Filomat 33:18 (2019), 5919–5933
https://doi.org/10.2298/FIL1918919Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Characterization of Graphs with Eigenvalue -1 of Multiplicity $n-4$ or $n-5$

Yuhong Yang[a], **Qiongxiang Huang**[a]

[a]*College of Mathematics and Systems Science, Xinjiang University, Urumqi, Xinjiang 830046, P.R. China*

Abstract. Petrović in [M. Petrović, On graphs with exactly one eigenvalue less than -1, J. Combin. Theory Ser. B 52 (1991) 102–112] determined all connected graphs with exactly one eigenvalue less than -1 and all minimal graphs with exactly two eigenvalues less than -1. By using these minimal graphs, in this paper, we determine all connected graphs having -1 as an eigenvalue with multiplicity $n-4$ or $n-5$.

1. Introduction

Throughout this paper all graphs are finite, simple and undirected. Let G be a graph. For $v \in V(G)$ and $X \subset V(G)$, let $N_G(v) = \{u \in V(G) \mid u \text{ is adjacent to } v\}$ be the neighborhood of v, $N_X(v) = N_G(v) \cap X$ be the set of neighbors of v in X and $G[X]$ be the subgraph induced by X. Conventionally, we denote the complete graph, cycle, path and complete bipartite graph by K_n, C_n, P_n and K_{n_1,n_2}, respectively.

Let G be a graph of order n with adjacency matrix $A = (a_{i,j})_{n\times n}$, where $a_{i,j} = 1$ if the vertex i is adjacent to j, written as $i \sim j$, and $a_{i,j} = 0$ otherwise. Clearly, A is real and symmetric, and so all its eigenvalues are real, which are labelled in non-increasing order as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. These eigenvalues are also called the *eigenvalues* of G. The multiplicity of λ_i is denoted by $m_G(\lambda_i)$ (or simply $m(\lambda_i)$), and the *nullity* of G is defined to be the multiplicity of 0 as an eigenvalue of G, i.e., $\eta(G) = m_G(0)$. Denoted by $p^-_{-1}(G)$ and $p^+_{-1}(G)$ the number of eigenvalues of G which are smaller and greater than -1, respectively. Thus $n = p^-_{-1}(G) + m_G(-1) + p^+_{-1}(G)$. It means that G has at most six distinct eigenvalues if $m_G(-1) \geq n-5$. The join of two graphs G and H , denoted by $G\nabla H$, is a graph obtained from G and H by joining each vertex of G to all vertices of H.

Connected graphs with few eigenvalues have aroused a lot of interests in the past several decades. One of the reason is that such graphs in general have pretty combinatorial properties and a rich structure [15]. This problem was perhaps first raised by Doob [18] in 1970. Over the past two decades, the investigations about this problem led to many results, we refer the reader to [2, 3, 7, 9, 10, 12–21, 24, 27] for details.

The graphs with $n-5 \leq \eta(G) = m_G(0) \leq n-2$ are explicitly characterized in [1, 5, 6, 8, 25, 26]. The graphs with $n-3 \leq m_G(-1) \leq n-1$ are also characterized in [4, 22]. In this paper, we also focus on the eigenvalue -1. Here, it is necessary to summarize the known results related to the eigenvalues -1.

Given an integer $i \geq 0$, let $\mathcal{G}_n([-1]^i)$ denote the set of all connected graphs on n vertices having eigenvalue -1 of multiplicity i. For $i = n-1$, we claim that $G \in \mathcal{G}_n([-1]^{n-1})$ if and only if $G \cong K_n$. Clearly, $K_n \in \mathcal{G}_n([-1]^{n-1})$.

2010 *Mathematics Subject Classification*. 05C50
Keywords. Canonical graph; Primitive graph; Eigenvalue; Multiplicity
Received: 05 February 2018; Revised: 24 December 2018; Accepted: 31 January 2019
Communicated by Francesco Belardo
Research supported by the National Natural Science Foundation of China (Grant nos. 11971274, 11671344, 11531011)
Email address: huangqx@xju.edu.cn (Qiongxiang Huang)

If $G \in \mathcal{G}_n([-1]^{n-1})$ and $G \ncong K_n$, then P_3 will be an induced subgraph of G, and so $\lambda_3(P_3) = -\sqrt{2} > \lambda_n(G) = -1$ by Interlacing Theorem, a contradiction. For $i = n-2$, according to the result of Cámara and Haemers [4], there are no graphs in $\mathcal{G}_n([-1]^{n-2})$. For $i = n-3$, by using a result of Oboudi [22] concerning the distribution of the third largest eigenvalue of graphs, we can easily deduce that $G \in \mathcal{G}_n([-1]^{n-3})$ if and only if $G \cong (K_a \cup K_b)\nabla K_{n-a-b}$ (see Lemma 2.2 below). In this paper, we continue to characterize the graphs in $\mathcal{G}_n([-1]^i)$ for large i.

Petrović in [23] characterized all connected graphs with exactly one eigenvalue less than −1, and also determined all minimal graphs with exactly two eigenvalues less than −1. By using these minimal graphs, in this paper, we explicitly characterize all graphs in $\mathcal{G}_n([-1]^{n-4})$ and $\mathcal{G}_n([-1]^{n-5})$. Concretely, for a connected graph G, we prove that $G \in \mathcal{G}_n([-1]^{n-4})$ if and only if its canonical graph (defined in next section) is isomorphic to one of $K_{1,3}$, P_4, C_4, P_5 or C_6; $G \in \mathcal{G}_n([-1]^{n-5})$ if and only if its canonical graph is isomorphic to one of H_1–H_{23} which are shown in Figure 2 and Figure 3.

2. Preliminaries

In this section, we will cite some lemmas and introduce some notions and symbols for latter use.

Lemma 2.1 (Interlacing Theorem). *Let G be a graph with n vertices and eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and H an induced subgraph of G with m vertices and eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. Then $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ where $i = 1, 2, \ldots, m$.*

Oboudi in [22] characterized the graphs with $\lambda_3 < 0$ where he gives a distribution of λ_3 in the following result.

Lemma 2.2 (Theorem 4.9, [22]). *Let G be a graph. Then $\lambda_3 \in \{-\sqrt{2}, -1, \frac{1-\sqrt{5}}{2}\} \cup (-0.59, -0.5) \cup (-0.496, \infty)$. Moreover, the following holds:*

(1) $\lambda_3 = -\sqrt{2}$ *if and only if* $G \cong P_3$.

(2) $\lambda_3 = -1$ *if and only if* $G \cong K_n$ *or* $G \cong K_s \cup K_{n-s}$ *or* $G \cong (K_a \cup K_b)\nabla K_{n-a-b}$*, where* $n, s, a, b > 0$ *are all integers and* $n > a + b$.

Let G be a graph of order n. For any $u, v \in V(G)$, we say that they have the relation ρ, denoted by $u\rho v$, if $u = v$, or $u \sim v$ and $N_G(u)\backslash v = N_G(v)\backslash u$. Clearly, ρ forms an equivalence relation on $V(G)$. Suppose that $V_1, V_2, \ldots, V_k$ are all distinct ρ-equivalence classes of $V(G)$, and $v_1, v_2, \ldots, v_k$ are the corresponding representatives, i.e. $v_i \in V_i = \{v \in V(G) \mid v\rho v_i\}$. The *canonical graph* G_c of G is defined as the graph with vertex set $\{V_1, V_2, \ldots, V_k\}$, and with an edge connecting V_i and V_j if $v_i \sim v_j$ in G. Obviously, $G_c \cong G[\{v_1, v_2, \ldots, v_k\}]$. A graph H is said to be *primitive* if $N_H(v)\backslash u \neq N_H(u)\backslash v$ whenever $u \sim v$ in H, and *imprimitive* otherwise. Obviously, the canonical graph G_c itself is primitive. By simple observation, we have

Lemma 2.3. *Let H be an induced subgraph of G. Then H is isomorphic to some induced subgraph of G_c if H is primitive. Particularly, $H \cong G_c$ if they have the same number of vertices.*

