{G}}$ is weakly connected. This not only assures the fixed point of the mapping g but its uniqueness too.

Theorem 3.6. *Retaining the hypothesis of the Theorem 3.5, additionally, we suppose that $\mathcal{H}_{\mathcal{G}}$ is weakly connected, then y' is the unique fixed point of g.*

Proof. Theorem 3.5 ensures that the g-PS $\{y_p\}$ with initial value y_0 converges to $y' \in Y$. Since $\mathcal{H}_{\mathcal{G}}$ is weakly connected, therefore $(y'Pgy')_{\mathcal{H}_{\mathcal{G}}}$ or $(gy'Py')_{\mathcal{H}_{\mathcal{G}}}$ and hence we have

$$\begin{aligned} b_G(y', gy') &\leq s\left[b_G(y', y_p) + b_G(y_p, gy')\right];\; y_p \in Y \\ &= s\left[b_G(y', y_p) + b_G(gy_{p-1}, gy')\right]. \end{aligned}$$

Utilizing $\mathbb{R}_2$, we obtain

$$b_G(y', gy') \leq s\Big[b_G(y', y_p) + c_1\, b_G(y_{p-1}, y') + c_2\, b_G(y_{p-1}, y_p) + c_3\, b_G(y', gy')\Big].$$

Again, since, $c_1 + c_2 + c_3 < \frac{1}{s}$, it follows that

$$b_G(y', gy') \leq \left(\frac{s}{1 - sc_3}\right)\left(b_G(y', y_p) + c_1\, b_G(y_{p-1}, y') + c_2\, b_G(y_{p-1}, y_p)\right) \to 0$$

as $p \to \infty$.
Hence $gy' = y'$, therefore y' is the fixed point of g.
For the uniqueness of fixed point, let y^* be another fixed point of g. Assume that $(y'Py^*)_{\mathcal{H}_{\mathcal{G}}}$, then there exists a sequence $\{y_j\}_{j=0}^{r}$ such that $y_0 = y'$, $y_r = y^*$ with $(y_j, y_{j+1}) \in \mathfrak{E}(\mathcal{H}_{\mathcal{G}})$, $j = 0, 1, ..., r-1$. Since g is Reich-graph contraction, repeated use of $\mathbb{R}_1$ gives $(g^p y_j, g^p y_{j+1}) \in \mathfrak{E}(\mathcal{H}_{\mathcal{G}})$, *for all* $p \in \mathbb{N}$. Making use of $\mathbb{R}_2$ and proceeding along with the same lines as done in the Theorem 3.5, we acquire

$$b_G(g^p y_j, g^p y_{j+1}) \leq \xi^p b_G(y_j, y_{j+1}),\ \xi \in [0, \frac{1}{s}).$$

Now utilizing (G_bM3), we have

$$\begin{aligned} b_G(g^p y', g^p y^*) &= b_G(g^p y_0, g^p y_r) \\ &\leq s[b_G(g^p y_0, g^p y_1) + b_G(g^p y_1, g^p y_r)] \\ &\leq s\Sigma_{k=1}^{r} s^{k-1} b_G(g^p y_{k-1}, g^p y_k) \\ &\leq s\xi^m \Sigma_{k=1}^{r} s^{k-1} b_G(y_{k-1}, y_k). \end{aligned} \tag{5}$$

Since $y', y^* \in Fix(g)$ implies that $g^p y' = y$ and $g^p y^* = y^*$. Proceeding limit $p \to \infty$, we obtain $y' = y^*$. Hence g possesses one and only one fixed point. □

We expound the following Example in order to make our results more lucid.

