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Semi-Parallel and Harmonic Surfaces in Semi-Euclidean 4-Space with Index 2

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Abstract. The present paper mainly deals with construction and investigation further properties of semi-parallel and harmonic surfaces. The first part of the study shall be devoted to investigate and present necessary and sufficient conditions of being semi-parallel surfaces by considering semi-parallelity condition $R(X,Y).h = 0$. In the light of the condition, the fact of a part of semi-parallel surfaces can be created by translation surfaces is captured. As a last result, we present that M must be a translation surface in the case of it is a harmonic surface.

1. Introduction

In [7] , J. Deprez initiated the study of semi- parallel, or semi symetric submanifolds. Denote by $\bar{R}$ the curvature tensor of the Vander Waerden-Bortoletti connection $\bar{\nabla}$ of M and h is the second fundamental form of M in $\mathbb{E}^{n+d}$. The submanifold M is called semi-parallel (or semi-symmetric) if $\bar{R}\cdot h = 0$. This notion is a direct generalization of parallel submanifolds, i.e. submanifolds for which $\bar{\nabla}h = 0$. In [6], J. Deprez showed the fact that the submanifold $M \subset \mathbb{E}^m$ is semiparallel implies that (M,g) is semi-symmetric. For more details on semi-symmetric spaces, we refer the readers to [12] and references therein. J. Deprez gave fundamental equalities and achived local classification of semi-parallel hypersurfaces in Euclidean space. By using these equalities several authors studied on semi parallelity of a surface in four dimensional Euclidean and semi-Euclidean spaces and some space forms, see [6-9, 11].

It is well known that the Gauss map plays important role in determining geometry of a submanifold. Denote by G the Gauss map of M, if $\Delta G = 0$, then M is called a harmonic surface. On the other hand harmonic functions have several properties in advanced analysis. Therefore, we aim to describe surfaces having harmonic Gauss map in locally cases. In Euclidean spaces Gauss map have been used to classify the surfaces by several authors, among the others we can refer some of them as [1], [8] and [13]. More detailed information about harmonic surfaces see [2-4].

The present paper is organized as follows: we first recall some basic concepts and the notations of submanifolds and we deal with geometrical properties of surfaces in semi- Euclidean space $\mathbb{E}^4_2$. The third section includes the first main results of the paper in which semi-parallelity conditions of a surface are presented and shown that some special translation surfaces are satisfies the semi-parallelity conditions in $\mathbb{E}^4_2$. The fourth which is the last section consists of the investigation of harmonic surfaces. As a result of this section, we get that harmonic surfaces in $\mathbb{E}^4_2$ are only translation surfaces.

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2. Preliminaries

Based on [5] and [6], we shall give some basic definitions, concepts and notions of Riemann submanifolds and some auxiliary results will be presented, which will be necessary to prove our main results.

Let $\iota : M \to \mathbb{E}^n$ be an immersion from an $m-$dimensional connected Riemannian manifold M into an $n-$dimensional Euclidean space $\mathbb{E}^n$. We denote by g the metric tensor of $\mathbb{E}^n$, which is also an induced metric on M. Let $\bar{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^n$ and ∇ be the induced connection on M. Then the Gaussian and Weingarten formulas are given by

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \bar{\nabla}_X N &= -A_N X + \nabla^{\perp}_X N, \end{aligned} \tag{1}$$

where X, Y are vector fields tangent to M and N normal to M. Moreover, h is the second fundamental form, $\nabla^{\perp}$ is a linear connection induced in the normal bundle $T^{\perp}M$, called normal connection and A_N is the shape operator in the direction of N that is related with h by

$$< h(X, Y), N >=< A_N X, Y > . \tag{2}$$

If the set $\{X_1, ..., X_m\}$ is a local basis for $\chi(M)$ and $\{N_1, ..., N_{n-m}\}$ is a orthonormal local basis for $\chi^{\perp}(M)$, then h can be written as

$$h = \sum_{\alpha=1}^{n-m} \sum_{i,j=1}^{m} h^{\alpha}_{ij} N_{\alpha}, \tag{3}$$

where

$$h^{\alpha}_{ij} =< h(X_i, X_j), N_{\alpha} > .$$

The covariant differentiation $\bar{\nabla}h$ of the second fundamental form h on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^{\perp}M$ of M is defined by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{4}$$

for any vector fields X, Y and Z tangent to M. Then we have the Codazzi equation

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z). \tag{5}$$

We denote the curvature tensor by R associated with ∇;

