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Hyponormality on General Bergman Spaces

Houcine Sadraoui[a]

[a]*Department of mathematics, College of Sciences, King Saud University, P. O Box 2455 Riyadh 11451, Saudi Arabia.*

Abstract. A bounded operator T on a Hilbert space is hyponormal if $T^*T - TT^*$ is positive. We give a necessary condition for the hyponormality of Toeplitz operators on weighted Bergman spaces, for a certain class of radial weights, when the symbol is of the form $f+\overline{g}$, where both functions are analytic and bounded on the unit disk. We give a sufficient condition when f is a monomial.

1. Introduction

Let $w(r)$ be a nonegative measurable function defined on $(0,1)$, and assume $0 < \int_0^1 rw(r)dr < \infty$. Define the Hilbert space $L^2_{a,w}$ to be the space of analytic functions on the unit disk U such that $\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^2 rw(r)\frac{drd\theta}{\pi} < \infty$. We set $\alpha_n = 2\int_0^1 r^{2n+1}w(r)dr$. Then $L^2_{a,w} = \left\{ f = \sum a_n z^n \text{ analytic on the unit disk such that } \|f\|^2 = \sum \alpha_n |a_n|^2 < \infty \right\}$ and its orthonormal basis is given by $e_n = \frac{z^n}{\sqrt{\alpha_n}}$. Toeplitz operators on $L^2_{a,w}$ are defined by $T_f(k) = P(fk)$, with f bounded measurable on U , k in $L^2_{a,w}$, and P the orthogonal projection on $L^2_{a,w}$. Hankel operators are defined by $H_f(k) = (I-P)(fk)$ where f and k are as before. A bounded operator on a Hilbert space is said to be hyponormal if $T^*T - TT^*$is positive. Unweighted Bergman spaces are considered in [2, 3, 11]. Hyponormality on the Hardy space was first considered in [4, 5]. The first results on hyponormality of Toeplitz operators on Bergman spaces are in [10] and the necessary condition is improved in [1]. All the known results on hyponormality on weighted Bergman spaces consider particular types of polynomials as a symbol. We cite for example [8] and [9]. In this work we consider hyponormality of Toeplitz operators on $L^2_{a,w}$. Under a condition on the weight we give a general necessary condition for the hyponormality of Toeplitz operators on $L^2_{a,w}$ with a symbol of the form $f+\overline{g}$, where f and g are bounded analytic on the the unit disk. We give sufficient conditions for hyponormality when f is a monomial and g is a polynomial. A necessary and sufficient condition for normality of $T_{f+\overline{g}}$, when f and g are analytic in an open set containing U , is also obtained as a consequence.

2. Basic properties of Toeplitz operators and equivalent forms of hyponormality

These properties are known on the Bergman space and they hold also for weighted Bergman spaces.We assume f, g are in $L^\infty(U)$. Then we have:

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Email address: sadrawi@ksu.edu.sa (Houcine Sadraoui)

1. $T_{f+g} = T_f + T_g$.
2. $T_f^* = T_{\overline{f}}$.
3. $T_{\overline{f}} T_g = T_{\overline{f}g}$ if f or g analytic on U.

The use of these properties leads to describing hyponormality in more than one form. These are easy to prove, and one of the forms uses Douglas lemma [6].

Proposition 2.1. *Let f, g be bounded and analytic on U. Then the following are equivalent:*

1. $T_{f+\overline{g}}$ *is hyponormal.*
2. $H_{\overline{g}}^* H_{\overline{g}} \leq H_{\overline{f}}^* H_{\overline{f}}$.
3. $\|(I-P)(\overline{g}k)\| \leq \|(I-P)(\overline{f}k)\|$ *for any k in $L^2_{a,w}$.*
4. $\|\overline{g}k\|^2 - \|P(\overline{g}k)\|^2 \leq \|\overline{f}k\|^2 - \|P(\overline{f}k)\|^2$ *for any k in $L^2_{a,w}$.*
5. $H_{\overline{g}} = L H_{\overline{f}}$ *where L is of norm less than or equal to one.*

We also need the following two lemmas. The symbols m, n, p etc denote nonnegative integers.