Proof. Suppose $V(H) = \{u_1, u_2, \ldots, u_h\} \subseteq V(G)$. We claim that any two adjacent vertices of H cannot have the relation ρ in G. Otherwise, assume that u_i and u_j are two adjacent vertices which are contained in the same ρ-equivalence class. Then u_i and u_j have the same neighbors in $V(G)\backslash\{u_i, u_j\}$, and so the same neighbors in $V(H)\backslash\{u_i, u_j\}$. This implies that H is imprimitive, a contradiction. Thus there are at least h different ρ-equivalence classes, and H is isomorphic to some induced subgraph of G_c. This proves the first part of the lemma, and the second part follows immediately. □

For a graph H with vertex set $V(H) = \{v_1, v_2, \ldots, v_k\}$ and complete graphs $K_{n_i} (i = 1, 2, \ldots, k)$, we can construct a graph Γ from H and K_{n_i} such that each v_i is replaced with K_{n_i}, and the vertices of K_{n_i} join that of K_{n_j} if v_iv_j is an edge of H. As usual, we write $\Gamma = H[K_{n_1}, K_{n_2}, \ldots, K_{n_k}]$. Such a graph is called the *generalized lexicographic product* of H (by $K_{n_1}, K_{n_2}, \ldots, K_{n_k}$). Obviously, each graph can be viewed as a generalized lexicographic product of its canonical graph, i.e., $G = G_c[K_{n_1}, K_{n_2}, \ldots, K_{n_k}]$. However the canonical graph of

$\Gamma = H[K_{n_1}, K_{n_2}, \ldots, K_{n_k}]$ is not necessary to be H. Clearly, the canonical graph of Γ is H if H is primitive. It implies that, to characterize a class of graphs, it suffices to characterize all canonical graphs in this class. The following result is useful.

Lemma 2.4 (Theorem 5, [23]). *If G_c is a canonical graph of a graph G, then $p^-_{-1}(G) = p^-_{-1}(G_c)$ and $p^+_{-1}(G) = p^+_{-1}(G_c)$.*

Corollary 2.5. *Let $G = G_c[K_{n_1}, K_{n_2}, \ldots, K_{n_k}]$, $n_1 + n_2 + \cdots + n_k = n$ and $1 \le i \le k$. Then $G \in \mathcal{G}_n([-1]^{n-i})$ if and only if $G_c \in \mathcal{G}_k([-1]^{k-i})$.*

Proof. By Lemma 2.4,

$$\begin{aligned} m_G(-1) &= n - p^-_{-1}(G) - p^+_{-1}(G) \\ &= n - p^-_{-1}(G_c) - p^+_{-1}(G_c) \\ &= n - k + m_{G_c}(-1) \end{aligned}$$

Thus $m_G(-1) = n - i$ if and only if $m_{G_c}(-1) = k - i$. □

Corollary 2.6. *A graph $G \in \mathcal{G}_n([-1]^{n-3})$ if and only if $G \cong (K_a \cup K_b)\nabla K_{n-a-b}$, where $n, a, b > 0$ are all integers and $n > a + b$.*

Proof. Let $G \in \mathcal{G}_n([-1]^{n-3})$. If $n = 3$, we have $G \cong P_3 = (K_1 \cup K_1)\nabla K_1$. Now suppose $n \ge 4$. By Lemma 2.2, we have $\lambda_3(G) \ge -1$. Also, we claim that $\lambda_n(G) < -1$, since otherwise G cannot contain P_3 as its induced subgraph by Interlacing Theorem, i.e., G must be isomorphic to K_n, a contradiction. Then we must have $\lambda_3(G) = -1$ due to $m_G(-1) = n - 3$, and so $G \cong (K_a \cup K_b)\nabla K_{n-a-b}$ again by Lemma 2.2.

Conversely, suppose $G \cong (K_a \cup K_b)\nabla K_{n-a-b}$. It is clear that P_3 ($\in \mathcal{G}_3([-1]^0)$) is just the canonical graph of G. Then, by Corollary 2.5, we may conclude that $G \in \mathcal{G}_n([-1]^{n-3})$. □

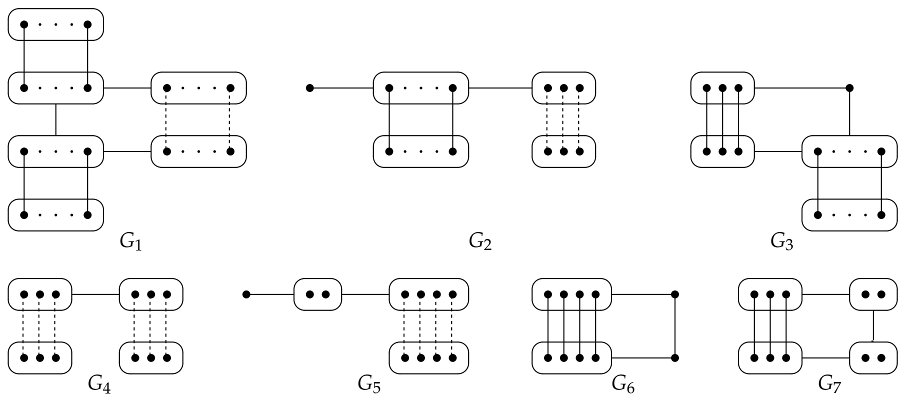

Figure 1: On graphs with exactly one eigenvalue less than −1.

Let G_1–G_7 be the graphs shown in Figure 1, in which ellipses denotes the independent sets; such two ellipses joining with exactly one full line denote a complete bipartite graph; such two ellipses joining with a sequence of k ($k \ge 1$) dotted parallel lines denote a complete bipartite graph on $k + k = 2k$ vertices with k edges of a perfect matching excluded; such two ellipses joining with a sequence of k ($k \ge 1$) full parallel lines denote a bipartite graph on $k + k = 2k$ vertices with k edges of a perfect matching.

Let G be a connected graph. By argument above, if $p^-_{-1}(G) = 0$, then G does not contain P_3 as an induced graph and so $G = K_n$, which means $p^-_{-1}(G) = 0$ if and only if $G = K_n$. The following elegant result characterizes the graph G with $p^-_{-1}(G) = 1$.

Lemma 2.7 (Theorem 7, [23]). *A connected graph $G \ne K_n$ has exactly one eigenvalue less than -1 if and only if its canonical graph G_c is an induced subgraph of any of the graphs $G_1 - G_7$ in Figure 1, so G_c is an bipartite graph.*

Lemma 2.8. *Let $G \in \mathcal{G}_n([-1]^i)$ have n vertices. If $0 \le i \le n - 4$ then $\lambda_3(G) > -1 > \lambda_n(G)$.*

Proof. First we prove $\lambda_3(G) > -1$. On the contrary, let $\lambda_3(G) \leq -1$. By Lemma 2.2, we get that

$$G \cong P_3,\ K_n \text{ or } (K_a \cup K_b)\nabla K_{n-a-b}.$$

However, $m_{P_3}(-1) = 0 > 3-4$, $m_{K_n}(-1) = n-1 > n-4$, and $m_{(K_a\cup K_b)\nabla K_{n-a-b}}(-1) = n-3 > n-4$, which are all contrary to $i \leq n-4$.

Next we show $-1 > \lambda_n(G)$. Obviously, $G \ncong K_n$ since $\lambda_3(G) > -1$. Thus G has an induced path P_3, which implies that $-\sqrt{2} = \lambda_3(P_3) \geq \lambda_n(G)$ by Lemma 2.1. Our result follows. □

3. The characterization of $\mathcal{G}_n([-1]^{n-4})$

Lemma 2.2 implies that $G \in \mathcal{G}_n([-1]^{n-3})$ if and only if $G \cong (K_a \cup K_b)\nabla K_{n-a-b}$ if and only if $G_c \cong P_3$. In this section, we will explicitly characterize the graphs in $\mathcal{G}_n([-1]^{n-4})$. It suffices to give all canonical graphs of $\mathcal{G}_n([-1]^{n-4})$.