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z \tag{6}$$

and denote the curvature tensor by $R^{\perp}$ associated with $\nabla^{\perp}$

$$R^{\perp}(X, Y)\eta = \nabla^{\perp}_Y \nabla^{\perp}_X \eta - \nabla^{\perp}_X \nabla^{\perp}_Y \eta - \nabla^{\perp}_{[X,Y]} \eta, \tag{7}$$

where η is normal vector field to M .

The well-known Gauss and Ricci equations are given by

$$< R(X, Y)Z, W >=< h(X, W), h(Y, Z) > - < h(X, Z), h(Y, W) >, \tag{8}$$

$$< \bar{R}(X,Y)\eta,\xi > - < R^{\perp}(X,Y)\eta,\xi > = < [A_{\eta},A_{\xi}]X,Y > \tag{9}$$

for any vector fields X, Y, Z, W are tangent to M and ξ, η are normal vector fields to M.

The Gaussian curvature of M is defined by

$$K = < h(X_1,X_1), h(X_2,X_2) > - \|h(X_1,X_2)\|^2 , \tag{10}$$

where the set $\{X_1, X_2\}$ is a linearly independent subset of $\chi(M)$.

The normal curvature K_N of M is defined by

$$K_N = \left\{ \sum_{1=\alpha<\beta}^{n-2} < R^{\perp}(X_1,X_2)N_{\alpha},N_{\beta} > \right\}^{1/2} , \tag{11}$$

where $\{N_\alpha, N_\beta\}$ is an orthonormal basis of $\chi^{\perp}(M)$. (11) directly implies that $K_N = 0$ if and only if $\nabla^{\perp}$ is a flat normal connection of M .

Further mean curvature vector $\vec{H}$ of M is defined by

$$\vec{H} = \frac{1}{2}\sum_{\alpha=1}^{2} tr(A_{N_\alpha})N_\alpha . \tag{12}$$

Let us consider product tensor $\bar{R}.h$ of the curvature tensor $\overline{R}$ with the second fundamental form h which is defined by

$$(\bar{R}(X,Y).h)(Z,T) = \bar{\nabla}_X(\bar{\nabla}_Y h(Z,T)) - \bar{\nabla}_Y(\bar{\nabla}_X h(Z,T)) - \bar{\nabla}_{[X,Y]}h(Z,T)), \tag{13}$$

for all X, Y, Z, T tangent to M.

Definition 2.1. *([7]) A surface M is called semi-parallel if $\bar{R}.h = 0$, i.e. $\bar{R}(X,Y).h = 0$.*

As a result of this definition, one can directly obtain that

$$(\bar{R}(X,Y).h)(Z,T) = R^{\perp}(X,Y)h(Z,T) - h(R(X,Y)Z,T) - h(Z,R(X,Y)T). \tag{14}$$

Lemma 2.2. *([7]) Let $M \subset \mathbb{E}^n$ be a smooth surface given with the patch X(u,v). Then the following equalities are hold;*

$$\left.\begin{aligned}
(\bar{R}(X_1,X_2).h)(X_1,X_1) = {} & \left(\sum_{\alpha=1}^{n-2} h_{11}^{\alpha}(h_{22}^{\alpha} - h_{11}^{\alpha} + 2K)\right) h(X_1,X_2) \\
& + \sum_{\alpha=1}^{n-2} h_{11}^{\alpha}h_{12}^{\alpha}(h(X_1,X_1) - h(X_2,X_2)) \\
(\bar{R}(X_1,X_2).h)(X_1,X_2) = {} & \left(\sum_{\alpha=1}^{n-2} h_{12}^{\alpha}(h_{22}^{\alpha} - h_{11}^{\alpha})\right) h(X_1,X_2) \\
& + \sum_{\alpha=1}^{n-2} h_{12}^{\alpha}h_{12}^{\alpha}(h(X_1,X_1) - h(X_2,X_2)) \\
(\bar{R}(X_1,X_2).h)(X_2,X_2) = {} & \left(\sum_{\alpha=1}^{n-2} h_{22}^{\alpha}(h_{22}^{\alpha} - h_{11}^{\alpha} - 2K)\right) h(X_1,X_2) \\
& + \sum_{\alpha=1}^{n-2} h_{22}^{\alpha}h_{12}^{\alpha}(h(X_1,X_1) - h(X_2,X_2))
\end{aligned}\right\} \tag{15}$$

Now, let us also give Deprez's result regarding with the classification of semi-parallel surfaces.