Example 3.7. Let $Y = \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\} \cup \{0\}$ be endowed with a graph $G = \mathcal{H}_{\mathcal{G}}$ such that $\mathfrak{U}(\mathcal{G}) = Y$ and

$$\mathfrak{E}(\mathcal{G}) = \Delta \cup \{(y_1, y_2) \in Y \times Y : (y_1 P y_2), y_2 \leq y_1\}.$$

Define the graphical b-metric b_G by

$$b_G(y_1, y_2) = \begin{cases} |y_1 - y_2|^2 & \text{if} \quad y_1 \neq y_2, \\ 0 & \text{if} \quad y_1 = y_2. \end{cases}$$

It is evident that (b_G, Y, s) is a graphical b-metric space with $s = 2$. Let the map $g : Y \to Y$ be defined by $gy^* = \frac{y^*}{2}$, for all $y^* \in Y$. One can easily find that there exists $y_0 = \frac{1}{2}$ such that $g(\frac{1}{2}) = \frac{1}{4} \in [\frac{1}{2}]^1_{\mathcal{H}_\mathcal{G}}$, i.e., $\left(\frac{1}{5}P\frac{1}{4}\right)_{\mathcal{H}_\mathcal{G}}$ and the mapping (1) is satisfied for $c_1 = 0.3, c_2 = 0.08$ and $c_3 = 0.09$, thus g is an Reich-graph contraction on Y. Figure 4 authenticates the domination of R.H.S. of Reich-graph mapping (1) over L.H.S. for $c_1 = 0.3, c_2 = 0.08$ and $c_3 = 0.09$.

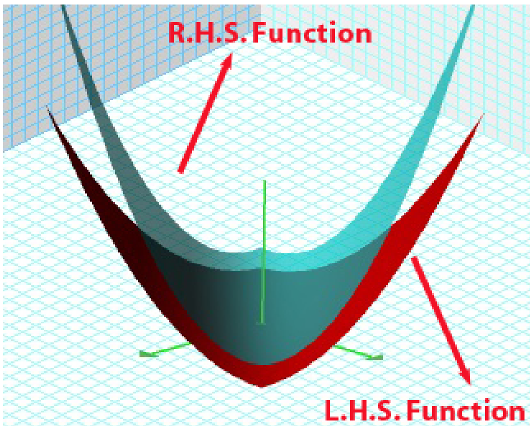

Figure 4: Validation Reich-graph contraction

By routine calculations, One can see that all the conditions of Theorem 3.6 are contented and 0 is the desired fixed point of the mapping g. Figure 5 exemplifies the weighted graph for $\mathfrak{U}'(\mathcal{H}_\mathcal{G}) = \{m, n, o, p, q, r, s, t\} \subseteq \mathfrak{U}(\mathcal{H}_\mathcal{G})$, where the value of $b_G(y_1, y_2)$ is equal to the weight of edge (y_1, y_2) and $\{m, n, o, p, q, r, s, t\} = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, 0\right\}$.

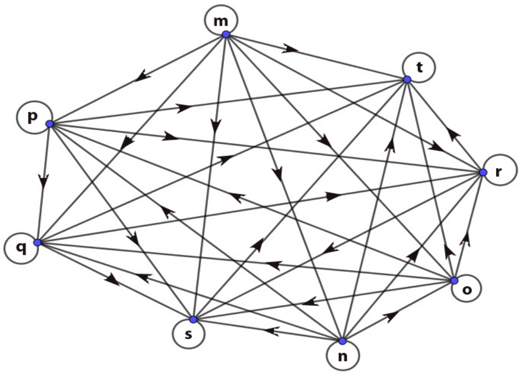

Figure 5: Weighted graph for $\mathfrak{U}'(\mathcal{G})$ where $b_G(y_1, y_2)$= weight of edge (y_1, y_2)

4. Applications

In this section we show the importance and applicability of the obtained results.