Lemma 2.2. *For m and n integers we have* $P(z^n\overline{z^m}) = \begin{cases} 0, & \text{if } n < m \\ \frac{\alpha_n}{\alpha_{n-m}} z^{n-m}, & \text{if } n \geq m \end{cases}$.

Proof. If $n < m$ we have $\left\langle P(z^n\overline{z^m}), z^p \right\rangle = \left\langle z^n\overline{z^m}, z^p \right\rangle = \langle z^n, z^{p+m}\rangle = 0$ for any integer p. Thus $P(z^n\overline{z^m}) = 0$. For $n \geq m$, $\left\langle P(z^n\overline{z^m}), z^p \right\rangle = \langle z^n, z^{p+m}\rangle = 0$ if $p \neq n-m$. So $P(z^n\overline{z^m}) = \lambda z^{n-m}$ and $\left\langle P(z^n\overline{z^m}), z^{n-m} \right\rangle = \lambda \|z^{n-m}\|^2 = \lambda\alpha_{n-m}$. Since $\left\langle P(z^n\overline{z^m}), z^{n-m} \right\rangle = \left\langle z^n\overline{z^m}, z^{n-m} \right\rangle = \langle z^n, z^n\rangle = \alpha_n$, we deduce that $\lambda = \frac{\alpha_n}{\alpha_{n-m}}$ and the result follows. □

Lemma 2.3. *For $f = \sum a_n z^n$ bounded and analytic on the unit disk .The matrix of $T_{\overline{f}}T_f - T_f T_{\overline{f}}$ in the orthonormal basis $\{e_n\}$ is given by*

$$\Lambda_{i,j} = \sum_{m \geq j-i, m\geq 0} \frac{a_{m+i-j}\overline{a_m}\alpha_{i+m}}{\sqrt{\alpha_i}\sqrt{\alpha_j}} - \sum_{i-j\leq m \leq i, 0\leq m} \frac{a_m \overline{a_{m+j-i}}\sqrt{\alpha_i}\sqrt{\alpha_j}}{\alpha_{i-m}}.$$

Proof. We have $T_f(e_j) = \frac{1}{\sqrt{\alpha_j}} \sum_n a_n z^{n+j}$ and $T_{\overline{f}}T_f(e_j) = \frac{1}{\sqrt{\alpha_j}} P(\sum_{m,n} a_n\overline{a_m} z^{n+j}\overline{z}^m) = \frac{1}{\sqrt{\alpha_j}} \sum_{m-n\leq j} a_n\overline{a_m} \frac{\alpha_{n+j}}{\alpha_{n+j-m}} z^{n+j-m}$ which can be written

$$T_{\overline{f}}T_f(e_j) = \frac{1}{\sqrt{\alpha_j}} \sum_{p\geq 0, m+p\geq j} \frac{\alpha_{m+p}}{\alpha_p} a_{m+p-j}\overline{a_m} z^p.$$

We deduce that

$$\left\langle T_{\overline{f}}T_f(e_j), e_i \right\rangle = \frac{1}{\sqrt{\alpha_i}\sqrt{\alpha_j}} \sum_{m\geq j-i, m\geq 0} a_{m+i-j}\overline{a_m}\alpha_{i+m}.$$

Similarly, we show that

$$\left\langle T_f T_{\overline{f}}(e_j), e_i \right\rangle = \sum_{i-j\leq m\leq i, 0\leq m} \frac{a_m\overline{a_{m+j-i}}\sqrt{\alpha_i}\sqrt{\alpha_j}}{\alpha_{i-m}}$$

and the proof is complete. □

Corollary 2.4. *The following holds*

$$\Lambda_{i+n,i+n+p} = \sum_{l\leq i+n} \frac{(\alpha_{i+n+p+l}\alpha_{i+n-l} - \alpha_{i+n}\alpha_{i+n+p})a_l\overline{a_{l+p}}}{\sqrt{\alpha_{i+n}}\sqrt{\alpha_{i+n+p}}\alpha_{i+n-l}} + \sum_{l>i+n} \frac{\alpha_{i+n+p+l}a_l\overline{a_{l+p}}}{\sqrt{\alpha_{i+n}}\sqrt{\alpha_{i+n+p}}}.$$