Theorem 2.3. *([7]) Let M a surface in n-dimensional Euclidean space $\mathbb{E}^n$. Then M is semi-parallel if and only if locally;*

i) M is equivalent to 2-sphere, or

ii)M has trivial normal connection, or

iii) M is an isotropic surface in $\mathbb{E}^5 \subset \mathbb{E}^n$ satisfying $\|H\|^2 = 3K$.

Let $G(n-m,n)$ denote the Grassmannian manifold consisting of all oriented $(n-m)-$planes through origin of $\mathbb{E}^n$. Let M be an oriented $m-$ dimensional submanifold of a Euclidean space E^n. The Gauss map $G : M \to G(n-m,n)$ of M is a smooth map which carries a point $p \in M$ into the oriented $m-$*plane* through the origin of $\mathbb{E}^n$ obtained by the parallel translation of the tangent space of M at p in $\mathbb{E}^n$.

Since $G(n-m,n)$ is canonically embedded in $\Lambda^{n-m}\mathbb{E}^n = \mathbb{E}^N$, $N = \binom{n}{n-m}$, the notion of the type of the Gauss map is naturally defined. If $\{e_{m+1}, e_{m+2}, ..., e_n\}$ is an oriented orthonormal normal frame on M, then the Gauss map $G : M \to G(n-m,n) \subset \mathbb{E}^N$ is given by $G(p) = (e_{m+1}\Lambda e_{m+2}\Lambda...\Lambda e_n)(p)$. The *inner product on* $\Lambda^m E^n$ is defined for $w_1 = u_1\Lambda u_2\Lambda...\Lambda u_{n-m}$, $w_2 = v_1\Lambda v_2\Lambda...\Lambda v_{n-m}$ by

$$< w_1, w_2 >= \det < u_i, v_j > . \tag{16}$$

For $n = 4$, an orthonormal basis of $\Lambda^2\mathbb{E}^4$ with respect to above inner product is the set

$$\{e_i\Lambda e_j \mid 1 \leq i < j \leq 4\}.$$

For any real function f on M, the *Laplacian of* f is defined by

$$\Delta f = -\sum_i \varepsilon_i(\bar{\nabla}_{e_i}\bar{\nabla}_{e_i} f - \bar{\nabla}_{\nabla_{e_i}e_i} f). \tag{17}$$

Let G be Gauss map of the surface M. If $\Delta G = 0$, then M is called a harmonic surface.

Definition 2.4. *([2]) A surface M defined as the sum of two space curves $\alpha(u) = (u, 0, f_3(u), f_4(u))$ and $\beta(v) = (0, v, g_3(v), g_4(v))$ is called a translation surface in $\mathbb{E}^4$. So, a translation surface is defined by a patch*

$$X(u,v) = (u, v, f_3(u) + g_3(v), f_4(u) + g_4(v)).$$

Let we define a semi-Euclidean metric with index two on $\mathbb{E}^4$ as

$$\mathbf{g} = 2(dx_1dx_3 + dx_2dx_4).$$

As a usual representation, $\mathbb{E}^4$ with above metric is denoted by $\mathbb{E}^4_2$. Now, we consider a non-degenerate surface in $\mathbb{E}^4_2$ and for this surface, a parametrization can be chosen as

$$\begin{aligned} &\xi : U \subset \mathbb{R}^2 \to \mathbb{E}^4_2, \\ &\xi(u,v) = (u, v, \bar{\xi}_1(u,v), \bar{\xi}_2(u,v)). \end{aligned} \tag{18}$$