Let Y=$C([0, K], \mathbb{R})$ be the set of real continuous functions on $[0, K]$. Consider $\mathcal{B} = \{v \in Y : \inf_{0\le r\le K} v(r) > 0 \text{ and } v(r) \le 1, r \in [0, K], K > 0\}$; and let the graph $\mathcal{G}$ be defined by $\mathfrak{U}(\mathcal{G}) = Y$ and

$$\mathfrak{E}(\mathcal{G}) = \Delta \cup \{(v, v^*) \in Y \times Y : v, v^* \in \mathcal{B}, v(r) \le v^*(r), \text{ for all } r \in [0, K]\}.$$

Define graphical metric $d_\mathcal{G} : Y \times Y \to \mathbb{R}$ as follows

$$d_G(v,v^*) = \begin{cases} 0, & if \quad v = v^*; \\ \sup_{0\le r\le K}\left\{\ln\left(\frac{1}{v(r)v^*(r)}\right)\right\}, & if \quad v,v^* \in \mathcal{G}, v \neq v^*; \\ 1, & \text{otherwise.} \end{cases} \tag{6}$$

for all $v, v^* \in Y$ is a $\mathcal{H}_{\mathcal{G}}$-complete graphical metric space. We consider the graphical b-metric space $b_{\mathcal{G}} : Y \times Y \to \mathbb{R}$ defined as follows

$$b_{\mathcal{G}}(v,v^*) = (d_{\mathcal{G}}(v,v^*))^q = \sup_{0\le r\le K} |v(r) - v^*(r)|^q. \tag{7}$$

Obviously, $(X, b_{\mathcal{G}})$ is complete graphical b-metric space with coefficient $s = 2^{q-1} > 1$.

4.1. An applicaton to the solution of ordinary differential equations:

Consider the following first-order periodic boundary value problem

$$\begin{cases} v'(r) = p(r, v(r)), & r \in J = [0,K] \\ v(0) = v(K) \end{cases} \tag{8}$$

Where $K > 0$ and $p : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Definition 4.1. *An element $\gamma \in Y$ is called a lower solution for the problem (8) if*

$$\begin{cases} \gamma(r) \le p(r,\gamma(r)) \quad , \quad t \in J = [0,K]; \\ \gamma(0) \le \gamma(K) \end{cases}$$

The problem (8) is equivalent to the integral equation

$$v(r) = \int_0^K \phi(r,w)[p(w,v(w)) + \lambda v(w)]dw, \tag{9}$$

where $\phi(r,w)$ is the Green's function given by

$$\phi(r,w) = \begin{cases} \frac{e^{\beta(K+w-r)}}{e^{\beta K}-1} & 0 \le w \le r \le K, \\ \frac{e^{\phi(w-r)}}{e^{\beta K}-1} & 0 \le r \le w \le K. \end{cases} \tag{10}$$

Let the function $g : Y \to Y$ is given by

$$gv(r) = \int_0^K \phi(r,w)[p(w,v(w)) + \lambda v(w)]dw, \tag{11}$$

Evidently, if $v \in C(J,\mathbb{R})$ is an fixed point of g then $v \in C^1(J,\mathbb{R})$ is a solution of the ordinary differential equation (8).

Theorem 4.2. *Consider the problem (8) and assume that*

(1) $p(w,.) : \mathbb{R} \to \mathbb{R}$ is increasing on (0,1], for every $w \in [0,K]$.
Additionally,

$$\inf_{0\le r\le K} \phi(r,w) > 0, \quad 1 \ge \phi(r,w)[p(w,1) + \beta];$$

(2) there exist $c_1 \in (0,1)$ *such that for* $u, u' \in X$ *with* $(u,u') \in \mathcal{G}$*, we have*

$$[u(r_1)u'(r_2)]^{\frac{c_1}{q}} \leq \Big[p(r_1, u(r_1)) + \beta u(r_1)\Big]\Big[p(r_2, u(r_2)) + \beta u(r_2)\Big]$$

for every $r_1, r_2 \in [0,K]$.
Then the existence of a lower solution for the periodic boundary value problem provides a solution for (8).