Proof. This follows from the previous lemma by putting $m = p+l$ in the first sum and $m = l$ in the second. □

3. The results

Denote by $(\theta_{i,j})$ the matrix of the, possibly unbounded, Toeplitz operator on H^2 with symbol $|f'|^2$. Our main result uses the following lemma, where C denotes a constant.

Lemma 3.1. *Let $f = \sum a_n z^n$ be bounded on U. Assume $f' \in H^2$ and (α_n) satisfies the following conditions:*

$$n^2\left|\frac{\alpha_{i+n+p+l}\alpha_{i+n-l} - \alpha_{i+n}\alpha_{i+n+p}}{\sqrt{\alpha_{i+n}}\sqrt{\alpha_{i+n+p}}\alpha_{i+n-l}}\right| \le Cl(l+p), \qquad l \le i+n \tag{1}$$

$$\frac{n^2(\alpha_{i+n+p+l}\alpha_{i+n-l} - \alpha_{i+n}\alpha_{i+n+p})}{\sqrt{\alpha_{i+n}}\sqrt{\alpha_{i+n+p}}\alpha_{i+n-l}} \underset{n\to\infty}{\longrightarrow} l(l+p) \tag{2}$$

Then

$$n^2\Lambda_{i+n,i+n+p} \underset{n\to\infty}{\longrightarrow} \theta_{i,i+p}.$$

Proof. We have $\sum_l l^2|a_l|^2 < \infty$ since $f' \in H^2$. Using the previous lemma we have

$$n^2\Lambda_{i+n,i+n+p} = \sum_{l\le i+n} \frac{n^2(\alpha_{i+n+p+l}\alpha_{i+n-l} - \alpha_{i+n}\alpha_{i+n+p})a_l\overline{a_{l+p}}}{\sqrt{\alpha_{i+n}}\sqrt{\alpha_{i+n+p}}\alpha_{i+n-l}} + \sum_{l>i+n} \frac{n^2\alpha_{i+n+p+l}a_l\overline{a_{l+p}}}{\sqrt{\alpha_{i+n}}\sqrt{\alpha_{i+n+p}}}.$$

Set $h_n(l) = \frac{n^2(\alpha_{i+n+p+l}\alpha_{i+n-l} - \alpha_{i+n}\alpha_{i+n+p})a_l\overline{a_{l+p}}}{\sqrt{\alpha_{i+n}}\sqrt{\alpha_{i+n+p}}\alpha_{i+n-l}}$. From (1) we have $|h_n(l)| \le (C/2)(l^2|a_l|^2 + (l+p)^2|a_{l+p}|^2) = I(l)$ and $\int_0^\infty I(l)dv(l) < \infty$, where v is the counting measure. Using (2) and the dominated convergence theorem we obtain

$$\lim_{n\to\infty}\sum_{l\le i+n} \frac{n^2(\alpha_{i+n+p+l}\alpha_{i+n-l} - \alpha_{i+n}\alpha_{i+n+p})a_l\overline{a_{l+p}}}{\sqrt{\alpha_{i+n}}\sqrt{\alpha_{i+n+p}}\alpha_{i+n-l}} = \sum l(l+p)a_l\overline{a_{l+p}}.$$

We also have, for $l > i+n$

$$\left|\frac{n^2\alpha_{i+n+p+l}a_l\overline{a_{l+p}}}{\sqrt{\alpha_{i+n}}\sqrt{\alpha_{i+n+p}}}\right| \le 1/2(l^2|a_l|^2 + (l+p)^2|a_{l+p}|^2).$$