Without loss of generality we can choose ξ as an orthogonal parametrization, i.e., $\bar{\xi}_{1v} = -\bar{\xi}_{2u}$. According to this parametrization, an orthonormal tangent frame to M is given by

$$\begin{aligned} e_1 &= \frac{1}{\|\xi_u\|}\xi_u \\ &= \frac{1}{\sqrt{2\xi_{1u}}}(1, 0, \bar{\xi}_{1u}, \bar{\xi}_{2u}), \\ e_2 &= \frac{1}{\|\xi_u\|}\xi_v \\ &= \frac{1}{\sqrt{2\xi_{2v}}}(0, 1, \bar{\xi}_{1v}, \bar{\xi}_{2v}), \end{aligned} \tag{19}$$

where $\xi_1 = \left|\bar{\xi}_1\right|$, $\xi_2 = \left|\bar{\xi}_2\right|$. The normal space of M is spanned by

$$\begin{aligned} n_1 &= \frac{1}{\sqrt{2\xi_{1u}}}(1,0,-\bar{\xi}_{1u},-\bar{\xi}_{2u}), \\ n_2 &= \frac{1}{\sqrt{2\xi_{2v}}}(0,1,-\bar{\xi}_{1v},-\bar{\xi}_{2v}), \end{aligned} \tag{20}$$

where

$$\begin{aligned} \mathbf{g}(e_1,e_1) &= -\mathbf{g}(n_2,n_2) = \varepsilon_1, \\ \mathbf{g}(e_2,e_2) &= -\mathbf{g}(n_1,n_1) = \varepsilon_2 \end{aligned} \tag{21}$$

and $\varepsilon_1 = \mp 1$, $\varepsilon_2 = \mp 1$. In addition, it is easily seen that $\{e_1,e_2,n_1,n_2\}$ is positively oriented frame along M.

By covariant differentiation with respect to e_1 and e_2, a straightforward calculation gives

$$\begin{aligned} &\bar{\nabla}_{e_1} e_1 = a\varepsilon_2 e_2 - b\varepsilon_1 n_1 - a\varepsilon_2 n_2, \\ &\bar{\nabla}_{e_1} e_2 = -a\varepsilon_1 e_1 + a\varepsilon_1 n_1 - c\varepsilon_2 n_2, \\ &\bar{\nabla}_{e_1} n_1 = -b\varepsilon_1 e_1 + a\varepsilon_2 e_2 - a\varepsilon_2 n_2, \\ &\bar{\nabla}_{e_1} n_2 = -a\varepsilon_1 e_1 - c\varepsilon_2 e_2 + a\varepsilon_1 n_1 \end{aligned} \tag{22}$$

and

$$\begin{aligned} &\bar{\nabla}_{e_2} e_1 = c\varepsilon_2 e_2 + a\varepsilon_1 n_1 - c\varepsilon_2 n_2, \\ &\bar{\nabla}_{e_2} e_2 = -c\varepsilon_1 e_1 + c\varepsilon_1 n_1 - d\varepsilon_2 n_2, \\ &\bar{\nabla}_{e_2} n_1 = a\varepsilon_1 e_1 + c\varepsilon_2 e_2 - c\varepsilon_2 n_2, \\ &\bar{\nabla}_{e_2} n_2 = -c\varepsilon_1 e_1 - d\varepsilon_2 e_2 + c\varepsilon_1 n_1, \end{aligned} \tag{23}$$

where a, b, c and d are Christoffel symbols given by

$$a = \frac{\xi_{2uu}}{2\sqrt{2}\xi_{1u}\sqrt{\xi_{2v}}}, \tag{24}$$

$$b = \frac{\xi_{1uu}}{2\sqrt{2}(\xi_{1u})^{\frac{3}{2}}}, \tag{25}$$

$$c = \frac{-\xi_{1vv}}{2\sqrt{2}\xi_{2v}\sqrt{\xi_{1u}}}, \tag{26}$$

$$d = \frac{\xi_{2vv}}{2\sqrt{2}(\xi_{2v})^{\frac{3}{2}}}. \tag{27}$$

By (3), second fundamental form of this structure is written as

$$h = \sum_{i,j,\alpha=1}^{2} h_{ij}^{\alpha} n_\alpha, \tag{28}$$

where

$$\begin{array}{ll} h_{11}^1 = -b\varepsilon_1 & h_{11}^2 = -a\varepsilon_2 \\ h_{12}^1 = h_{21}^1 = a\varepsilon_1 & h_{12}^2 = h_{21}^2 = -c\varepsilon_2 \\ h_{22}^1 = c\varepsilon_1 & h_{22}^2 = -d\varepsilon_2 \end{array} . \tag{29}$$