Proof. It is easy to verify that, g is well defined. On account of the hypothesis of the underlying Theorem, for $u, u' \in Y$ with $(u,u') \in \mathcal{G}$, we get

$$\begin{aligned}
&\left(\ln\left(\frac{1}{gu(r)gu'(r)}\right)\right)^q \\
&= \left(\ln\left(\frac{1}{\int_0^K\int_0^K \phi(r,r_1)\phi(r,r_2)[p(r_1,u(r_1)) + \beta u(r_1)][p(r_2,u'(r_2)) + \beta u'(r_2)]\ dr_1\, dr_2}\right)\right)^q \\
&\leq \left(\ln\left(\frac{1}{\int_0^K\int_0^K \phi(r,r_1)\phi(r,r_2)[u(r_1)u'(r_2)]^{\frac{c_1}{q}}\ dr_1\, dr_2}\right)\right)^q \\
&\leq \left(\ln\left(\frac{1}{\inf_{0\leq r\leq K}\ [u(r)u'(r)]^{\frac{c_1}{q}} \int_0^K\int_0^K \phi(r,r_1)\phi(r,r_2)\ dr_1\, dr_2}\right)\right)^q \\
&\leq \left(\ln\left(\frac{1}{\inf_{0\leq r\leq K}\ [u(r)u'(r)]^{\frac{c_1}{q}} \int_0^K \Big[\int_0^r \frac{e^{\beta(K+r_1-r)}}{e^{\beta K}-1}dr_1 + \int_r^K \frac{e^{\beta(r_1-r)}}{e^{\beta K}-1}dr_1\Big]\phi(r,r_2)\ dr_2}\right)\right)^q \\
&\leq \left(\ln\left(\frac{1}{\beta \inf_{0\leq r\leq K}\ [u(r)u'(r)]^{\frac{c_1}{q}} \int_0^K \Big[\int_0^t \frac{e^{\beta(K+r_2-r)}}{e^{\beta K}-1}dr_2 + \int_r^K \frac{e^{\beta(r_2-r)}}{e^{\beta K}-1}dr_2\Big]}\right)\right)^q \\
&\leq \left(\ln\left(\frac{1}{\beta^2\ \inf_{0\leq r\leq K} [u(r)u'(r)]^{\frac{c_1}{q}}}\right)\right)^q \\
&\leq c_1\Big(d_G(u(r), u(r))\Big)^q \\
&= c_1 b_G\Big(u(r), u'(r)\Big) \\
&\leq c_1 b_G\Big(u(r), u'(r)\Big) + c_2 b_G\Big(u(r), gu(r)\Big) + c_3 b_G\Big(u'(r), gu'(r)\Big),
\end{aligned}$$

where $c_1 + c_2 + c_3 < \frac{1}{s}$.

Hence, we have

$$\begin{aligned}
b_G(gu(r), gu'(r)) = (d_G(gu(r), gu'(r)))^q &= \sup_{0\leq r\leq K}\left(\ln\left(\frac{1}{gu(r)gu'(r)}\right)\right)^q \\
&\leq c_1 b_G\Big(u(r), u'(r)\Big) + c_2 b_G\Big(u(r), gu(r)\Big) + c_3 b_G\Big(u'(r), gu'(r)\Big).
\end{aligned}$$

Thus the contractive condition of Theorem 3.6 is satisfied. Further, for each $u, u' \in Y$ such that $(u,u') \in \mathfrak{E}(\mathcal{G})$, we obtain that $u, u' \in \mathcal{B}$ and $u(r) \leq u'(r)$ for all $r \in [0,K]$. Moreover, by the condition $\mathbb{R}_2$, we obtain $\inf_{0\leq r\leq K} g(u)(r) > 0$,

$$g(u)(r) = \int_0^K \phi(r,w)[p(w,u(w)) + \beta u(w)]\, dw \leq \int_0^K \phi(r,w)[p(w,1) + \lambda]\, dw \leq 1$$

and

$$\begin{aligned} g(u)(r) &= \int_0^K \phi(r,w)[p(w,u(w)) + \beta u(w)]\,dw \\ &\leq \int_0^K \phi(r,w)[p(w,u'(w)) + \beta u'(w)]\,dw \\ &= g(u')(r). \end{aligned}$$