By the dominated convergence theorem we see that

$$\sum_{l>i+n} \frac{n^2\alpha_{i+n+p+l}a_l\overline{a_{l+p}}}{\sqrt{\alpha_{i+n}}\sqrt{\alpha_{i+n+p}}} \underset{n\to\infty}{\longrightarrow} 0\,.$$

The result follows since $\theta_{i,i+p} = \sum_l l(l+p)a_l\overline{a_{l+p}}$. □

Remark 3.2. *Examples of weights satisfying conditions (1) and (2) of the previous lemma are : $w(r) = r^{2s}$, $s > -\frac{1}{2}$, $w(r) = |\log r|$, and $w(r) = 1 - r^2$.*

From now on we assume (α_n) satisfies the hypotheses of the previous lemma. We state our main result.

Theorem 3.3. *Let f and g be bounded analytic functions on U, and assume $f' \in H^2$. If $T_{f+\overline{g}}$ is hyponormal on $L^2_{a,w}$ then $g' \in H^2$and $|g'| \le |f'|$ a.e on the unit circle.*

Proof. Denote by $(\Gamma_{i,j})$ the matrix of $T_{\overline{g}}T_g - T_g T_{\overline{g}}$ and put $g = \sum_n b_n z^n$. Hyponormality of $T_{f+\overline{g}}$ leads to the inequality $n^2\Gamma_{i+n,i+n} \leq n^2\Lambda_{i+n,i+n}$. We deduce that

$$\sum_{l\leq i+n} \frac{n^2(\alpha_{i+n+l}\alpha_{i+n-l} - (\alpha_{i+n})^2)|b_l|^2}{\alpha_{i+n}\alpha_{i+n-l}} \leq \sum_{l\leq i+n} \frac{n^2(\alpha_{i+n+l}\alpha_{i+n-l} - (\alpha_{i+n})^2)|a_l|^2}{\alpha_{i+n}\alpha_{i+n-l}} + \sum_{l>i+n} \frac{n^2\alpha_{i+n+l}|a_l|^2}{\alpha_{i+n}}.$$

Write the left hand side sum as an integral $\int u_n(l)dv(l)$. By Fatou's lemma, condition (2) of the previous lemma and taking the limit on both sides we get

$$\sum l^2|b_l|^2 \leq \sum l^2|a_l|^2.$$

Thus $g' \in H^2$. From the previous lemma we deduce that $n^2(\Lambda_{i+n,i+n+p} - \Gamma_{i+n,i+n+p}) \underset{n\to\infty}{\to} \theta_{i,i+p} - \phi_{i,i+p}$ where $(\phi_{i,j})$ is the matrix of the Hardy space Toeplitz operator $T_{|g'|^2}$. Hyponormality leads to the positivity of $T_{|f'|^2-|g'|^2}$, and a property of Toeplitz forms [7] implies that $|g'| \leq |f'|$ a.e on the unit circle. The proof is complete. □

Corollary 3.4. *Let f and g be analytic and univalent in an open set containing U. Then $T_{f+\overline{g}}$ is normal if and only if $g = cf + d$ for some constants c and d with $|c| = 1$.*

Proof. if $g = cf + d$ with $|c| = 1$, it is easy to see that $T_{f+\overline{g}}$ is normal. Conversely if $T_{f+\overline{g}}$ is normal then $|g'| = |f'|$ on the circle and a maximum modulus argument shows that $g' = cf'$ with $|c| = 1$. Thus $g = cf + d$. □

We now find a sufficient condition for hyponormality when $f = z^q$. We begin with the case $g = \lambda z^p$. We set $\mu_1 = \min\left\{\sqrt{\frac{\alpha_{i+p}}{\alpha_{i+q}}},\ 0 \leq i < q\right\}$, $\mu_2 = \min\left\{\sqrt{\frac{\alpha_{i+p}\alpha_{i-q}}{\alpha_{i+q}\alpha_{i-q}-\alpha_i^2}},\ q \leq i < p\right\}$ and $\mu_3 = \inf\left\{\sqrt{\frac{(\alpha_{i+p}\alpha_{i-p}-\alpha_i^2)\alpha_{i-q}}{(\alpha_{i+q}\alpha_{i-q}-\alpha_i^2)\alpha_{i-p}}},\ p \leq i\right\}$.