Corollary 2.5. *M is a totally geodesic surface in* $\mathbb{E}^4_2$ *if and only if* $a = b = c = d = 0$*, i.e.,*

$$\begin{aligned} \xi_1(u,v) &= -\lambda v + \lambda_1 u + \lambda_2, \\ \xi_2(u,v) &= \lambda u + \mu_1 v + \mu_2, \end{aligned}$$

for some real constants $\lambda, \lambda_1, \lambda_2, \mu_1, \mu_2$.

The induced covariant differentiation on M can be stated as

$$\left.\begin{aligned} \nabla_{e_1} e_1 &= a\varepsilon_2 e_2, \\ \nabla_{e_1} e_2 &= -a\varepsilon_1 e_1, \\ \bar{\nabla}_{e_2} e_1 &= c\varepsilon_2 e_2, \\ \bar{\nabla}_{e_2} e_2 &= -c\varepsilon_1 e_1. \end{aligned}\right\} \tag{30}$$

$$\left.\begin{aligned} \nabla^{\perp}_{e_1} n_1 &= -a\varepsilon_2 n_2, \\ \nabla^{\perp}_{e_1} n_2 &= a\varepsilon_1 n_1, \end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned} \nabla^{\perp}_{e_2} n_1 &= -c\varepsilon_2 n_2, \\ \nabla^{\perp}_{e_2} n_2 &= c\varepsilon_1 n_1, \end{aligned}\right\} \tag{32}$$

where the equalities (31) and (32) define the normal connection on M.

Lemma 2.6. *Let M be a surface in* $\mathbb{E}^4_2$ *with patch (18). Then we have its shape operators as*

$$A_{n_1} = \begin{bmatrix} -b\varepsilon_1 & a\varepsilon_1 \\ a\varepsilon_2 & c\varepsilon_2 \end{bmatrix}, \qquad A_{n_2} = \begin{bmatrix} -a\varepsilon_1 & -c\varepsilon_1 \\ -c\varepsilon_2 & -d\varepsilon_2 \end{bmatrix}. \tag{33}$$

With an elemantary calculation we get that Gauss and Ricci equations of M are identical and

$$\frac{a_v}{\sqrt{2\xi_{2v}}} - \frac{c_u}{\sqrt{2\xi_{1u}}} = (2a^2 + bc)\varepsilon_1 + (2c^2 - ad)\varepsilon_2. \tag{34}$$

Codazzi equations of M are

$$\frac{b_v}{\sqrt{2\xi_{2v}}} + \frac{a_u}{\sqrt{2\xi_{1u}}} = ab\varepsilon_1 - ac\varepsilon_2, \tag{35}$$

$$\frac{c_v}{\sqrt{2\xi_{2v}}} - \frac{d_u}{\sqrt{2\xi_{1u}}} = ac\varepsilon_1 + cd\varepsilon_2. \tag{36}$$

3. Semi Parallel Surfaces

In this section we investigate semi-parallelity of a surface with patch (18). By considering (3), (4) and (14) we determine semi-parallelity conditions of M.

We suppose that M is a semi-parallel surface, i.e., $\bar{R}.h = 0$. The equalities (15) and elementary calculations yield

$$\begin{aligned} (a^2d - abc - a^3 - ac^2)\varepsilon_1 - (2ad^2 + 2ac^2)\varepsilon_1\varepsilon_2 + 2abc + 2a^3 &= 0 \\ (bc^2 + b^2c - abd + a^2b)\varepsilon_2 - (2bc^2 + 2a^2c)\varepsilon_1\varepsilon_2 + 2adc + 2c^3 &= 0 \\ (acd - c^2b - a^2c - c^3)\varepsilon_1 - (abd + bc^2 + acd + c^3)\varepsilon_1\varepsilon_2 + b^2c + a^2b + bc^2 + a^2c &= 0 \\ (a^2d - abc - a^3 - ac^2)\varepsilon_2 + (-bcd - a^2d + abc + a^3)\varepsilon_1\varepsilon_2 + ad^2 + c^2d - a^2d - ac^2 &= 0 \\ (ad^2 - a^2d - bcd - dc^2)\varepsilon_1 + (2a^2d + 2ac^2)\varepsilon_1\varepsilon_2 - 2abc - 2a^3 &= 0 \\ (acd - c^2b - a^2c - c^3)\varepsilon_2 + (2bc^2 + 2a^2c)\varepsilon_1\varepsilon_2 - 2adc - 2c^3 &= 0\ . \end{aligned}$$

