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On the Hermite-Hadamard Inequalities for $h-$Convex Functions on Balls and Ellipsoids

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Abstract. In this paper, we establish some Hermite-Hadamard type inequalities for $h-$ convex function on high-dimensional balls and ellipsoids, which extend some known results. Some mappings connected with these inequalities and related results are also obtained.

1. Introduction

The concept of $h-$convexity was first introduced by Varošanec [16] in 2007, and then has been studied extensively by many mathematicians, see e.g. [2, 9, 10, 13] and the references therein.

Definition 1. *Let $h : [0, 1] \to [0, \infty)$ be a given function. We say that $f : \mathcal{D} \to \mathbb{R}$, where $\mathcal{D}$ is a convex subset of $\mathbb{R}^n$, is $h-$convex if for any $X, Y \in \mathcal{D}$ and $\alpha \in [0, 1]$,*

$$f(\alpha X + (1 - \alpha) Y) \le h(\alpha) f(X) + h(1 - \alpha) f(Y). \tag{1}$$

This notion unifies and generalizes the known classes of the usual convex functions, $s-$convex functions (in the second sense) [3], $P-$functions [14] and Godunova-Levin functions [8], which are obtained by putting in (1)

$$h(\alpha) = \alpha,\ h(\alpha) = \alpha^s\ (0 < s \le 1),\ h(\alpha) = 1,$$

and

$$h(\alpha) = \begin{cases} 1/\alpha, & 0 < \alpha \le 1, \\ 0, & \alpha = 0, \end{cases}$$

respectively.

Convexity and its generalizations are very important both in pure mathematics and in applications. One of the significant application involved in convex type functions is the following well-known Hermite-Hadamard inequality.

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Theorem A. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

In 1999, Dragomir and Fitzpatrick [7] proved the variant of Hermite-Hadamard's inequality which holds for $s-$convex functions in the second sense.

Theorem B. [7] *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a nonnegative $s-$convex function in the second sense with $0 < s < 1$. Then*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}.$$

In 2008, Sarikaya, Saglam and Yildririm obtained the following analogue inequalities for $h-$convex functions.

Theorem C. [15] *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be an $h-$convex function on $[a,b]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le [f(a)+f(b)]\int_0^1 h(x)dx.$$

At the meantime, there is an extensive literature devoted to develop Hermite-Hadamard's type inequalities to higher-dimensions. For example, some inequalities for convex type functions on rectangles can be found in [1, 6, 11], and on disks can be found in [4, 5]. In this paper, we mainly deal with analogue inequalities for h-convex functions on balls and ellipsoids. Compared to the methods employed on rectangles, which used on balls (ellipsoids) are rather technical.

In the sequel, unless otherwise specified, $\mathbb{R}^n$ denotes the Euclidean space of dimension n and $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$, $d\sigma(x)$ is the usual surface measure ($n \ge 3$) or the arc length ($n = 2$) in general. $B_n(C,r)$ and $\delta_n(C,r)$ are the $n-$dimensional ball and its sphere respectively centered at the point $C = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ with radius $r > 0$. $E_n(C,R)$ denotes the $n-$dimensional ellipsoid centered at the point $C = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ with semiaxises $R = (r_1, r_2, \cdots, r_n)$, that is

$$\frac{(x_1-c_1)^2}{r_1^2} + \frac{(x_2-c_2)^2}{r_2^2} + \cdots + \frac{(x_n-c_n)^2}{r_n^2} \le 1, \quad 0 < r_1, r_2, \ldots, r_n < \infty,$$

and $S_n(C,R)$ is the sphere of $E_n(C,R)$. It is well known that

$$|B_n(C,r)| = \frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)}, \quad |\delta_n(C,r)| = \frac{n\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(\frac{n}{2}+1)}, \tag{2}$$

$$|E_n(C,R)| = \frac{\pi^{\frac{n}{2}} r_1 \cdots r_n}{\Gamma(\frac{n}{2}+1)}, \quad |S_n(C,tR)| = t^{n-1}|S_n(C,R)|, \quad t > 0, \tag{3}$$

where $\Gamma(\cdot)$ denotes the Gamma function and $tR = (tr_1, tr_2, \cdots, tr_n)$.

Throughout the paper, we also assume that the function h in Definition 1 is always Lebesgue integrable on the interval $[0,1]$ and satisfies $h\left(\frac{1}{2}\right) > 0$.

Now we recall some known results. In 2000, Dragomir [4] proved the Hermite-Hadamard type inequality of convex functions on the disk in $\mathbb{R}^2$.

Theorem D. [4] *Let $f : B_2(C,r) \to \mathbb{R}$ be a convex function on the disk $B_2(C,r)$. Then*

$$f(C) \le \frac{1}{\pi r^2}\int_{B_2(C,r)} f(X)dX \le \frac{1}{2\pi r}\int_{\delta_2(C,r)} f(X)d\sigma(X).$$

Furthermore, Dragomir extended the proceeding result from the disk in $\mathbb{R}^2$ to the ball in $\mathbb{R}^3$ in the same year and obtained the following similar result.

Theorem E. [5] *Let $f : B_3(C, r) \to \mathbb{R}$ be a convex function on $B_3(C, r)$. Then*

$$f(C) \le \frac{1}{|B_3(C, r)|} \int_{B_3(C,r)} f(X)dX \le \frac{1}{|\delta_3(C, r)|} \int_{\delta_3(C,r)} f(X)d\sigma(X).$$

In 2014, Matłoka [12] generalized Theorem D for $h-$convex functions on disks and established the corresponding Hermite-Hadamard inequality.

Theorem F. [12] *Let $f : B_2(C, r) \to \mathbb{R}$ be an $h-$convex function on $B_2(C, r)$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f(C) \le \frac{1}{\pi r^2} \int_{B_2(C,r)} f(X)dX \le \mathcal{F}(r) \frac{1}{2\pi r} \int_{\delta_2(C,r)} f(X)d\sigma(X), \tag{4}$$

where

$$\mathcal{F}(r) = \frac{2}{r^2} \int_0^r th\left(\frac{t}{r}\right)dt \left(1 + \frac{2\int_0^r th(1-\frac{t}{r})dt}{\frac{r^2}{2h(\frac{1}{2})} - 2\int_0^r th(1-\frac{t}{r})dt}\right).$$

As a consequence of Theorem F, the author obtained the variant Hermite-Hadamard inequality for s-convex functions.