Theorem 3.1. *A graph $G \in \mathcal{G}_n([-1]^{n-4})$ if and only if its canonical graph G_c is isomorphic to one of $K_{1,3}$, P_4, C_4, P_5 or C_6.*

Proof. By Lemma 2.8, $\lambda_3 > -1 > \lambda_n$. Thus the spectrum of G can be written as $Spec(G) = [\lambda_1^1, \lambda_2^1, \lambda_3^1, -1^{n-4}, \lambda_n^1]$, where $\lambda_1 > \lambda_2 \geq \lambda_3 > -1$, $\lambda_4 = \cdots = \lambda_{n-1} = -1$ and $-1 > \lambda_n$. In accordance with ρ-partition, we have $G = G_c[K_{n_1}, K_{n_2}, \ldots, K_{n_k}]$. From Lemma 2.4, G_c also has exactly three eigenvalues more than -1 and one eigenvalue less than -1. From Lemma 2.7, G_c is a bipartite graph and then the spectrum of G_c is symmetric about 0. Thus we may assume that $Spec(G_c) = [\mu_1^1, \mu_2^1, \mu_3^1, (-1)^{k-4}, \mu_k^1]$, where $\mu_1 \geq \mu_2 \geq \mu_3 > -1$, $\mu_4 = \cdots = \mu_{k-1} = -1$ and $-1 > \mu_k = -\mu_1$. Clearly, $k \geq 4$. Additionally, if $k \geq 8$, then $\mu_4 = -\mu_{k-3} = 1$, a contradiction. Next we consider $k = 4, 5, 6, 7$.

If $k = 4$, then $1 > \mu_2 = -\mu_3 > -1$. Since $K_{1,3}$, P_4 and C_4 are the only three connected bipartite graphs of 4 vertices, their spectra $Spec(K_{1,3}) = [\sqrt{3}, 0^2, -\sqrt{3}]$, $Spec(P_4) = [1.618, 0.618, -0.618, -1.618]$ and $Spec(C_4) = [2, 0, 0, -2]$ meet with the requirement. Thus $G_c \cong K_{1,3}$, P_4 or C_4.

If $k = 5$, then $\mu_2 = -\mu_4 = 1$ and $\mu_3 = 0$. We find that P_5 is the only bipartite graph of 5 vertices whose spectrum $Spec(P_5) = [1.73, 1, 0, -1, -1.73]$ meets with the requirement. Thus $G_c \cong P_5$.

If $k = 6$, then $\mu_2 = -\mu_5 = 1$ and $\mu_3 = -\mu_4 = 1$. Similarly, we find that C_6, with $Spec(C_6) = [2^1, 1^2, -1^2, -2^1]$, is the only bipartite graph of 6 vertices as our required, and so $G_c \cong C_6$.

If $k = 7$, then $\mu_4 = 0$, which contradicts $\mu_4 = -1$.

Conversely, each canonical graph G_c, which is isomorphic to one of $K_{1,3}$, P_4, C_4, P_5, C_6, has spectrum of the form $Spec(G_c) = [\lambda_1^1, \lambda_2^1, \lambda_3^1, (-1)^{k-4}, \lambda_k]$, where $k = 4, 5$ or 6, $\lambda_1 \geq \lambda_2 \geq \lambda_3 > -1$, and $-1 > \lambda_k$. Hence $G_c \in \mathcal{G}_k([-1]^{k-4})$. It follows that $G \in \mathcal{G}_n([-1]^{n-4})$ by Corollary 2.5.

The proof is complete. □

By Theorem 3.1 and Corollary 2.5, we have the following result immediately.

Corollary 3.2. *A graph $G \in \mathcal{G}_n([-1]^{n-4})$ if and only if $G = H[K_{n_1}, K_{n_2}, \ldots, K_{n_k}]$ where H is isomorphic to one of $K_{1,3}$, P_4, C_4, P_5, C_6 and $n_1 + n_2 + \cdots + n_k = n \geq 4$.*

It is worth mentioning that Corollary 3.2 gives some classes of graphs with a few eigenvalues. In fact, for $G \in \mathcal{G}_n([-1]^{n-4})$, we see that G has at most five distinct eigenvalues and $d(G) \leq 4$. Especially, $K_{1,3}[K_{n_1}, K_{n_2}, K_{n_3}, K_{n_4}]$ and $C_4[K_{n_1}, K_{n_2}, K_{n_3}, K_{n_4}]$ are two classes of graphs. Each of them has at most five distinct eigenvalues and $d(G) = 2$.

4. The characterization of $\mathcal{G}_n([-1]^{n-5})$

Recall that $\mathcal{G}_n([-1]^{n-5})$ is the set of all connected graphs on n vertices in which each graph has eigenvalue -1 of multiplicity $n-5$, where $n \geq 5$. Clearly, each $G \in \mathcal{G}_n([-1]^{n-5})$ has at most six distinct eigenvalues. Denote by $\mathcal{G}_n^1([-1]^{n-5})$ the connected graphs with spectra $\{\lambda_1^1, \lambda_2^1, \lambda_3^1, \lambda_4^1, -1^{n-5}, \lambda_n^1\}$ where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq$

$\lambda_4 > -1 > \lambda_n$. Similarly, denote by $\mathcal{G}_n^2([-1]^{n-5})$ the connected graphs with spectra $\{\lambda_1^1, \lambda_2^1, \lambda_3^1, -1^{n-5}, \lambda_{n-1}^1, \lambda_n^1\}$, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 > -1 > \lambda_{n-1} \geq \lambda_n$. By Lemma 2.8, $\mathcal{G}_n([-1]^{n-5})$ is the disjoint union of $\mathcal{G}_n^1([-1]^{n-5})$ and $\mathcal{G}_n^2([-1]^{n-5})$.

Firstly, we characterize the graphs in $\mathcal{G}_n^1([-1]^{n-5})$. By using the software SageMath 8.0, we can find all bipartite graphs on 5–8 vertices such that they have four eigenvalues greater than −1 and one eigenvalue smaller than −1, then they are H_1–H_{11} (see Figure 2), whose spectra are listed in Table 1. From which it is clear that H_1–$H_{11} \in \mathcal{G}_n^1([-1]^{n-5})$ are all primitive. We will show that they are exactly all canonical graphs of $\mathcal{G}_n^1([-1]^{n-5})$.

Graph	Spectrum	Graph	Spectrum
H_1	$[2^1, 0^3, -2^1]$	H_7	$[1.93^1, 1^1, 0.52^1, -0.52^1, -1^1, -1.93^1]$
H_2	$[1.85^1, 0.77^1, 0^1, -0.77^1, -1.85^1]$	H_8	$[2.41^1, 1^1, 0.41^1, -0.41^1, -1^1, -2.41^1]$
H_3	$[2.14^1, 0.66^1, 0^1, -0.66^1, -2.14^1]$	H_9	$[2^1, 1^2, 0, -1^2, -2^1]$
H_4	$[2.45^1, 0^3, -2.45^1]$	H_{10}	$[2.65^1, 1^2, 0^1, -1^2, -2.65^1]$
H_5	$[2.24^1, 1^1, 0^2, -1^1, -2.24]$	H_{11}	$[3^1, 1^3, -1^3, -3^1]$
H_6	$[2^1, 1^1, 0^2, -1^1, -2^1]$		

Table 1: The spectra of H_1–H_{11}.

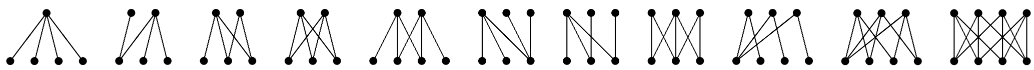

Figure 2: The canonical graphs of $\mathcal{G}_n^1([-1]^{n-5})$.

Theorem 4.1. *A graph $G \in \mathcal{G}_n^1([-1]^{n-5})$ if and only if its canonical graph G_c is isomorphic to one of $H_1, H_2, \ldots, H_{11}$.*

Proof. Let $G \in \mathcal{G}_n^1([-1]^{n-5})$. Then $G = G_c[K_{n_1}, K_{n_2}, \ldots, K_{n_k}]$ and $G_c \in \mathcal{G}_k([-1]^{k-5})$ by Corollary 2.5 and so $k \geq 5$. From Lemma 2.4, the canonical graph G_c also has four eigenvalues greater than −1 and one eigenvalue less than −1. Hence the spectrum of G_c can be written by $Spec(G_c) = [\mu_1^1, \mu_2^1, \mu_3^1, \mu_4^1, (-1)^{k-5}, \mu_k^1]$, where $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4 > -1$, $\mu_5 = \cdots = \mu_{k-1} = -1$ and $-1 > \mu_k = -\mu_1$. From Lemma 2.7, G_c is a bipartite graph, and then the spectrum of G_c is symmetric about 0. Thus, if $k \geq 10$, then $\mu_5 = -\mu_{k-4} = 1$, a contradiction. Next we consider $k = 5, 6, 7, 8, 9$.