Proposition 3.5. *Assume $p > q$. The operator $T_{z^q+\lambda\overline{z^p}}$ is hyponormal if and only if $|\lambda| \leq \lambda_{p,q} = \min\{\mu_1, \mu_2, \mu_3\}$.*

Proof. In this case hyponormality is equivalent to $|\lambda|^2 H^*_{\overline{z^p}} H_{\overline{z^p}} \leq H^*_{\overline{z^q}} H_{\overline{z^q}}$. A computation shows that the matrix of $H^*_{\overline{z^m}} H_{\overline{z^m}}$ is diagonal and its diagonal term is given by:

$$D_i = \frac{\alpha_{i+m}}{\alpha_i}\ if\ m > i,\ \ D_i = \frac{\alpha_{i+m}}{\alpha_i} - \frac{\alpha_i}{\alpha_{i-m}}\ if\ m \leq i.$$

Hyponormality is thus equivalent to the following inequalities:

$$i)\ |\lambda|^2\frac{\alpha_{i+q}}{\alpha_i} \leq \frac{\alpha_{i+p}}{\alpha_i} \quad 0 \leq i < q$$

$$ii)\ |\lambda|^2\left(\frac{\alpha_{i+q}}{\alpha_i} - \frac{\alpha_i}{\alpha_{i-q}}\right) \leq \frac{\alpha_{i+p}}{\alpha_i} \quad q \leq i < p$$

$$iii)\ |\lambda|^2\left(\frac{\alpha_{i+q}}{\alpha_i} - \frac{\alpha_i}{\alpha_{i-q}}\right) \leq \frac{\alpha_{i+p}}{\alpha_i} - \frac{\alpha_i}{\alpha_{i-p}} \quad p \leq i$$

Obviously inequality i) is equivalent to $|\lambda| \leq \mu_1 = \min\left\{\sqrt{\frac{\alpha_{i+p}}{\alpha_{i+q}}}, 0 \leq i \leq q\right\}$, and ii) is equivalent to $|\lambda| \leq \mu_2 = \min\{\sqrt{\frac{\alpha_{i+p}\alpha_{i-q}}{\alpha_{i+q}\alpha_{i-q}-\alpha_i^2}}, q \leq i < p\}$. The last inequality is equivalent to $|\lambda| \leq \mu_3 = \inf\{\sqrt{\frac{(\alpha_{i+p}\alpha_{i-p}-\alpha_i^2)\alpha_{i-q}}{(\alpha_{i+q}\alpha_{i-q}-\alpha_i^2)\alpha_{i-p}}}, p \leq i\}$. Thus hyponormality of $T_{z^q+\lambda\overline{z^p}}$ is equivalent to $|\lambda| \leq \mu_{p,q} = \min\{\mu_1, \mu_2, \mu_3\}$. □

Remark 3.6. *If $p = q$ then clearly hyponormality of $T_{z^q+\lambda\overline{z^p}}$ is equivalent to $|\lambda| \leq 1$.Thus if $p \geq q$,from the previous theorem $|\mu_{p,q}| \leq \frac{q}{p}$.*

In the following proposition we assume $q \geq 2$ (the case $q = 1$ being trivial). We set $\tau_1 = \min\{\sqrt{\frac{\alpha_{i+q}}{\alpha_{i+p}}},\ 0 \leq i < p\}$, $\tau_2 = \min\{\sqrt{\frac{\alpha_{i+q}\alpha_{i-p}}{\alpha_{i-p}\alpha_{i+p}-\alpha_i^2}},\ p \leq i < q\}$ and $\tau_3 = \inf\{\sqrt{\frac{(\alpha_{i+q}\alpha_{i-q}-\alpha_i^2)\alpha_{i-p}}{(\alpha_{i+p}\alpha_{i-p}-\alpha_i^2)\alpha_{i-q}}},\ q \leq i\}$.