Theorem G. [12] *Let $f : B_2(C, r) \to \mathbb{R}$ be an $s-$convex function in the second sense on $B_2(C, r)$ with $0 < s < 1$. Then*

$$\frac{2^s}{2} f(C) \le \frac{1}{\pi r^2} \int_{B_2(C,r)} f(X)dX \le \frac{1}{\pi r} \frac{2^s(s+1)}{2^s(s+1)(s+2)-4} \int_{\delta_2(C,r)} f(X)d\sigma(X). \tag{5}$$

Remark 1. *Taking the changing of variable $\frac{t}{r} = v$ in Theorem F, we have*

$$\mathcal{F}(r) = \frac{2\int_0^1 vh(v)dv}{1 - 4h(\frac{1}{2})\int_0^1 vh(1-v)dv},$$

which implies that $\mathcal{F}(r)$ is independent of the radius r.

Remark 2. *There was a mistake in Theorem F. The condition*

$$1 - 4h\left(\frac{1}{2}\right)\int_0^1 th(1-t)dt > 0$$

is necessary for the second inequality in (4). *We will prove the assertion by contradiction. Suppose that*

$$1 - 4h\left(\frac{1}{2}\right)\int_0^1 th(1-t)dt \le 0.$$

Then $F(r) \le 0$. Choosing $f > 0$, we yield that

$$\int_{B_2(C,r)} f(X)dX > 0, \quad \int_{\delta_2(C,r)} f(X)d\sigma(X) > 0,$$

which is a contradiction with $F(r) \le 0$.

According to proceeding argument, the second inequality in (5) *of Theorem G is valid under the additional assumption of*

$$2^s(s+1)(s+2) > 4.$$

With these motivations, one of the purposes of this paper is to establish analogues of Hermite-Hadamard inequalities for $h-$convex functions on $n-$dimensional convex bodies—–balls and ellipsoids. Now we are in a position to state our results.

Theorem 1. *Let $f: B_n(C,r) \to \mathbb{R}$ be an $h-$convex function on $B_n(C,r)$. Suppose that h satisfies*

$$1 - 2nh\left(\frac{1}{2}\right)\int_0^1 t^{n-1}h(1-t)dt > 0. \tag{6}$$

Then

$$\frac{1}{2h\left(\frac{1}{2}\right)}f(C) \le \frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)dX \le \frac{\mathcal{K}(n)}{|\delta_n(C,r)|}\int_{\delta_n(C,r)} f(X)d\sigma(X), \tag{7}$$

where

$$\mathcal{K}(n) = \frac{n\int_0^1 t^{n-1}h(t)dt}{1 - 2nh\left(\frac{1}{2}\right)\int_0^1 t^{n-1}h(1-t)dt}. \tag{8}$$

It is not difficult to see that (6) is always true if $h(t) = t$. In fact, we have

$$1 - 2nh\left(\frac{1}{2}\right)\int_0^1 t^{n-1}h(1-t)dt = 1 - n\int_0^1 (t^{n-1} - t^n)dt = \frac{n}{n+1} > 0.$$

On the other hand, a direct calculation shows that $\mathcal{K}(n) = 1$. These observations imply that

Corollary 1. *If $f: B_n(C,r) \to \mathbb{R}$ be a convex function, then*

$$f(C) \le \frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)dX \le \frac{1}{|\delta_n(C,r)|}\int_{\delta_n(C,r)} f(X)d\sigma(X).$$

Particularly, Corollary 1 reduces to Theorem D and Theorem E if $n = 2$ and $n = 3$ respectively.

If $h(t) = t^s$, $0 < s < 1$, then integration by parts tells us that

$$\begin{aligned}\int_0^1 t^{n-1}h(1-t)dt &= \int_0^1 t^{n-1}(1-t)^s dt = \int_0^1 (t-1)^{n-1}t^s dt \\ &= \frac{(n-1)!}{(s+1)(s+2)\cdots(s+n)}.\end{aligned} \tag{9}$$

Combining (9) and Theorem 1, we arrive at the Hermite-Hadamard inequality of $s-$convex functions on the ball.

Corollary 2. *Let $f: B_n(C,r) \to \mathbb{R}$ be an $s-$convex function in the second sense on $B_n(C,r)$. If $0 < s < 1$ and it satisfies*

$$2^s(s+1)(s+2)\cdots(s+n) > 2n!, \tag{10}$$

then

$$\frac{2^s}{2}f(C) \le \frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)dX \le \frac{\mathcal{K}_1}{|\delta_n(C,r)|}\int_{\delta_n(C,r)} f(X)d\sigma(X),$$

where

$$\mathcal{K}_1 = \frac{n2^s(s+1)(s+2)\cdots(s+n-1)}{2^s(s+1)(s+2)\cdots(s+n) - 2n!}. \tag{11}$$

Furthermore, we will extend the above results to more general convex sets, i.e. ellipsoids.

Theorem 2. *Let $f : E_n(C,R) \to \mathbb{R}$ be an $h-$convex function on the ellipsoid $E_n(C,R)$. Suppose that h satisfies* (6). *Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f(C) \leq \frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \leq \frac{\mathcal{K}_1(n)}{|\delta_n(0,1)|}\int_{\delta_n(0,1)} f\left(\widetilde{X}\right)d\sigma\left(X'\right), \tag{12}$$

where $\mathcal{K}(n)$ is as in Theorem 1,

$$X' = (x_1', x_2', ..., x_n') \in \delta_n(0,1), \;\; \widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, ..., \widetilde{x}_n) \;\; \textit{and} \;\; \widetilde{x}_j = r_j x_j' + c_j, \;\; j = 1,2,...,n.$$

Furthermore, if $f \geq 0$, we have

$$\frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \leq \frac{\tilde{\mathcal{F}}(R)}{|S_n(C,R)|}\int_{S_n(C,R)} f(X)d\sigma(X), \tag{13}$$

where

$$\tilde{\mathcal{F}}(R) = \frac{|S_n(C,R)|}{r^{n-1}}\frac{\Gamma(\frac{n}{2}+1)}{n\pi^{\frac{n}{2}}}\mathcal{K}(n), \tag{14}$$

and

$$r = \min\{r_1,\ r_2, \ldots,\ r_n\}.$$

It follows from Theorem 2 and the similar arguments as in Corollary 1 and Corollary 2 that

Corollary 3. *If $f : E_n(C,R) \to \mathbb{R}$ be a convex function, then*

$$f(C) \leq \frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \leq \frac{\Gamma(\frac{n}{2}+1)}{n\pi^{\frac{n}{2}}}\int_{\delta_n(0,1)} f(\widetilde{X})d\sigma(X'),$$

where $\widetilde{X}$ are as in Theorem 2.