If $k = 5$, then $1 > \mu_2 = -\mu_4 > -1$ and $\mu_3 = 0$. From Table 1 it is clear that H_1, H_2, H_3 and H_4 are the only four bipartite graphs on 5 vertices with this property. Hence $G_c \cong H_1, H_2, H_3, H_4$.

If $k = 6$, then $\mu_2 = -\mu_5 = 1$ and $1 > -\mu_3 = \mu_4 > -1$. From Table 1 we find that H_5, H_6, H_7 and H_8 are the only four bipartite graphs on 6 vertices satisfying this property. Hence $G_c \cong H_5, H_6, H_7$ or H_8.

If $k = 7$, then $\mu_2 = -\mu_6 = 1$, $\mu_3 = -\mu_5 = 1$ and $\mu_4 = 0$. Similarly, H_9 and H_{10} in Table 1 are the only two bipartite graphs on 7 vertices we needed. Hence $G_c \cong H_9$ or H_{10}.

If $k = 8$, then $\mu_2 = -\mu_7 = 1$, $\mu_3 = -\mu_6 = 1$ and $\mu_4 = -\mu_5 = 1$. We have $G_c \cong H_{11}$ in Table 1.

If $k = 9$, then $\mu_5 = 0$, which is impossible.

Conversely, it is clear from Table 1 that each of H_1–H_{11} belongs to $\mathcal{G}_k([-1]^{k-5})$ where $k = 5, 6, 7$ or 8. By Corollary 2.5, $G = G_c[K_{n_1}, K_{n_2}, \ldots, K_{n_k}] \in \mathcal{G}_n^1([-1]^{n-5})$ for each $G_c \cong H_i$ and $1 \geq i \geq 11$ where $n = n_1 + n_2 + \cdots + n_k$.

The proof is complete. □

It remains to characterize those graphs in $\mathcal{G}_n^2([-1]^{n-5})$. By the software SageMath 8.0, we can find all graphs on 5–7 vertices satisfying the following properties:

(1) they are connected non-bipartite.
(2) they are graphs belonging to $\mathcal{G}_n^2([-1]^{n-5})$ (that is, $p_G^+(-1) = 3$ and $p_G^-(-1) = 2$).
(3) they are primitive.

Graph	Spectrum	Graph	Spectrum
H_{12}	$[2.48^1, 0.69^1, 0^1, -1.17^1, -2^1]$	H_{18}	$[3.78^1, 0.71^1, 0^1, -1^1, -1.49^1, -2^1]$
H_{13}	$[2.94^1, 0.62^1, -0.46^1, -1.47^1, -1.62^1]$	H_{19}	$[4.20^1, 1^1, 0.55^1, -1^2, -1.75^1, -2^1]$
H_{14}	$[2.69^1, 0.33^1, 0^1, -1.27^1, -1.75^1]$	H_{20}	$[2.81^1, 1^1, 0.53^1, -1^1, -1.34^1, -2^1]$
H_{15}	$[3.24^1, 0^2, -1.24^1, -2^1]$	H_{21}	$[3.22^1, 1^1, 0.11^1, -1^1, -1.53^1, -1.81^1]$
H_{16}	$[2.30^1, 0.62^1, 0^1, -1.30^1, -1.62^1]$	H_{22}	$[3.59^1, 0.62^1, 0.16^1, -1^1, -1.62^1, -1.75^1]$
H_{17}	$[2^1, 0.62^2, -1.62^2]$	H_{23}	$[3.65^1, 1^2, -1^2, -1.65^1, -2^1]$

Table 2: The spectra of H_{12}–H_{23}.

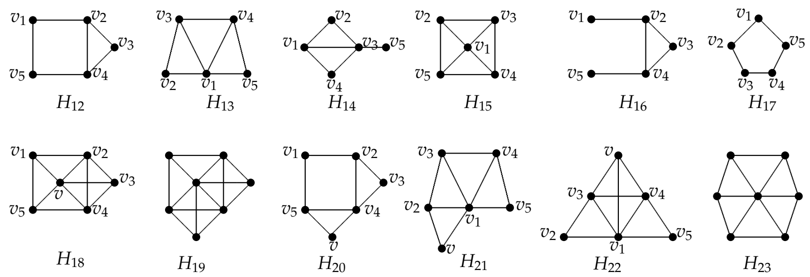

Figure 3: The canonical graphs of $\mathcal{G}_n^2([-1]^{n-5})$.

All these graphs are H_{12}–H_{23} shown in Figure 3, and their spectra are listed in Table 2. In what follows, we will give a series of lemmas and theorems to show that $G \in \mathcal{G}_n^2([-1]^{n-5})$ if and only if G_c is isomorphic to one of the graphs H_{12}–H_{23}.

One can directly verify the following result by Interlacing Theorem.

Lemma 4.2. *Let $G \in \mathcal{G}_n^2([-1]^{n-5})$ and $n \geq m \geq 6$. If H is an induced subgraph of G on m vertices with eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{m-1} \geq \mu_m$, then $\mu_4 = \ldots = \mu_{m-2} = -1$.*

Lemma 4.3 (Theorem 8, [23]). *If a graph G has exactly two eigenvalues less than -1, then G contains at least one induced graph which is isomorphic to one of M_1–M_{12} (see Figure 4) or H_{12}–H_{17} (see Figure 3).*

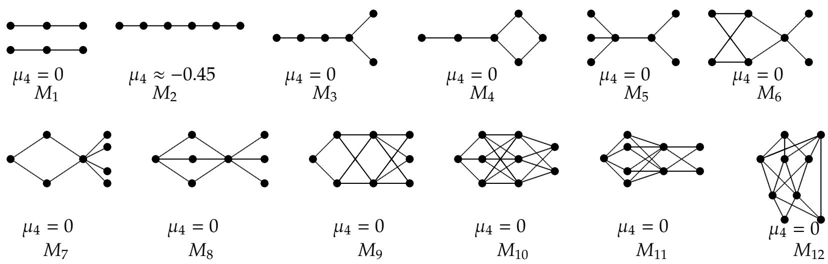

Figure 4: The minimal graphs M_1–M_{12}.

Lemma 4.4. *The graphs H_{12}–H_{17} displayed in Figure 3 are exactly six minimal graphs in $\mathcal{G}_n^2([-1]^{n-5})$ (it means that any $G \in \mathcal{G}_n^2([-1]^{n-5})$ contains at least one induced subgraph which is isomorphic to one of H_{12}–H_{17}), where $n \geq 5$.*

Proof. Let $G \in \mathcal{G}_n^2([-1]^{n-5})$. Then G contains exactly two eigenvalues less than -1. By Lemma 4.3, G contains at least one induced graph which is isomorphic to one of M_1–M_{12} (see Figure 4) or H_{12}–H_{17} (see Figure 3). On the other aspect, let H be any induced subgraph of G, where $n = |V(G)| \geq m = |V(H)| \geq 6$. By Lemma 4.2 we have $\mu_4(H) = -1$. However, the fourth largest eigenvalues of the graphs M_1–M_{12} are all not equal to -1 (see Figure 4). Hence M_1–M_{12} should be eliminated. Indeed, H_{12}–H_{17} are the six minimal graphs belonging to $\mathcal{G}_n^2([-1]^{n-5})$ (see Table 2). □

In terms of Lemma 4.4, we will give a series of lemmas and theorems that exhaust all canonical graphs of $\mathcal{G}_n^2([-1]^{n-5})$ that contain at least one induced subgraph which is isomorphic to one of H_{12}–H_{17}. This leads to the final characterization of the graphs in $\mathcal{G}_n^2([-1]^{n-5})$ for any $n \geq 5$. First, we give a lemma that is frequently used later on.