On the other hand, existence of lower solution of the problem (8) ensures that there is a path from γ to $g(\gamma)$ of length 1 i.e. $g(\gamma) \in [\gamma]^1_{\mathcal{H}_G}$, so that the condition (b) of Theorem 3.6 are also satisfied. Therefore, Theorem 3.6 gurantees that g has a unique fixed point and hence the problem (8) possesses a unique solution in Y. □

4.2. Application to existence of solution of integral equation:

Now we invoke our reults to find the existence of solution of following integral equation for an unknown function v:

$$v(r) = \int_0^K \psi(r,w)p(w,v(w))\,dw, \tag{12}$$

where $K > 0$, $p : [0,K] \times \mathbb{R} \to \mathbb{R}$ and $\psi : [0,T] \times [0,T] \to [0,\infty]$ are continuous functions.
Consider the mapping $g : Y \to Y$ is defined by

$$gv(r) = \int_0^K \psi(r,w)p(w,v(w))\,dw,$$

then v is a solution of integral equation (12) if and only if it is an fixed point of g. A function $\beta \in Y$ with ($Y = C([0,K],\mathbb{R})$) is called a lower solution of (12) if

$$\int_0^K \psi(r,w)p(w,\beta(w))\,dw \geq \beta(r), r \in [0,K].$$

Theorem 4.3. *Consider the problem (12) and assume that the following assumptions hold:*
(1) $p(c,.) : \mathbb{R} \to \mathbb{R}$ *is increasing on (0,1], for every* $c \in [0,K]$. *Further*

$$\inf_{0\leq r\leq K} \psi(r,c) > 0, \quad \psi(r,c)p(c,1) \leq 1;$$

(2) there exist $c_1 \in (0,1)$ *and* $\gamma \in [1,\infty)$ *such that for* $v, v' \in Y$ *with* $(v,v') \in \mathcal{E}(\mathcal{G})$, *we have*

$$p(c,v(c))p(h,v'(h)) \geq [v(c)v'(h)]^{\frac{c_1}{q}}$$

and

$$\int_0^K \int_0^K \psi(r,c)\psi(r,h)\,dc\,dh \geq \gamma, \quad r \in [0,K]$$

for every $c,h \in [0,K]$.
Then the integral equation (12) *has a unique solution.*

Proof. Theorem can be proved on the similar lines as done in the Theorem 4.2. Hence, for the sake of brevity, we omit it. □

Open Problems:

- Consider the nonlocal wave interaction in electromagnetic wave problems governing an integro-differential equation of the form

$$\frac{d^2m}{dr^2} + \alpha^2 m + \int_0^1 K(|t - t'|)m(t')dt' = 0, \ 0 < t < \infty$$

  subject to boundary conditions $m(0) = 1$ and $\lim_{t\to\infty} m(t) = 0$.
  Whether the existence of solution of the above integro-differential equation can be derived from results established in this note?

- Establish analogue results of Edelstein [10], Hardy-Roger [11], Meir-Keelar [19], type contractions in the underlying space.

- Can we extend the results proved in this article to the recently introduced graphical rectangular b-metric spaces [27]?

5. Conclusions

In this article, we proposed analogous results of Reich type contractions equipped with graph structure. We proved that every Reich type contraction is Reich-graph contraction but the inverse implication is not true in general. We obtained the fixed point result by dropping the property S as used in [7, 24]. Obtained results are validated by appropriate examples endowed with suitable graphs. Applications to the solutions of ordinary differential equations and integral equations are also entrusted to manifest the viability of the obtained results.

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Not Applicable

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Competing interests
The authors declare that they have no competing interests.

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