Especially, if f is a nonnegative convex function on $E_n(C,R)$, then

$$\frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \leq \frac{\Gamma(\frac{n}{2}+1)}{n\pi^{\frac{n}{2}}r^{n-1}}\int_{S_n(C,R)} f(X)d\sigma(X).$$

If taking $h(t) = t^s$, we derive from Theorem 2 and Corollary 2 that

Corollary 4. *Let $f : E_n(C,R) \to \mathbb{R}$ be an $s-$convex function in the second sense on the ellipsoid $E_n(C,R)$ and $\mathcal{K}_1$ be the constant defined by* (11). *If $0 < s < 1$ and* (10) *holds, then*

$$\frac{2^s}{2}f(C) \leq \frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \leq \mathcal{K}_2\int_{\delta_n(0,1)} f\left(\widetilde{X}\right)d\sigma\left(X'\right),$$

where $\widetilde{X}$ are as in Theorem 2 and

$$\mathcal{K}_2 = \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}}\frac{2^s n(s+1)(s+2)\cdots(s+n-1)}{2^s(s+1)(s+2)\cdots(s+n)-2n!} = \frac{\Gamma(\frac{n}{2}+1)}{n\pi^{\frac{n}{2}}}\mathcal{K}_1.$$

Furthermore, if $f \geq 0$, we have

$$\frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \leq \frac{\tilde{\mathbf{F}}(R)}{|S_n(C,R)|}\int_{S_n(C,R)} f(X)d\sigma(X),$$

where

$$\tilde{\mathbf{F}}(R) = \frac{\Gamma(\frac{n}{2}+1)|S_n(C,R)|}{n\pi^{\frac{n}{2}}r^{n-1}}\frac{2^s n(s+1)(s+2)\cdots(s+n-1)}{2^s(s+1)(s+2)\cdots(s+n)-2n!} = \frac{|S_n(C,R)|}{r^{n-1}}\frac{\Gamma(\frac{n}{2}+1)}{n\pi^{\frac{n}{2}}}\mathcal{K}_1.$$

The second purpose in this paper is to provide some applications of the Hermite-Hadamard inequalities for $h-$convex functions. In [4] and [5], Dragomir studied some properties of the mappings connected to the Hermite-Hadamard type inequality of convex function on disks and balls . In [12], Matłoka considered the similar mappings connected to the $h-$convex function on disks.

Theorem H. [12] *Define the mapping $\mathfrak{H} : [0,1] \to \mathbb{R}$ by*

$$\mathfrak{H}(t) = \frac{1}{\pi r^2}\int_{B_2(C,r)} f(tX + (1-t)C)dX.$$

If f is an $h-$convex function on the disk $B_2(C,r)$, then
(i) *the function $\mathfrak{H}$ is an $h-$convex function on* $[0,1]$,
(ii) *for any $t \in (0,1]$,*

$$\frac{f(C)}{2h\left(\frac{1}{2}\right)} \leq \mathfrak{H}(t) \leq \mathfrak{H}(1)\left[h(t) + 2h\left(\frac{1}{2}\right)h(1-t)\right].$$

Theorem I. [12] *Define the mapping $\mathfrak{G} : [0,1] \to \mathbb{R}$ by*

$$\mathfrak{G}(t) = \begin{cases} \dfrac{1}{2\pi tr}\displaystyle\int_{\delta_2(C,tr)} f(X)d\sigma(X), & t \in (0,1], \\ f(C), & t = 0. \end{cases}$$

If f is an $h-$convex function on the disk $B_2(C,r)$, then
(i) *the function $\mathfrak{G}$ is an $h-$convex function on* $[0,1]$,
(ii) *for any $t \in (0,1]$, $\mathfrak{H}(t) \leq \mathcal{F}(tr)\mathfrak{G}(t)$,*
(iii) *for any $t \in (0,1]$,*

$$\frac{f(C)}{2h\left(\frac{1}{2}\right)\mathcal{F}(tr)} \leq \mathfrak{G}(t) \leq \mathfrak{G}(1)\left[h(t) + 2h\left(\frac{1}{2}\right)h(1-t)\mathcal{F}(r)\right].$$

Remark 3. *According to Remark 1 and using the notation in* (8)*, we can rewrite* (ii) *and* (iii) *in Theorem I as the following explicit forms, respectively,*
(ii') *for any $t \in (0,1]$, $\mathfrak{H}(t) \leq \mathcal{K}(2)\mathfrak{G}(t)$,*
(iii') *for any $t \in (0,1]$,*

$$\frac{f(C)}{2h\left(\frac{1}{2}\right)\mathcal{K}(2)} \leq \mathfrak{G}(t) \leq \mathfrak{G}(1)\left[h(t) + 2h\left(\frac{1}{2}\right)h(1-t)\mathcal{K}(2)\right].$$

Remark 4. *There was a mistake in Theorem I. By checking the proof of Theorem I in [12] and the statement of Remark 2, the condition*

$$1 - 4h\left(\frac{1}{2}\right)\int_0^1 th(1-t)dt > 0$$

is necessary for (ii) *and* (iii) *in Theorem I.*

Now, we will prove some properties of these two mappings assuming that the function f is $h-$convex on ellipsoids. Correspondingly, the associated properties of balls are also obtained.