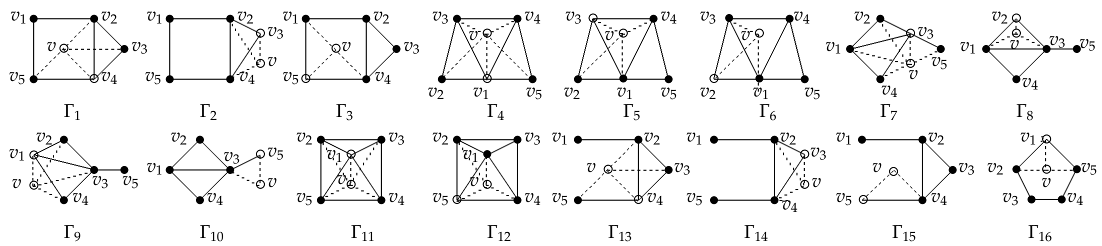

Figure 5: Γ_1–Γ_{16} (some connected graphs on 6 vertices with $\lambda_4 = -1$).

Lemma 4.5. *Let $G \in \mathcal{G}_n^2([-1]^{n-5})$. Then the canonical graph G_c has 6 vertices if and only if G_c is isomorphic to one of H_{18}, H_{20}, H_{21} or H_{22} (shown in Figure 3), in which H_{18} and H_{20} contain induced H_{12}; H_{21} and H_{22} contain induced H_{13}; H_{18} contains induced H_{15}; H_{22} contains induced H_{16}.*

Proof. From Figure 3 and Table 2, it is clear that H_{18}, H_{20}, H_{21} and H_{22} are primitive and belong to $\mathcal{G}_n^2([-1]^{n-5})$ for $n = 6$. The sufficiency follows.

Let $G \in \mathcal{G}_n^2([-1]^{n-5})$ and its canonical graph G_c has 6 vertices. By Lemma 2.4 and Lemma 4.2, we get $\mu_4(G_c) = -1$. By Lemma 4.4, G_c contains at least one induced graph which is isomorphic to one of H_{12}–H_{17}. By using Table A3 in [11] (one can also use software SageMath 8.0 under the restriction of $\mu_4(G_c) = -1$), we find that there are only twenty connected graphs on 6 vertices belonging to $\mathcal{G}_n^2([-1]^{n-5})$, in which Γ_1–Γ_{16} are shown in Figure 5 and others are H_{18}, H_{20}, H_{21} and H_{22} in Figure 3. From which we choose, according to Lemma 4.4, the primitive graphs that contain one of H_{12}–H_{17} as their induced subgraphs. It is clear from Figure 5 that $\Gamma_1, \Gamma_2, \Gamma_3$ are generalized lexicographic products of H_{12} (where the vertices satisfying $v\rho v_i$ are labelled as hollow dots, the edges connecting v and H_{12} are labelled as dotted lines, and the following is similar), Γ_i ($i = 4, 5, 6$) are the products of H_{13}, Γ_i ($i = 7, 8, 9, 10$) are the products of H_{14}, Γ_i ($i = 11, 12$) are the products of H_{15}, Γ_i ($i = 13, 14, 15$) are the products of H_{16}, and Γ_{16} is a product of H_{17}. Hence all the Γ_i are imprimitive and will be excluded. The remainders H_{18}, H_{20}, H_{21} and H_{22} are the only primitive graphs containing one of H_{12}–H_{17} as induced subgraph. In fact, H_{18} and H_{20} contains H_{12}; H_{21} and H_{22} contain H_{13}; H_{18} contains H_{15}; H_{22} contains H_{16}.

The proof is complete. □

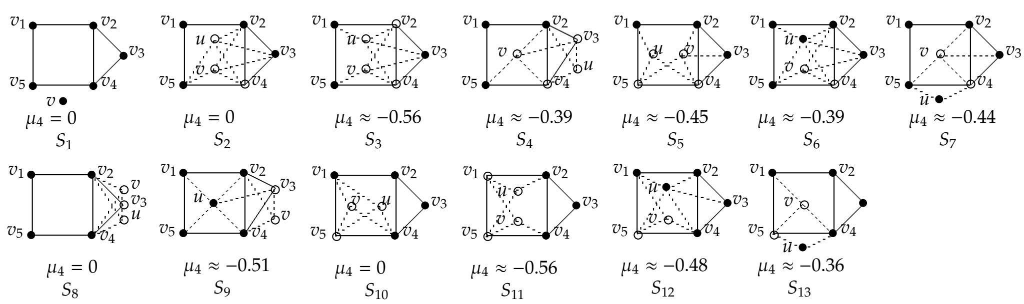

Figure 6: Forbidden subgraphs $\mu_4 \neq -1$.

Lemma 4.6. *Let $G_c \in \mathcal{G}_n^2([-1]^{n-5})$ contain an induced subgraph which is isomorphic to H_{12} and $H_v = G_c[V(H_{12}) \cup \{v\}]$ for $v \in V(G_c) \backslash V(H_{12})$. Then $H_v \cong H_{18}$ or H_{20} (shown in Figure 3).*

Proof. The graph H_v has six vertices and $\mu_4(H_v) = -1$ by Lemma 4.2. Additionally, H_v will be connected since otherwise $H_v \cong S_1$ (see Figure 6) but $\mu_4(S_1) = 0$. By using the Table A3 in [11] (also can using software SageMath 8.0 under the restriction of $\mu_4(G_c) = -1$), we find that there are only five connected graphs on 6 vertices whose fourth largest eigenvalues equal -1 and each of them contains an induced subgraph which is isomorphic to H_{12}, in which $\Gamma_1, \Gamma_2, \Gamma_3$ are shown in Figure 5 and others H_{18}, H_{20}. Thus we have $H_v \cong \Gamma_1$, Γ_2, Γ_3, H_{18} or H_{20}. It suffices to eliminate the graphs: Γ_1–Γ_3.

If $H_v \cong \Gamma_1$, then $v_4 \rho v$ in Γ_1 (see Figure 5). Since G_c is primitive, v_4 and v has no relation ρ in G_c, and so $N_{G_c}(v_4)\backslash v \neq N_{V(G_c)}(v)\backslash v_4$. Since ρ is symmetric, we may assume that G_c has another vertex $u \sim v_4$ but $u \nsim v$. Thus $H_u = G_c[V(H_{12}) \cup \{u\}] \in \{\Gamma_1, \Gamma_2, \Gamma_3, H_{18}, H_{20}\}$ by above arguments, where we regard u as v in these graphs. Now $H_{v,u} = G_c[V(H_{12}) \cup \{v, u\}]$ consists of two induced subgraphs which are isomorphic to Γ_1 and H_u, respectively. Clearly, $H_{v,u}$ will be S_2 or S_3 if H_u takes Γ_1 (where $H_v = H_u = \Gamma_1$ corresponds to S_2; $H_u \cong \Gamma_1 \cong H_v$ corresponds to S_3). Similarly, H will be S_4, S_5, S_6 and S_7 if H_u takes $\Gamma_2, \Gamma_3, H_{18}$ and H_{20}, respectively. However, S_2, S_3, S_4, S_5, S_6 and S_7 are all forbidden induced subgraphs of G_c because their fourth largest eigenvalues are not equal to -1.

If $H_v \cong \Gamma_2$, then $v_3 \rho v$ in Γ_2 (see Figure 5). Similarly, there exists some u with $u \sim v_3$ but $u \nsim v$, and then $H_u = G_c[V(H_{12}) \cup \{u\}] \in \{\Gamma_1, \Gamma_2, \Gamma_3, H_{18}, H_{20}\}$. Again we consider $H_{v,u} = G_c[V(H_{12}) \cup \{v, u\}]$. Clearly, $H_u \cong \Gamma_3$ or H_{20} cannot appear in $H_{v,u}$ since $u \nsim v_3$ in Γ_3 and H_{20} (but $u \sim v_3$ in $H_{v,u}$). Additionally, $\{H_v, H_u\} \neq \{\Gamma_1, \Gamma_2\}$ as above. Thus $H_u \in \{\Gamma_2, H_{18}\}$, and H will be S_8 and S_9 if H_u takes Γ_2 and H_{18}, respectively. However, S_8 and S_9 are all forbidden induced subgraphs of G_c.

If $H_v \cong \Gamma_3$, then $v_5 \rho v$ in Γ_3 (see Figure 5). Similarly, there exists some u with $u \sim v_5$ but $u \nsim v$, and then $H_u = G_c[V(H_{12}) \cup \{u\}] \in \{\Gamma_3, H_{18}, H_{20}\}$ (Γ_1, Γ_2 will be abandoned as above). Thus $H = G_c[V(H_{12}) \cup \{v, u\}]$ will be S_{10} or S_{11} if H_u takes Γ_3; H will be S_{12}, S_{13} if H_u takes H_{18} and H_{20}, respectively. However, S_{10}, S_{11}, S_{12} and S_{13} are all forbidden induced subgraphs of G_c.