Proposition 3.7. *Assume $p < q$ then $T_{z^q+\lambda\overline{z^p}}$ is hyponormal if and only if $|\lambda| \leq \sigma_{p,q} = \min\{\tau_1, \tau_2, \tau_3\}$.*

The proof, being similar to the proof given above, is omitted. We set $\sigma_{q,q} = 1$. Note that hyponormality of $T_{f+\overline{g}}$ implies that $\|g\| \leq \|f\|$. In particular $\sigma_{p,q} \leq \sqrt{\frac{\alpha_q}{\alpha_p}}$. In what follows we give a sufficient condition for the hyponormality of $T_{z^q+\overline{g}}$. We denote by B_1 the unit ball of $L^{2\perp}_{a,w}$.

Definition 3.8. *For $f \in L^2_{a,w}$, set*

$$G_f = \left\{g \in L^2_{a,w},\ \sup\{|<\overline{g}\,k, u>|,\ u \in B_1\} \leq \sup\{|<\overline{f}\,k, u>|,\ u \in B_1\}\ \textit{for any}\ k \in H^\infty\right\}.$$

By the density of H^∞ in $L^2_{a,w}$ we see that $g \in G_f$ is equivalent to $T_{f+\overline{g}}$ is hyponormal. We list the properties of G_f in the following proposition:

Proposition 3.9. *Let $f \in L^2_{a,w}$, the following holds:*
i) G_f is convex and balanced.
ii) If $g \in G_f$ and c is a constant the $g + c \in G_f$.
iii) $f \in G_f$.
iv) G_f is weakly closed.

Proof. i), ii) and iii) follow from the definition of G_f. For the proof of iv) assume (g_i) is a net in G_f such that $g_i \to g$. We have for $v \in B_1$ and $k \in H^\infty$, $|<\overline{g_i}\,k, v>| \leq \sup\{|<\overline{f}\,k, u>|,\ u \in B_1\}$. Taking the limit we get $|<\overline{g}\,k, v>| \leq \sup\{|<\overline{f}\,k, u>|,\ u \in B_1\}$ for any $v \in B_1$. Taking the supremum on the left hand side we get: $\sup\{|<\overline{g}\,k, u>|,\ u \in B_1\} \leq \sup\{|<\overline{f}\,k, u>|,\ u \in B_1\}$ for any $k \in H^\infty$. This completes the proof. □

Corollary 3.10. *Assume (λ_n) is a sequence of complex numbers satisfying $\sum|\lambda_n| \leq 1$. Then $T_{z^q+\sum_1^q \lambda_m\sigma_{m,q}\overline{z^m}+\sum_{q+1}^\infty \lambda_m\mu_{m,q}\overline{z^m}}$ is hyponormal.*

Proof. Set $g_N = \sum_1^q \lambda_m\sigma_{m,q}z^m + \sum_{q+1}^N \lambda_m\mu_{m,q}z^m$ for $N \geq q+1$ and let $h = \sum_n h_n z^n$ be in $L^2_{a,w}$. We have the following inequalities for $M > N \geq q+1$

$$|\langle g_M - g_N, h\rangle| \leq \sum_N^M |\lambda_m||h_m||\alpha_m| \leq (\sum_N^M |\lambda_m|^2|\alpha_m|)^{1/2}(\sum_N^M |h_m|^2|\alpha_m|)^{1/2}.$$

Thus (g_N) converges weakly and a similar argument shows that the limit is $\sum_1^q \lambda_m\sigma_{m,q}z^m + \sum_{q+1}^\infty \lambda_m\mu_{m,q}z^m$. The result follows from the previous proposition. □

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