Theorem 3. *Define the mapping* $\tilde{\mathfrak{H}} : [0,1] \to \mathbb{R}$ *by*

$$\tilde{\mathfrak{H}}(t) = \frac{1}{|E_n(C,R)|} \int_{E_n(C,R)} f(tX + (1-t)C)dX.$$

If f is an $h-$convex function on the ellipsoid $E_n(C,R)$, then
(i) *the function $\tilde{\mathfrak{H}}$ is an $h-$convex function on* $[0,1]$,
(ii) *for any* $t \in (0,1]$,

$$\frac{f(C)}{2h\left(\frac{1}{2}\right)} \le \tilde{\mathfrak{H}}(t) \le \tilde{\mathfrak{H}}(1)\left[h(t) + 2h\left(\frac{1}{2}\right)h(1-t)\right]. \tag{15}$$

As a consequence of the proceeding theorem, we have the following results.

Corollary 5. *Define the mapping* $\tilde{\mathbf{H}} : [0,1] \to \mathbb{R}$ *by*

$$\tilde{\mathbf{H}}(t) = \frac{1}{|B_n(C,r)|} \int_{B_n(C,r)} f(tX + (1-t)C)dX.$$

If f is an $h-$convex function on the ball $B_n(C,r)$, then the mapping $\tilde{\mathbf{H}}$ enjoys the same properties as $\tilde{\mathfrak{H}}$ in Theorem 3.

If we choose $n = 2$ in Corollary 5, then it reduces to Theorem H.

Theorem 4. *Define the mapping* $\tilde{\mathbf{G}} : [0,1] \to \mathbb{R}$ *by*

$$\tilde{\mathbf{G}}(t) = \begin{cases} \dfrac{1}{|\delta_n(C,tr)|} \displaystyle\int_{\delta_n(C,tr)} f(X)d\sigma(X), & t \in (0,1], \\ f(C), & t = 0. \end{cases}$$

If f is an $h-$convex function on the ball $B_n(C,r)$ and (6) *holds, then*
(i) *the function $\tilde{\mathbf{G}}(t)$ is an $h-$convex function on* $[0,1]$,
(ii) *for any $t \in (0,1]$, $\tilde{\mathbf{H}}(t) \le \mathcal{K}(n)\tilde{\mathbf{G}}(t)$,*
(iii) *for any* $t \in (0,1]$,

$$\frac{f(C)}{2h\left(\frac{1}{2}\right)\mathcal{K}(n)} \le \tilde{\mathbf{G}}(t) \le \tilde{\mathbf{G}}(1)\left[h(t) + 2h\left(\frac{1}{2}\right)h(1-t)\mathcal{K}(n)\right]. \tag{16}$$

By virtue of Remark 3 and Remark 4, it is obviously that Theorem 4 generalizes Theorem I.

Theorem 5. *Define the mapping* $\tilde{\mathfrak{G}} : [0,1] \to \mathbb{R}$ *by*

$$\tilde{\mathfrak{G}}(t) = \begin{cases} \dfrac{1}{|S_n(C,tR)|} \displaystyle\int_{S_n(C,tR)} f(X)d\sigma(X), & t \in (0,1], \\ f(C), & t = 0. \end{cases}$$

If f is an $h-$convex function on the ellipsoid $E_n(C,R)$ and (6) *holds, then*
(i) *the function $\tilde{\mathfrak{G}}(t)$ is an $h-$convex function on* $[0,1]$,
(ii) *when $f \ge 0$, for any $t \in (0,1]$, $\tilde{\mathfrak{H}}(t) \le \tilde{\mathcal{F}}(R)\tilde{\mathfrak{G}}(t)$,*
(iii) *when $f \ge 0$, for any* $t \in (0,1]$,

$$\frac{f(C)}{2h\left(\frac{1}{2}\right)\tilde{\mathcal{F}}(R)} \le \tilde{\mathfrak{G}}(t) \le \tilde{\mathfrak{G}}(1)\left[h(t) + 2h\left(\frac{1}{2}\right)h(1-t)\tilde{\mathcal{F}}(R)\right], \tag{17}$$

where $\tilde{\mathcal{F}}(R)$ is defined by (14), *i.e.*

$$\tilde{\mathcal{F}}(R) = \frac{|S_n(C,R)|}{r^{n-1}} \frac{\Gamma(\frac{n}{2}+1)}{n\pi^{\frac{n}{2}}}\mathcal{K}(n) \ \ \textit{and} \ \ r = \min\{r_1, r_2, \ldots, r_n\}.$$

2. Proof of The Theorems

2.1. Proof of Theorem 1

(i) A changing of variables yields that

$$\int_{B_n(C,r)} f(X)dX = \int_{B_n(C,r)} f(2C-X)dX.$$

Since $f(C) = f\left(\frac{X}{2}+\frac{2C-X}{2}\right)$,then

$$\begin{aligned} f(C) &= \frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f\left(\frac{X}{2}+\frac{2C-X}{2}\right)dX \\ &\le \frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} \left[h\left(\frac{1}{2}\right)f(X)+h\left(\frac{1}{2}\right)f(2C-X)\right]dX \\ &= \frac{2h\left(\frac{1}{2}\right)}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)dX. \end{aligned}$$

In this way we obtain the first part of (7).

(ii) The translation invariance of Lebesgue measure shows that

$$\frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(C)dX = \frac{1}{|B_n(0,r)|}\int_{B_n(0,r)} f(X+C)dX. \tag{18}$$

Taking the spherical change of the unit sphere $\delta_n(0,1)$

$$X' = \begin{cases} x_1' = \cos\varphi_1, \\ x_2' = \sin\varphi_1\cos\varphi_2, \\ x_3' = \sin\varphi_1\sin\varphi_2\cos\varphi_3, \\ \vdots \\ x_{n-1}' = \sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\cos\varphi_{n-1}, \\ x_n' = \sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\sin\varphi_{n-1}, \end{cases} \tag{19}$$

where

$$0 \le \varphi_1, \ldots, \varphi_{n-2} \le \pi,\ 0 \le \varphi_{n-1} \le 2\pi,$$

we have

$$\begin{aligned} &\int_{B_n(0,r)} f(X+C)dX \\ =\ &\int_0^r\int_{\delta_n(0,1)} f(tX'+C)\,t^{n-1}d\sigma(X')dt \\ =\ &\int_0^r\int_{\delta_n(0,1)} f\left(\frac{t}{r}(rX'+C)+\left(1-\frac{t}{r}\right)C\right)t^{n-1}d\sigma(X')dt \\ \le\ &\int_0^r\int_{\delta_n(0,1)} \left[h\left(\frac{t}{r}\right)f(rX'+C)+h\left(1-\frac{t}{r}\right)f(C)\right]t^{n-1}d\sigma(X')dt \\ =\ &\left(\int_0^r t^{n-1}h\left(\frac{t}{r}\right)dt\right)\left(\int_{\delta_n(0,1)} f(rX'+C)\,d\sigma(X')\right) \\ &+f(C)|\delta_n(0,1)|\int_0^r h\left(1-\frac{t}{r}\right)t^{n-1}dt. \end{aligned} \tag{20}$$