The proof is complete. □

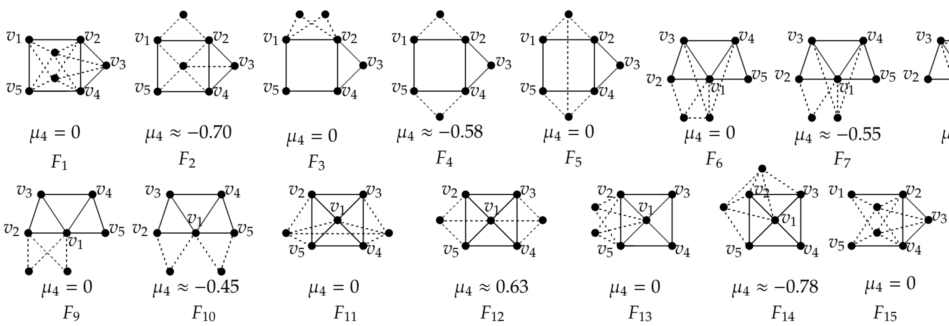

Figure 7: Forbidden subgraphs $\mu_4 \neq -1$.

Theorem 4.7. *A graph $G \in \mathcal{G}_n^2([-1]^{n-5})$ contains an induced subgraph which is isomorphic to H_{12} if and only if its canonical graph G_c is isomorphic to one of H_{12}, H_{18}, H_{19} or H_{20} (see Figure 3).*

Proof. Assume that $G_c \cong H_{12}$, H_{18}, H_{19} or H_{20}. Then G_c has an induced subgraph which is isomorphic to H_{12} since each of H_{18}, H_{19} and H_{20} has an induced subgraph which is isomorphic to H_{12}. Consequently, G contains an induced subgraph which is isomorphic to H_{12}.

Conversely, suppose that G contains an induced graph which is isomorphic to H_{12}. Since H_{12} is primitive, by Lemma 2.3 G_c has induced H_{12}, and $G_c \cong H_{12}$ if $|V(G_c)| = 5$. Assume that $|V(G_c)| > 5$. By Lemma 4.6, $H_v = G_c[V(H_{12}) \cup \{v\}] \in \{H_{18}, H_{20}\}$ for $v \in V(G_c)\backslash V(H_{12})$. It is all right if $G_c \cong H_v$. Otherwise, there exists $u \in V(G_c)\backslash V(H_v)$ such that $H_u = G_c[V(H_{12}) \cup \{u\}] \in \{H_{18}, H_{20}\}$ again by Lemma 4.6. We will distinguish the following cases.

Case 1. If $H_v \cong H_{18} \cong H_u$ then $N_{H_v}(v) = V(H_{12}) = N_{H_u}(u)$ (see H_{18} in Figure 3). If $v \nsim u$ then $H = G_c[V(H_{12}) \cup \{v, u\}] \cong F_1$ (see Figure 7), but $\mu_4(F_1) \neq -1$. Thus $v \sim u$ and so $v \rho u$ in H. Since G_c is primitive, we have $N_{G_c}(v)\backslash u \neq N_{G_c}(u)\backslash v$. Thus we may assume that G_c has a vertex $w \sim v$ but $w \nsim u$. Again we have $H_w = G_c[V(H_{12}) \cup \{w\}] \in \{H_{18}, H_{20}\}$ and so $H_w \cong H_{20}$ due to $w \nsim u$. Thus, $N_{H_w}(w) = \{v_1, v_2\}$ or $\{v_4, v_5\}$. Then $G_c[V(H_{12}) \cup \{w, u\}] \cong F_2$ (see Figure 7), however $\mu_4(F_2) \neq -1$, a contradiction.

Case 2. If $H_v \cong H_{20} \cong H_u$ then $N_{H_v}(v), N_{H_u}(u) = \{v_1, v_2\}$ or $\{v_4, v_5\}$ (see H_{20} in Figure 3). We first assume that $N_{H_v}(v) = \{v_1, v_2\} = N_{H_u}(u)$. Then $v \sim u$, since otherwise $G_c[V(H_{12}) \cup \{v, u\}] \cong F_3$ (see Figure 7), but $\mu_4(F_3) \neq -1$. Similarly as in Case 1, G_c has a vertex $w \sim v$ but $w \nsim u$. Obviously, $H_w = G_c[V(H_{12}) \cup \{w\}] \in \{H_{18}, H_{20}\}$. If $H_w \cong H_{18}$, then $G_c[V(H_{12}) \cup \{w, u\}] \cong F_2$ (see Figure 7), but $\mu_4(F_2) \neq -1$. If $H_w \cong H_{20}$, then

$$G_c[V(H_{12}) \cup \{w, u\}] \cong \begin{cases} F_3, & \text{if } N_{H_w}(w) = \{v_1, v_2\} \\ F_4, & \text{if } N_{H_w}(w) = \{v_4, v_5\} \end{cases} \quad \text{(see } F_1, F_2 \text{ in Figure 7)}$$

which are impossible since F_3 and F_4 are all forbidden subgraphs of G_c.

By symmetry (see H_{20} in Figure 3), the case of $N_{H_v}(v) = \{v_4, v_5\} = N_{H_u}(u)$ is equivalent to that of $N_{H_v}(v) = \{v_1, v_2\} = N_{H_u}(u)$ in above discussion. It remains to consider $N_{H_v}(v) = \{v_1, v_2\}$ and $N_{H_u}(u) = \{v_4, v_5\}$. Clearly,

$$G_c[V(H_{12}) \cup \{v, u\}] \cong \begin{cases} F_4, & \text{if } v \nsim u \\ F_5, & \text{if } v \sim u \end{cases} \quad \text{(see } F_4, F_5 \text{ in Figure 7)}$$

which are impossible since F_4 and F_5 are forbidden subgraphs of G_c.

Case 3. If $H_v \cong H_{18}$ and $H_u \cong H_{20}$ then $G_c[V(H_{12}) \cup \{v, u\}] \cong G_c$. Since otherwise, G_c has another vertex $w \neq v, u$ such that $H_w = G_c[V(H_{12}) \cup \{w\}] \cong H_{18}$ or H_{20} by Lemma 4.6. However, the case of $H_w \cong H_{18} \cong H_v$ is eliminated as in Case 1 and the case of $H_w \cong H_{20} \cong H_u$ is eliminated as in Case 2. Now, if $v \nsim u$ then $G_c[V(H_{12}) \cup \{v, u\}] \cong F_2$ (see Figure 7), but $\mu_4(F_2) \neq -1$; if $v \sim u$ then $G_c = G_c[V(H_{12}) \cup \{v, u\}] \cong H_{19}$ (see Figure 3), as required.

The proof is complete. □

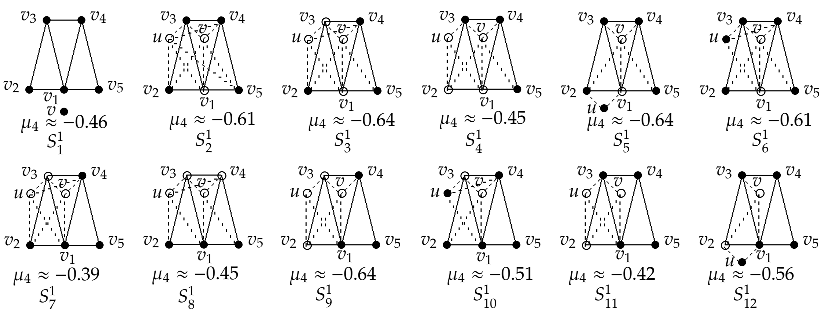

Figure 8: Forbidden subgraphs $\mu_4 \neq -1$.