Theorem 3.1. *Let M be a surface given with the surface patch in (18). Then M is a semi parallel surface if and only if:*

i) For $\varepsilon_1 = \varepsilon_2 = 1$*, a and c must be zero or equivalently to this, following equalities are satisfied,*

$$\begin{aligned} \xi_1(u,v) &= -\lambda v - A(u), \\ \xi_2(u,v) &= \lambda u + B(v), \end{aligned}$$

where $A'(u) < 0$ *and* $B'(v) > 0$*,*

ii) For $\varepsilon_1 = -\varepsilon_2 = 1$*, either a and c must be zero or equivalently to this, following equalities are satisfied,*

$$\begin{aligned} \xi_1(u,v) &= -\lambda v - A(u), \\ \xi_2(u,v) &= \lambda u + B(v) \text{ where } A'(u) < 0,\ B'(v) < 0 \end{aligned}$$

or a and d must be zero and $b = -c$ *or equivalently to these, following equalities are satisfied,*

$$\begin{aligned} \xi_1(u,v) &= -\lambda_1 \frac{v^2}{2} - \lambda_2 v - \frac{3c_1}{8\lambda_1}(\lambda_1 u + r_1)^{\frac{8}{3}} + c_2, \\ \xi_2(u,v) &= \lambda_1 uv + \lambda_2 u + r_1 v + r_2 \end{aligned}$$

where $\lambda_1, \lambda_2, c_1, c_2, r_1, r_2$ *are real constants and* $(u,v) \in (\frac{-r_1}{\lambda_1}, \infty) \times \mathbb{R}$ *,* $c_1 < 0$.

iii) For $-\varepsilon_1 = \varepsilon_2 = 1$*, either a and c must be zero or equivalently to these, following equalities are satisfied,*

$$\begin{aligned} \xi_1(u,v) &= -\lambda v - A(u), \\ \xi_2(u,v) &= \lambda u + B(v), \text{ where } A'(u) > 0,\ B'(v) > 0 \end{aligned}$$

or $a = 0, b = -c$ *equivalently,*

$$\begin{aligned} \xi_1(u,v) &= \frac{\alpha}{\lambda} v + (\frac{\mu \lambda^{\frac{1}{3}}}{k})^2 u + m, \\ \xi_2(u,v) &= -\frac{\alpha}{\lambda} u - \frac{1}{k(kv+l)} - \frac{\beta}{\alpha}, \end{aligned}$$

where $\alpha, \beta, \lambda, k, l, m, \mu$ *are real constants* $(u,v) \in (-\infty, 0) \times \mathbb{R}$.

iv) For $\varepsilon_1 = \varepsilon_2 = -1$ *, a and c must be zero or equivalently to thesethe, following equalities are satisfied,*

$$\begin{aligned} \xi_1(u,v) &= -\lambda v - A(u), \\ \xi_2(u,v) &= \lambda u + B(v), \end{aligned}$$

where $A'(u) > 0$ *and* $B'(v) < 0$.

Example 3.2. *A translation surface with coordinate patch*

$$\xi(u,v) = (u, 0, -A(u), \lambda u) + (0, v, -\lambda v, B(v))$$

which is given in [2], is a semi- parallel surface in $\mathbb{E}^4_2$.

4. Harmonic Surfaces

In this last section, we focus on harmonic surfaces and their properties.

Lemma 4.1. *([8]) Let M be an n- dimensional submanifold of Euclidean space* $\mathbb{E}^{n+2}$*. Then the Laplacian of the Gauss map* $G = e_{n+1} \Lambda e_{n+2}$ *is given by*

$$\begin{aligned} \Delta G = & \|h\|^2 G + 2\sum_{j<k} R^{\perp}(e_j, e_k; e_{n+1}, e_{n+2}) e_j \Lambda e_k \\ & + n\sum_{j=1}^{n} \omega_{n+2}^{n+1}(e_j) e_j \Lambda H + \nabla(trA_{n+1}) \Lambda e_{n+2} - \nabla(trA_{n+2}) \Lambda e_{n+1}, \end{aligned}$$

where $\|h\|^2$ *is the squared length of the second fundamental form,* $R^{\perp}$ *is normal curvature tensor and* $\nabla(trA_r)$ *is gradient of* trA_r.