On the other hand, the change of variable formula tells us that

$$\frac{1}{r^n}\int_0^r t^{n-1}h\left(\frac{t}{r}\right)dt = \int_0^1 t^{n-1}h(t)dt, \tag{21}$$

$$\frac{1}{r^n}\int_0^r t^{n-1}h\left(1-\frac{t}{r}\right)dt = \int_0^1 t^{n-1}h(1-t)dt, \tag{22}$$

and

$$\int_{\delta_n(0,1)} f\left(rX'+C\right)d\sigma\left(X'\right) = \frac{1}{r^{n-1}}\int_{\delta_n(C,r)} f(X)d\sigma(X). \tag{23}$$

Then, by (2), (18)-(23) and the first inequality in (7),

$$\begin{aligned}
&\frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)dX \\
\leq\ & n\int_0^1 t^{n-1}h(t)dt\frac{1}{|\delta_n(C,r)|}\int_{\delta_n(C,r)} f(X)d\sigma(X) + n\int_0^1 t^{n-1}h(1-t)dt f(C) \\
\leq\ & n\int_0^1 t^{n-1}h(t)dt\frac{1}{|\delta_n(C,r)|}\int_{\delta_n(C,r)} f(X)d\sigma(X) \\
&+2n\int_0^1 t^{n-1}h(1-t)dt h\left(\frac{1}{2}\right)\frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)dX.
\end{aligned}$$

Recalling

$$1-2nh\left(\frac{1}{2}\right)\int_0^1 t^{n-1}h(1-t)dt > 0,$$

we have

$$\frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)dX \leq \frac{n\int_0^1 t^{n-1}h(t)dt}{1-2nh(\frac{1}{2})\int_0^1 t^{n-1}h(1-t)dt}\frac{1}{|\delta_n(C,r)|}\int_{\delta_n(C,r)} f(X)d\sigma(X),$$

which completes the proof. □

2.2. *Proof of Theorem 2*

Since the proof for the left part of (12) follows the same procedure as in Theorem 1 (i), we omit the details. Now, we will focus on proving the right part of (12). Let $X' = \left(x'_1, x'_2, \ldots, x'_n\right)$ be the spherical transformation of the unit sphere $\delta_n(0,1)$ defined by (19). Suppose that

$$\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n) \text{ and } \widetilde{x}_j = r_j x'_j + c_j,\ j = 1, 2, \ldots, n.$$

That is

$$\widetilde{X} = \begin{cases}
\widetilde{x}_1 = r_1\cos\varphi_1 + c_1, \\
\widetilde{x}_2 = r_2\sin\varphi_1\cos\varphi_2 + c_2, \\
\widetilde{x}_3 = r_3\sin\varphi_1\sin\varphi_2\cos\varphi_3 + c_3, \\
\vdots \\
\widetilde{x}_{n-1} = r_{n-1}\sin\varphi_1\sin\varphi_2\sin\varphi_3\cdots\sin\varphi_{n-2}\cos\varphi_{n-1} + c_{n-1}, \\
\widetilde{x}_n = r_n\sin\varphi_1\sin\varphi_2\sin\varphi_3\cdots\sin\varphi_{n-2}\sin\varphi_{n-1} + c_n,
\end{cases}$$

where

$$0 \le \varphi_1 \ \ldots, \varphi_{n-2} \le \pi, \ 0 \le \varphi_{n-1} \le 2\pi.$$

Thus, for any $X = (x_1, x_2, \ldots, x_n) \in E_n(C, R)$, there is $0 \le t \le 1$ such that

$$X = t\widetilde{X} + (1-t)C := X(t, \varphi_1, \ldots, \varphi_{n-1}).$$

It is not difficult to check that the Jacobian of the transformation X is

$$\begin{aligned} J_1(t, \varphi_1, \ldots, \varphi_{n-1}) &:= \left|\frac{\partial (x_1, x_2, \ldots, x_n)}{\partial (t, \varphi_1, \ldots, \varphi_{n-1})}\right| = \left|\det\begin{pmatrix} \frac{\partial x_1}{\partial t} & \frac{\partial x_2}{\partial t} & \cdots & \frac{\partial x_n}{\partial t} \\ \frac{\partial x_1}{\partial \varphi_1} & \frac{\partial x_2}{\partial \varphi_1} & \cdots & \frac{\partial x_n}{\partial \varphi_1} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_1}{\partial \varphi_{n-1}} & \frac{\partial x_2}{\partial \varphi_{n-1}} & \cdots & \frac{\partial x_n}{\partial \varphi_{n-1}} \end{pmatrix}\right| \\ &= r_1 r_2 \cdots r_n t^{n-1} (\sin\varphi_1)^{n-2} \cdots (\sin\varphi_{n-3})^2 (\sin\varphi_{n-2}). \end{aligned} \tag{24}$$

We infer from (24) that

$$\begin{aligned} &\int_{E_n(C,R)} f(X)dX \\ = \ & \int_0^1 \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f\left(t\widetilde{X} + (1-t)C\right) J_1(t, \varphi_1, \ldots, \varphi_{n-1}) d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1 dt \\ \le \ & r_1 r_2 \cdots r_n \int_0^1 t^{n-1} h(t) dt \\ & \times \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f\left(\widetilde{X}\right) (\sin\varphi_1)^{n-2} \cdots (\sin\varphi_{n-2}) d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1 \\ & + r_1 r_2 \cdots r_n f(C) \int_0^1 t^{n-1} h(1-t) dt \\ & \times \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} (\sin\varphi_1)^{n-2} \cdots (\sin\varphi_{n-2}) d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1 \\ = \ & r_1 r_2 \cdots r_n \int_0^1 t^{n-1} h(t) dt \int_{\delta_n(0,1)} f\left(\widetilde{X}\right) d\sigma(X') \\ & + r_1 r_2 \cdots r_n f(C) |\delta_n(0,1)| \int_0^1 t^{n-1} h(1-t) dt. \end{aligned}$$