Lemma 4.8. *Let $G_c \in \mathcal{G}_n^2([-1]^{n-5})$ contain an induced subgraph which is isomorphic to H_{13} and $H_v = G_c[V(H_{13}) \cup \{v\}]$ for $v \in V(G_c)\backslash V(H_{13})$. Then $H_v \cong H_{21}$ or H_{22}.*

Proof. The graph H_v has six vertices and $\mu_4(H_v) = -1$ by Lemma 4.2. Additionally, H_v will be connected, since otherwise $H_v \cong S_1^1$ (see Figure 8) but $\mu_4(S_1^1) \approx -0.46$. By using the Table A3 in [11] (also can using software SageMath 8.0 under the restriction of $\mu_4(G_c) = -1$), we find that there are five connected graphs on 6 vertices whose fourth largest eigenvalues equal -1 and each of them contains an induced subgraph which is isomorphic to H_{13}, in which Γ_4, Γ_5 and Γ_6 are shown in Figure 5 and others H_{21}, H_{22}. Thus we have $H_v \cong \Gamma_4, \Gamma_5, \Gamma_6, H_{21}$ or H_{22}. It suffices to eliminate the graphs: Γ_4–Γ_6.

If $H_v \cong \Gamma_4$, then $v_1 \rho v$ in Γ_4 (see Figure 5). Since G_c is primitive, we have $N_{G_c}(v_1)\backslash v \neq N_{G_c}(v)\backslash v_1$. Thus we may assume that there exists $u \sim v_1$ but $u \nsim v$. We have $H_u = G_c[V(H_{13}) \cup \{u\}] \in \{\Gamma_4, \Gamma_5, \Gamma_6, H_{21}, H_{22}\}$ by above arguments. Now $H_{v,u} = G_c[V(H_{13}) \cup \{v, u\}]$ contains $H_v \cong \Gamma_4$ and H_u as its induced subgraphs. From

Figure 5 and Figure 8, clearly, $H_{v,u}$ will be $S_2^1, S_3^1, S_4^1, S_5^1$ and S_6^1 if H_u takes $\Gamma_4, \Gamma_5, \Gamma_6, H_{21}$ and H_{22}, respectively. However, $S_2^1, S_3^1, S_4^1, S_5^1$ and S_6^1 are all forbidden induced subgraphs of G_c.

If $H_v \cong \Gamma_5$, then $v_3 \rho v$ in Γ_5 (see Figure 5). Similarly, there exists some u with $u \sim v_3$ but $u \nsim v$, and then $H_u = G_c[V(H_{13}) \cup \{u\}] \in \{\Gamma_4, \Gamma_5, \Gamma_6, H_{21}, H_{22}\}$. Thus $H_{v,u} = G_c[V(H_{13}) \cup \{v, u\}]$ has $H_v \cong \Gamma_5$ and H_u as its induced subgraphs. First $H_u \neq H_{21}$ since $u \nsim v_3$ in H_{21}. Additionally, $\{H_v, H_u\} \neq \{\Gamma_4, \Gamma_5\}$ as above. It is clear from Figure 8 that $H_{v,u}$ will be S_7^1 or S_8^1 if H_u takes Γ_5 (where $H_v = H_u = \Gamma_5$ corresponds S_7^1; $H_v, H_u \cong \Gamma_5$ corresponds S_8^1) and $H_{v,u}$ will be S_9^1 and S_{10}^1 if H_u takes Γ_6 and H_{22}, respectively. However, S_7^1, S_8^1, S_9^1, and S_{10}^1 are all forbidden induced subgraphs of G_c.

If $H_v \cong \Gamma_6$, then $v_2 \rho v$ in Γ_6 (see Figure 5). Similarly, there exists some u with $u \sim v_2$ but $u \nsim v$, and then $H_u = G_c[V(H_{13}) \cup \{u\}] \in \{\Gamma_4, \Gamma_5, \Gamma_6, H_{21}, H_{22}\}$. Clearly, $H_u \neq H_{22}$ since $u \nsim v_2$ in H_{22}. Γ_4, Γ_5 will be abandoned as above. Thus $H = G_c[V(H_{13}) \cup \{v, u\}]$ will be S_{11}^1 and S_{12}^1 if H_u takes Γ_6 and H_{21}, respectively. However, S_{11}^1 and S_{12}^1 are all forbidden induced subgraphs of G_c.

The proof is complete. □

Theorem 4.9. *A graph* $G \in \mathcal{G}_n^2([-1]^{n-5})$ *contains an induced subgraph which is isomorphic to* H_{13} *if and only if* $G_c \cong H_{13}, H_{21}, H_{22}$ *or* H_{23}.

Proof. Assume that $G_c \cong H_{13}$, H_{21}, H_{22} or H_{23}. Obviously, G_c contains an induced subgraph which is isomorphic to H_{13} since each of H_{21}, H_{22} and H_{23} has an induced subgraph which is isomorphic to H_{13}. Consequently, G contains an induced subgraph which is isomorphic to H_{13}.

Conversely, assume that G contains an induced subgraph which is isomorphic to H_{13}. Since H_{13} is primitive, from Lemma 2.3 we know that G_c also has an induced subgraph isomorphic to H_{13}, and $G_c \cong H_{13}$ if $|V(G_c)| = 5$. If $|V(G_c)| \geq 6$ then, by Lemma 4.8, $H_v = G_c[V(H_{13}) \cup \{v\}] \in \{H_{21}, H_{22}\}$ for each $v \in V(G_c)\backslash V(H_{13})$. If $|V(G_c)| > 6$, then G_c has another vertex $u \neq v$ such that $H_u = G_c[V(H_{13}) \cup \{u\}] \in \{H_{21}, H_{22}\}$. We will distinguish the following cases.

Case 1. Assume that $H_v \cong H_{21}$ and $H_u \cong H_{22}$. We have $N_{H_v}(v) = \{v_1, v_2\}$ or $\{v_1, v_5\}$, and $N_{H_u}(u) = \{v_1, v_3, v_4\}$. Thus

$$G_c[V(H_{13}) \cup \{v, u\}] \cong \begin{cases} F_6, & \text{if } v \sim u \\ F_7, & \text{if } v \nsim u \end{cases} \quad \text{(see } F_6, F_7 \text{ in Figure 7)}$$

which are impossible since F_6 and F_7 are forbidden subgraphs.

Case 2. Assume that $H_v \cong H_{22} \cong H_u$. We have $N_{H_v}(v) = \{v_1, v_3, v_4\} = N_{H_u}(u)$. If $v \nsim u$ then $H_{v,u} = G_c[V(H_{13}) \cup \{v, u\}] \cong F_8$ (see Figure 7), but $\mu_4(F_8) \neq -1$. Thus $v \sim u$ and so $v \rho u$ in $H_{v,u}$. Since G_c is primitive, $N_{G_c}(v)\backslash u \neq N_{G_c}(u)\backslash v$. Thus we may assume that there exists $w \sim v$ but $w \nsim u$. Again we have $H_w = G_c[V(H_{13}) \cup \{w\}] \in \{H_{21}, H_{22}\}$ and so $H_w \cong H_{21}$ due to $w \nsim u$. Thus $N_{H_w}(w) = \{v_1, v_2\}$ or $\{v_1, v_5\}$. Then $G_c[V(H_{13}) \cup \{w, u\}] \cong F_7$ (see Figure 7), however $\mu_4(F_7) \neq -1$, a contradiction.

Case 3. Assume that $H_v \cong H_{21} \cong H_u$. Then $N_{H_v}(v), N_{H_u}(u) = \{v_1, v_2\}$ or $\{v_1, v_5\}$. By the symmetry of $\{v_1, v_5\}$ and $\{v_1, v_2\}$ in H_v or H_u, $N_{H_v}(v) = \{v_1, v_5\} = N_{H_u}(u)$ is equivalent to $N_{H_v}(v) = \{v_1, v_2\} = N_{H_u}(u)$. We only need to consider the following two subcases.

If $N_{H_v}(v) = \{v_1, v_2\} = N_{H_u}(u)$, then $v \sim u$ since otherwise $G_c[V(H_{13}) \cup \{v, u\}] \cong F_9$ (see Figure 7), but $\mu_4(F_9) \neq -1$. Similarly as in Case 2, there exists some w with $w \sim v$ and $w \nsim u$ such that $H_w = G_c[V(H_{13}) \cup \{w\}] \in \{H_{21}, H_{22}\}$. If $H_w \cong H_{22}$ then we turn to Case 1. If $H_w \cong H_{21}$, then

$$G_c[V(H_{13}) \cup \{w, u\}] \cong \begin{cases} F_9, & \text{if } N_{H_{21}}(w) = \{v_1, v_2\} \\ F_{10}, & \text{if } N_{H_{21}}(w) = \{v_1, v_5\} \end{cases} \quad \text{(see } F_9, F_{10} \text{ in Figure 7)}$$

However, F_9 and F_{10} are forbidden subgraphs of G_c, a contradiction.