Using (12) and (33), we immediately have

$$H=\frac{1}{2}[(\varepsilon_1 b-\varepsilon_2 c)n_1+(\varepsilon_1 a+\varepsilon_2 d)n_2],$$

$$\nabla(trA_{n_1})=[\frac{b_u}{p}+\varepsilon_1\varepsilon_2\frac{c_u}{p}]e_1+[\frac{c_v}{q}-\varepsilon_1\varepsilon_2\frac{b_v}{q}]e_2,$$

$$\nabla(trA_{n_2})=[-\frac{a_u}{p}-\varepsilon_1\varepsilon_2\frac{d_u}{p}]e_1-[\frac{d_v}{q}+\varepsilon_1\varepsilon_2\frac{a_v}{q}]e_2,$$

where $p=\sqrt{2\xi_{1u}}$ and $q=\sqrt{2\xi_{2v}}$. Furthermore, considering (22) we obtain

$$w_{34}(e_1)=-a\varepsilon_2,\qquad w_{34}(e_2)=-c\varepsilon_2,$$

$$R^{\perp}(e_1,e_2;n_1,n_2)=-a^2\varepsilon_1-c^2\varepsilon_2-bc\varepsilon_1+ad\varepsilon_2,$$

$$||h||^2=-3a^2\varepsilon_2-3c^2\varepsilon_1-b^2\varepsilon_1-d^2\varepsilon_2.$$

Thus Laplacian of the surface M can be stated as

$$\begin{aligned}\Delta G \;=\;& (-2a^2\varepsilon_2-2c^2\varepsilon_1-2bc\varepsilon_2+2ad\varepsilon_1)e_1\Lambda e_2\\ &+(\frac{a_u}{p}+\varepsilon_1\varepsilon_2\frac{d_u}{p}+ab\varepsilon_2-ac\varepsilon_1)e_1\Lambda n_1\\ &+(-\frac{b_u}{p}+\varepsilon_1\varepsilon_2\frac{c_u}{p}+a^2\varepsilon_2+ad\varepsilon_1)e_1\Lambda n_2\\ &+(\frac{d_v}{q}+\varepsilon_1\varepsilon_2\frac{a_v}{q}+bc\varepsilon_1-c^2\varepsilon_2)e_2\Lambda n_1\\ &+(\frac{c_v}{q}-\varepsilon_1\varepsilon_2\frac{b_v}{q}+ac\varepsilon_1+cd\varepsilon_2)e_2\Lambda n_2\\ &+(-3a^2\varepsilon_2-3c^2\varepsilon_1-b^2\varepsilon_1-d^2\varepsilon_2)n_1\Lambda n_2.\end{aligned} \tag{37}$$

Corollary 4.2. *The surface M is harmonic if and only if one of the following statements is hold*
(i) For $\varepsilon_1=\varepsilon_2$, it is totally geodesic,
(ii) For $\varepsilon_1=-\varepsilon_2=1$, by considering fundamental equations (Gauss, Ricci and Codazzi) we have

$$a^2-c^2+bc+ad=0, \tag{38}$$

$$3a^2-3c^2-b^2+d^2=0. \tag{39}$$

From (38) and (39) it is easily seen that $a^2-c^2=-(bc+ad)$. If it replaces in Gauss equation, from (34), (35) and (36) we get the following system;

$$\begin{aligned}
\frac{a_u}{p} - \frac{d_u}{p} &= ab + ac, \\
\frac{b_u}{p} + \frac{c_u}{p} &= -a^2 + ad, \\
\frac{d_v}{q} - \frac{a_v}{q} &= -bc - c^2, \\
\frac{c_v}{q} + \frac{b_v}{q} &= -ac + cd, \\
\frac{a_v}{q} - \frac{c_u}{p} &= a^2 - c^2, \\
\frac{b_v}{q} + \frac{a_u}{p} &= ab + ac, \\
\frac{c_v}{q} - \frac{d_u}{p} &= ac - cd.
\end{aligned}$$