With the aid of (2), (3) and the inequality

$$f(C) \le \frac{2h(\frac{1}{2})}{|E_n(C,R)|} \int_{E_n(C,R)} f(X)dX,$$

we deduce that

$$\begin{aligned} \int_{E_n(C,R)} f(X)dX \ \le \ & r_1 r_2 \cdots r_n \int_0^1 t^{n-1} h(t) dt \int_{\delta_n(0,1)} f\left(\widetilde{X}\right) d\sigma(X') \\ & + 2nh\left(\frac{1}{2}\right) \int_0^1 t^{n-1} h(1-t) dt \int_{E_n(C,R)} f(X)dX. \end{aligned}$$

Then, by (6),

$$\frac{1}{|E_n(C,R)|} \int_{E_n(C,R)} f(X)dX \le \frac{\mathcal{K}(n)}{|\delta_n(0,1)|} \int_{\delta_n(0,1)} f\left(\widetilde{X}\right) d\sigma(X').$$

This proves the right part of (12).

Now we turn to prove inequality (13). Let

$$\begin{aligned} A_i(\varphi_1,\varphi_2,\ldots,\varphi_{n-1}) &:= \left|\frac{\partial\left(\widetilde{x_1},\ldots,\widetilde{x_{i-1}},\widetilde{x_{i+1}},\ldots,\widetilde{x_n}\right)}{\partial\left(\varphi_1,\varphi_2,\ldots,\varphi_{n-1}\right)}\right| \\ &= \left|\det\begin{pmatrix} \frac{\partial\widetilde{x_1}}{\partial\varphi_1} & \cdots & \frac{\partial\widetilde{x_{i-1}}}{\partial\varphi_1} & \frac{\partial\widetilde{x_{i+1}}}{\partial\varphi_1} & \cdots & \frac{\partial\widetilde{x_n}}{\partial\varphi_1} \\ \frac{\partial\widetilde{x_1}}{\partial\varphi_2} & \cdots & \frac{\partial\widetilde{x_{i-1}}}{\partial\varphi_2} & \frac{\partial\widetilde{x_{i+1}}}{\partial\varphi_2} & \cdots & \frac{\partial\widetilde{x_n}}{\partial\varphi_2} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial\widetilde{x_1}}{\partial\varphi_{n-1}} & \cdots & \frac{\partial\widetilde{x_{i-1}}}{\partial\varphi_{n-1}} & \frac{\partial\widetilde{x_{i+1}}}{\partial\varphi_{n-1}} & \cdots & \frac{\partial\widetilde{x_n}}{\partial\varphi_{n-1}} \end{pmatrix}\right|. \end{aligned} \tag{25}$$

Comparing with the Jacobian of transformation of the unit sphere, we easily see that

$$\begin{aligned} J_2(\varphi_1,\ldots,\varphi_{n-1}) &:= \sqrt{\sum_{i=1}^{n} A_i^2(\varphi_1,\varphi_2,\ldots,\varphi_{n-1})} \\ &\geq r^{n-1}(\sin\varphi_1)^{n-2}\cdots(\sin\varphi_{n-3})^2(\sin\varphi_{n-2}), \end{aligned} \tag{26}$$

where $r = \min\{r_1, r_2, \ldots, r_n\}$. Since $f \geq 0$,

$$\begin{aligned} &\int_{S_n(C,R)} f(X)d\sigma(X) \\ = &\int_{\varphi_1=0}^{\pi}\cdots\int_{\varphi_{n-2}=0}^{\pi}\int_{\varphi_{n-1}=0}^{2\pi} f\left(\widetilde{X}\right) J_2(\varphi_1,\ldots,\varphi_{n-1})d\varphi_{n-1}d\varphi_{n-2}\cdots d\varphi_1 \\ \geq\ & r^{n-1}\int_{\varphi_1=0}^{\pi}\cdots\int_{\varphi_{n-2}=0}^{\pi}\int_{\varphi_{n-1}=0}^{2\pi} f\left(\widetilde{X}\right)(\sin\varphi_1)^{n-2}\cdots(\sin\varphi_{n-3})^2(\sin\varphi_{n-2})d\varphi_{n-1}d\varphi_{n-2}\cdots d\varphi_1, \end{aligned}$$

which yields that

$$\int_{\delta_n(0,1)} f\left(\widetilde{X}\right)d\sigma\left(X'\right) \leq \frac{1}{r^{n-1}}\int_{S_n(C,R)} f(X)d\sigma(X). \tag{27}$$

By combing (12) and (27) we finish the proof of Theorem 2. □

2.3. *Proof of Theorem 3*

(i) Let $t_1, t_2 \in [0,1]$, and $\alpha, \beta \geq 0, \alpha+\beta = 1$. Then

$$\begin{aligned} &\tilde{\mathfrak{H}}(\alpha t_1+\beta t_2) \\ = &\frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f\left(\alpha\left[t_1X+(1-t_1)C\right]+\beta\left[t_2X+(1-t_2)C\right]\right)dX \\ \leq &\frac{h(\alpha)}{|E_n(C,R)|}\int_{E_n(C,R)} f(t_1X+(1-t_1)C)dX + \frac{h(\beta)}{|E_n(C,R)|}\int_{E_n(C,R)} f(t_2X+(1-t_2)C)dX \\ = &h(\alpha)\tilde{\mathfrak{H}}(t_1)+h(\beta)\tilde{\mathfrak{H}}(t_2), \end{aligned}$$

which means that $\tilde{\mathfrak{H}}$ is an $h-$convex function on $[0,1]$.