Now, if we put

$$\begin{array}{cccc} \frac{a_u}{p} = x & \frac{b_u}{p} = y & \frac{c_u}{p} = z & \frac{d_u}{p} = w \\ \frac{a_v}{q} = \bar{x} & \frac{b_v}{q} = \bar{y} & \frac{c_v}{q} = \bar{z} & \frac{d_u}{p} = \bar{w} \end{array},$$

then we obtain the following linear equation system in order (x,x̄,y,ȳ,z,z̄,w,w̄)

$$\left[\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & : & ab+ac \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & : & -a^2+ad \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & : & -bc-c^2 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & : & cd-ac \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & : & a^2-c^2 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & : & ab+ac \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & ac-cd
\end{array}\right].$$

After some elementary operations, this system turns to

$$\left[\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & : & ab+ac \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & : & bc+c^2 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & : & ad-a^2 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & : & cd-ac \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & : & 2c^2+bc-a^2 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & cd-ac \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & : & -ac+cd
\end{array}\right].$$

Consequently, this system is compatible if and only if $d - a = 0$ or $c = 0$. Then we will consider following two cases:

i) If $a = d$, then M is a totally geodesic surface in $\mathbb{E}^4_2$, i.e. M becomes a plane or a part of plane.

ii) If $c = 0$, either M is a totally geodesic surface in $\mathbb{E}^4_2$ or M has following parametrization

$$\begin{aligned}
\xi_1 &= \lambda v + f(u), \\
\xi_2 &= -\lambda u + g(v),
\end{aligned}$$

such that, f and g satisfy the following differential equations

$$\begin{aligned}
f'' \mp \beta f'^{3/2} &= 0, \\
g'' \mp \beta g'^{3/2} &= 0.
\end{aligned}$$

For $-\varepsilon_1 = \varepsilon_2 = 1$, by considering fundamental equations (Gauss, Ricci and Codazzi) we have

$$\left.\begin{array}{l} c^2 - a^2 - bc - ad = 0, \\ 3c^2 - 3a^2 + b^2 - d^2 = 0. \end{array}\right\} \tag{40}$$

From first equation of 40 it is easily seen that $a^2 - c^2 = -(bc + ad)$. If it replaces in Gauss equation,from (34), (35) and (36) we get the following system;

$$\begin{aligned} \frac{a_u}{p} - \frac{d_u}{p} &= -ab - ac, \\ \frac{b_u}{p} + \frac{c_u}{p} &= -ad + a^2, \\ -\frac{a_v}{q} + \frac{d_v}{q} &= bc + c^2, \\ \frac{b_v}{q} + \frac{c_v}{q} &= ac - cd, \\ \frac{a_v}{q} - \frac{c_u}{p} &= bc + ad, \\ \frac{a_u}{p} + \frac{b_v}{q} &= -ab - ac, \\ \frac{c_v}{q} - \frac{d_u}{p} &= cd - ac. \end{aligned}$$

Now, as in previous case, if we put

$$\begin{array}{llll} \frac{a_u}{p} = x, & \frac{b_u}{p} = y, & \frac{c_u}{p} = z, & \frac{d_u}{p} = w, \\ \frac{a_v}{q} = \bar{x}, & \frac{b_v}{q} = \bar{y}, & \frac{c_v}{q} = \bar{z}, & \frac{d_u}{p} = \bar{w}, \end{array}$$

then we obtain the following linear equation system:

$$\left[\begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & : & -ab - ac \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & : & a^2 - ad \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & : & bc + c^2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & : & ac - cd \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & : & bc + ad \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & : & \text{-}ab - ac \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & cd - ac \end{array}\right]$$

After some elementary operations, this system turns to the following system,

$$\left[\begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & : & \text{-}ab - ac \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & : & \text{-}bc - c^2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & : & a^2 - ad \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & : & ac - cd \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & : & \text{-}2bc - c^2 - ad \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & cd - ac \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & : & ac - cd \end{array}\right]$$

This system is compatible if and only if $d - a = 0$ or $c = 0$. Thus we see that this case is same with previous case.

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