(ii) For any fixed $t \in (0,1]$, taking the substitution $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$, where $\eta_i = tx_i + (1-t)c_i$, we have

$$\tilde{\mathfrak{H}}(t) = \frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(tX+(1-t)C)dX$$

$$\begin{aligned} &= \frac{1}{|E_n(C,R)|}\int_{E_n(C,tR)} f(\eta)\left|\frac{\partial(x_1,x_2,\ldots,x_n)}{\partial(\eta_1,\eta_2,\ldots,\eta_n}\right| d\eta \\ &= \frac{1}{t^n|E_n(C,R)|}\int_{E_n(C,tR)} f(\eta)d\eta \\ &= \frac{1}{|E_n(C,tR)|}\int_{E_n(C,tR)} f(X)dX. \end{aligned} \tag{28}$$

Then Theorem 2 gives us that

$$\frac{1}{2h(\frac{1}{2})}f(C) \le \tilde{\mathfrak{H}}(t).$$

In this way the first part of the inequality (15) is proved.

By the $h-$convexity of f on the ellipsoid and the left-side of (12), we have

$$\begin{aligned} \tilde{\mathfrak{H}}(t) &\le \frac{h(t)}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX + h(1-t)f(C) \\ &\le \frac{h(t)}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX + \frac{2h(1-t)h\left(\frac{1}{2}\right)}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX \\ &= \left[h(t)+2h\left(\frac{1}{2}\right)h(1-t)\right]\frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX. \end{aligned}$$

And the definition of $\tilde{\mathfrak{H}}$ implies that

$$\tilde{\mathfrak{H}}(1) = \frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)dX.$$

Therefore,

$$\tilde{\mathfrak{H}}(t) \le \tilde{\mathfrak{H}}(1)\left[h(t)+2h\left(\frac{1}{2}\right)h(1-t)\right],$$

which completes the proof. □

2.4. *Proof of Theorem 4*

Due to (2) and the spherical transformation given by (19), we can deduce that

$$\widetilde{\mathbf{G}}(t) = \frac{1}{|\delta_n(0,1)|}\int_{\delta_n(0,1)} f\left(trX'+C\right)d\sigma\left(X'\right). \tag{29}$$

(i) Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \ge 0,\ \alpha+\beta = 1$. Then, by (29),

$$\begin{aligned} &\widetilde{\mathbf{G}}(\alpha t_1+\beta t_2) \\ &= \frac{1}{|\delta_n(0,1)|}\int_{\delta_n(0,1)} f\left(\alpha\left(t_1 rX'+C\right)+\beta\left(t_2 rX'+C\right)\right)d\sigma\left(X'\right) \\ &\le \frac{1}{|\delta_n(0,1)|}\int_{\delta_n(0,1)} \left[h(\alpha)f\left(\left(t_1 rX'+C\right)+h(\beta)f\left(t_2 rX'+C\right)\right)\right]d\sigma\left(X'\right) \\ &= h(\alpha)\widetilde{\mathbf{G}}(t_1)+h(\beta)\widetilde{\mathbf{G}}(t_2). \end{aligned}$$

This means that $\widetilde{\mathbf{G}}$ is $h-$convex on $[0,1]$.

(ii) As a special case of (28), we easily to see that

$$\widetilde{\mathbf{H}}(t)=\frac{1}{|B_n(C,tr)|}\int_{B_n(C,tr)} f(X)dX. \tag{30}$$

Thus, according to Theorem 1,

$$\widetilde{\mathbf{H}}(t)\le\frac{\mathcal{K}(n)}{|\delta_n(C,tr)|}\int_{\delta_n(C,tr)} f(X)d\sigma(X)=\mathcal{K}(n)\widetilde{\mathbf{G}}(t) \tag{31}$$

holds for all $t\in(0,1]$.

(iii) With the aid of (30), (31) and the left part of (7), we can arrive at

$$\frac{f(C)}{2h\left(\frac{1}{2}\right)}\le\widetilde{\mathbf{H}}(t)\le\mathcal{K}(n)\widetilde{\mathbf{G}}(t) \tag{32}$$

for all $t\in(0,1]$. Especially,

$$f(C)\le 2h\left(\frac{1}{2}\right)\mathcal{K}(n)\widetilde{\mathbf{G}}(1). \tag{33}$$

On the other hand, (29) provides us that

$$\begin{aligned}\widetilde{\mathbf{G}}(t) &= \frac{1}{|\delta_n(0,1)|}\int_{\delta_n(0,1)} f\left(t\left(rX'+C\right)+(1-t)C\right)d\sigma\left(X'\right)\\ &\le \frac{1}{|\delta_n(0,1)|}\int_{\delta_n(0,1)} \left[h(t)f((rX'+C)+h(1-t)f(C)\right]d\sigma\left(X'\right)\\ &= h(t)\widetilde{\mathbf{G}}(1)+h(1-t)f(C)\\ &\le \widetilde{\mathbf{G}}(1)\left[h(t)+2h\left(\frac{1}{2}\right)h(1-t)\mathcal{K}(n)\right],\end{aligned} \tag{34}$$

where the last inequality is obtained by (33). By combining (32) and (34) we finish the proof. □

2.5. *Proof of Theorem 5*

For any fixed $t\in(0,1]$, we know that the surface of the ellipsoid can be presented as follows, $X=(x_1,x_2,\ldots,x_n)\in S_n(C,tR)$,

$$X(t,\varphi_1,\varphi_2,\ldots,\varphi_{n-1}):=\begin{cases}x_1=tr_1\cos\varphi_1+c_1,\\ x_2=tr_2\sin\varphi_1\cos\varphi_2+c_2,\\ x_3=tr_3\sin\varphi_1\sin\varphi_2\cos\varphi_3+c_3,\\ \vdots\\ x_{n-1}=tr_{n-1}\sin\varphi_1\sin\varphi_2\sin\varphi_3\cdots\sin\varphi_{n-2}\cos\varphi_{n-1}+c_{n-1},\\ x_n=tr_n\sin\varphi_1\sin\varphi_2\sin\varphi_3\cdots\sin\varphi_{n-2}\sin\varphi_{n-1}+c_n,\end{cases} \tag{35}$$

where $0\le\varphi_1,\cdots,\varphi_{n-2}\le\pi, 0\le\varphi_{n-1}\le 2\pi$. Let

$$B_i(t,\varphi_1,\varphi_2,\ldots,\varphi_{n-1}) := \left|\frac{\partial(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)}{\partial(\varphi_1,\varphi_2,\ldots,\varphi_{n-1})}\right|$$

$$= \left|\det\begin{pmatrix} \frac{\partial x_1}{\partial \varphi_1} & \cdots & \frac{\partial x_{i-1}}{\partial \varphi_1} & \frac{\partial x_{i+1}}{\partial \varphi_1} & \cdots & \frac{\partial x_n}{\partial \varphi_1} \\ \frac{\partial x_1}{\partial \varphi_2} & \cdots & \frac{\partial x_{i-1}}{\partial \varphi_2} & \frac{\partial x_{i+1}}{\partial \varphi_2} & \cdots & \frac{\partial x_n}{\partial \varphi_2} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial x_1}{\partial \varphi_{n-1}} & \cdots & \frac{\partial x_{i-1}}{\partial \varphi_{n-1}} & \frac{\partial x_{i+1}}{\partial \varphi_{n-1}} & \cdots & \frac{\partial x_n}{\partial \varphi_{n-1}} \end{pmatrix}\right|. \tag{36}$$

It is clear that

$$B_i(t, \varphi_1, \varphi_2, \ldots, \varphi_{n-1}) = t^{n-1} A_i(\varphi_1, \varphi_2, \ldots, \varphi_{n-1})$$

and

$$J_3(t, \varphi_1, \varphi_2, \ldots, \varphi_{n-1}) := \sqrt{\sum_{i=1}^{n} B_i^2(t, \varphi_1, \varphi_2, \ldots, \varphi_{n-1})} = t^{n-1} J_2(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}),$$

where $A_i(\varphi_1, \varphi_2, \ldots, \varphi_{n-1})$ and $J_2(\varphi_1, \varphi_2, \ldots, \varphi_{n-1})$ are presented by (25) and (26) respectively.

In the sequential of the paper, without confusion, we sometimes rewrite the notation $X(t, \varphi_1, \varphi_2, \ldots, \varphi_{n-1})$ by $X(t)$ and $J_2(\varphi_1, \varphi_2, \ldots, \varphi_{n-1})$ by J_2 for the sake of convenience. Therefore, for any $t \in (0, 1]$, we have

$$\begin{aligned} & \tilde{\mathfrak{G}}(t) \\ = & \frac{1}{|S_n(C, tR)|} \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(X(t)) J_3(t, \varphi_1, \varphi_2, \ldots, \varphi_{n-1}) d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1 \\ = & \frac{1}{|S_n(C, R)|} \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(X(t)) J_2(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}) d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1. \end{aligned}$$

(i) Let $t_1, t_2 \in [0, 1]$ and $\alpha, \beta \geq 0,\ \alpha + \beta = 1$. Then

$$\begin{aligned} & \tilde{\mathfrak{G}}(\alpha t_1 + \beta t_2) \\ = & \frac{1}{|S_n(C, R)|} \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(X(\alpha t_1 + \beta t_2)) J_2 d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1 \\ = & \frac{1}{|S_n(C, R)|} \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(\alpha X(t_1) + \beta X(t_2)) J_2 d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1 \\ \leq & \frac{h(\alpha)}{|S_n(C, R)|} \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(X(t_1)) J_2 d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1 \\ & + \frac{h(\beta)}{|S_n(C, R)|} \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(X(t_2)) J_2 d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1 \\ = & h(\alpha) \tilde{\mathfrak{G}}(t_1) + h(\beta) \tilde{\mathfrak{G}}(t_2). \end{aligned}$$

This concludes the proof of (i).

(ii) For any given $t \in (0, 1]$, the identity (28) tells us that

$$\tilde{\mathfrak{H}}(t) = \frac{1}{|E_n(C, tR)|} \int_{E_n(C,tR)} f(X) dX.$$

Since $f \geq 0$, by Theorem 2, we can claim that

$$\frac{1}{|E_n(C, tR)|} \int_{E_n(C,tR)} f(X) dX \leq \frac{\tilde{\mathcal{F}}(tR)}{|S_n(C, tR)|} \int_{S_n(C,tR)} f(X) d\sigma(X).$$

That is

$$\tilde{\mathfrak{H}}(t) \leq \tilde{\mathcal{F}}(tR) \tilde{\mathfrak{G}}(t), \quad t \in (0, 1],$$

where

$$\tilde{\mathcal{F}}(tR) = \frac{|S_n(C,tR)|}{(tr)^{n-1}} \frac{\Gamma(\frac{n}{2}+1)}{n\pi^{\frac{n}{2}}} \mathcal{K}(n).$$

On the other hand, by (3), we have

$$\tilde{\mathcal{F}}(tR) = \tilde{\mathcal{F}}(R).$$

This observation yields that

$$\tilde{\mathfrak{H}}(t) \leq \tilde{\mathcal{F}}(R)\tilde{\mathfrak{G}}(t)$$

for all $t \in (0,1]$. We finish the proof of (ii).

(iii) Since the inequality

$$\frac{f(C)}{2h(\frac{1}{2})} \leq \tilde{\mathfrak{H}}(t)$$

is easily reached by (15) and (ii), next we will pay more attention to proving the right part of (17).

Because of

$$\tilde{\mathfrak{G}}(t) = \frac{1}{|S_n(C,R)|} \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(X(t)) J_2 d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1$$

and the $h-$convexity of f, we have

$$\begin{aligned}
\tilde{\mathfrak{G}}(t) &= \frac{1}{|S_n(C,R)|} \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(tX(1)+(1-t)C) J_2 d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1 \\
&\leq \frac{1}{|S_n(C,R)|} \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} [h(t)f(X(1)) + h(1-t)f(C)] J_2 d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1 \\
&= \frac{h(t)}{|S_n(C,R)|} \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(X(1)) J_2 d\varphi_{n-1} d\varphi_{n-2} \cdots d\varphi_1 + h(1-t)f(C) \\
&\leq h(t)\tilde{\mathfrak{G}}(1) + 2\tilde{\mathcal{F}}(R) h\left(\frac{1}{2}\right) h(1-t)\tilde{\mathfrak{G}}(1) \\
&= \tilde{\mathfrak{G}}(1)\left[h(t) + 2\tilde{\mathcal{F}}(R) h\left(\frac{1}{2}\right) h(1-t)\right],
\end{aligned}$$

which completes the proof. □

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