If $N_{H_v}(v) = \{v_1, v_2\}$ and $N_{H_u}(u) = \{v_1, v_5\}$, then $v \sim u$ since otherwise $G_c[V(H_{13}) \cup \{v, u\}] \cong F_{10}$ (see Figure 7), but $\mu_4(F_{10}) \neq -1$, and so $H_{v,u} = G_c[V(H_{12}) \cup \{v, u\}] \cong H_{23}$ (see Figure 3). If $G_c \cong H_{v,u}$, there is nothing to do. Otherwise, G_c has another vertex $w \neq v, u$ such that $H_w = G_c[V(H_{13}) \cup \{w\}] \in \{H_{21}, H_{22}\}$ by Lemma 4.8. First let $H_w \cong H_{21}$. Then $N_{H_w}(w) = \{v_1, v_2\}$ or $\{v_1, v_5\}$. If the former occurs then $N_{H_w}(w) = \{v_1, v_2\} = N_{H_v}(v)$; if the later occurs then $N_{H_w}(w) = \{v_1, v_5\} = N_{H_u}(u)$. The both are impossible by the above arguments. Next let $H_w \cong H_{22}$. Then we turn to Case 1 since $H_v \cong H_{21}$.

The proof is complete. □

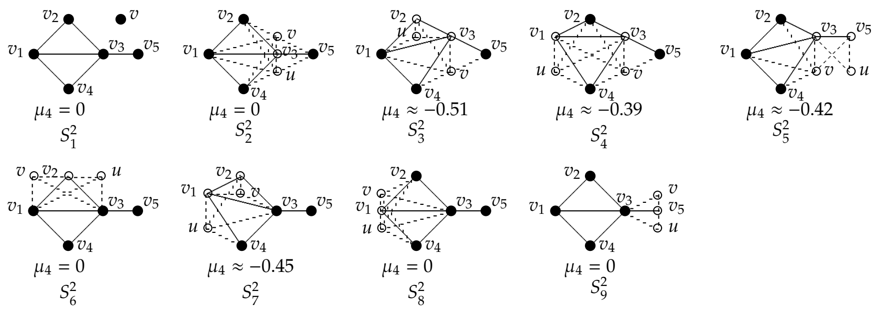

Figure 9: Forbidden subgraphs $\mu_4 \neq -1$.

Theorem 4.10. *A graph $G \in \mathcal{G}_n^2([-1]^{n-5})$ contains an induced subgraph which is isomorphic to H_{14} if and only if its canonical graph $G_c \cong H_{14}$ (see in Figure 3).*

Proof. The sufficiency is obvious. We show the necessity. Since H_{14} is primitive and G contains an induced subgraph which is isomorphic to H_{14}, by Lemma 2.3, G_c also has an induced subgraph which is isomorphic to H_{14} and $G_c \cong H_{14}$ if $|V(G_c)| = 5$. For $|V(G_c)| \geq 6$, let $H_v = G_c[V(H_{14}) \cup \{v\}]$ for $v \in V(G_c) \backslash V(H_{14})$. Thus $\mu_4(H_v) = -1$ by Lemma 4.2. Additionally, H_v will be connected, since otherwise $H_v \cong S_1^2$ (see Figure 9) but $\mu_4(S_1^2) = 0$. By using the Table A3 in [11] (also can using software SageMath 8.0 under the restriction of $\mu_4(G_c) = -1$), we find that Γ_7–Γ_{10}, shown in Figure 5, are the only four connected graphs of 6 vertices whose fourth largest eigenvalue is equal to -1 and each of them contains an induced subgraph which is isomorphic to H_{14}. Thus we have $H_v \in \{\Gamma_7, \Gamma_8, \Gamma_9, \Gamma_{10}\}$. Clearly H_v is imprimitive (in fact, $v_3 \rho v$ in Γ_7, $v_2 \rho v$ in Γ_8, $v_1 \rho v$ in Γ_9, $v_5 \rho v$ in Γ_{10}(see Figure 5)). However, since G_c is primitive, H_v must be a proper induced subgraph of G_c. There exists $u \neq v$ such that $H_u = G_c[V(H_{14}) \cup \{u\}] \in \{\Gamma_7, \Gamma_8, \Gamma_9, \Gamma_{10}\}$ for $u \in V(G_c) \backslash V(H_v)$ by the above arguments. Now $H_{v,u} = G_c[V(H_{14}) \cup \{v, u\}]$ contains two induced subgraphs $H_u, H_v \in \{\Gamma_7, \Gamma_8, \Gamma_9, \Gamma_{10}\}$. On the other hand, since $v_2 \rho v$ in Γ_8, we may take $u \sim v_2$ and $u \nsim v$. Thus $H_{v,u}$ can not contain two induced subgraphs isomorphic to Γ_8 or Γ_{10} simultaneously because $u \nsim v_2$ in Γ_{10}. Similarly, $H_{v,u}$ can not contain two induced subgraphs isomorphic to Γ_9 or Γ_{10} simultaneously because $v_1 \rho v$ in Γ_9 but $v_1 \nsim u$ in Γ_{10}. Furthermore, from Figure 9, $H_{v,u}$ will be $S_2^2, S_3^2, S_4^2, S_5^2, S_6^2, S_7^2, S_8^2$ and S_9^2 if $\{H_v, H_u\}$ equals $\{\Gamma_7, \Gamma_7\}$, $\{\Gamma_7, \Gamma_8\}$, $\{\Gamma_7, \Gamma_9\}$, $\{\Gamma_7, \Gamma_{10}\}$, $\{\Gamma_8, \Gamma_8\}$, $\{\Gamma_8, \Gamma_9\}$, $\{\Gamma_9, \Gamma_9\}$ and $\{\Gamma_{10}, \Gamma_{10}\}$, respectively. However, $S_2^2, S_3^2, S_4^2, S_5^2, S_6^2, S_7^2, S_8^2$ and S_9^2 are all forbidden induced subgraphs of G_c.

The proof is complete. □

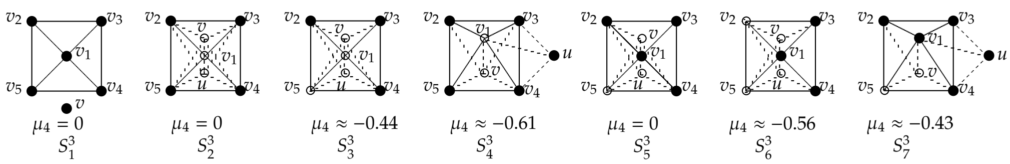

Figure 10: Forbidden subgraphs $\mu_4 \neq -1$.

Lemma 4.11. *Let $G_c \in \mathcal{G}_n^2([-1]^{n-5})$ contain an induced subgraph which is isomorphic to H_{15} and $H_v = G_c[V(H_{15}) \cup \{v\}]$ for $v \in V(G_c) \backslash V(H_{15})$. Then $H_v \cong H_{18}$.*

Proof. The graph H_v has six vertices and $\mu_4(H_v) = -1$ by Lemma 4.2. Additionally, H_v will be connected, since otherwise $H_v \cong S_1^3$ (see Figure 10) but $\mu_4(S_1^3) = 0$. By using the Table A3 in [11] (also can using software SageMath 8.0 under the restriction of $\mu_4(G_c) = -1$), we find that Γ_{11}, Γ_{12} and H_{18} are only three connected graphs on 6 vertices whose fourth largest equals -1 and contain an induced subgraph which is isomorphic to H_{15}. Thus $H_v \in \{\Gamma_{11}, \Gamma_{12}, H_{18}\}$. It suffices to eliminate the graphs Γ_{11}, Γ_{12}.

If $H_v \cong \Gamma_{11}$, then $v_1 \rho v$ in Γ_{11} (see Figure 5). Since G_c is primitive, we may assume that there exists another vertex $u \sim v_1$ but $u \nsim v$. Let $H_u = G_c[V(H_{15}) \cup \{u\}]$. We have $H_u \in \{\Gamma_{11}, \Gamma_{12}, H_{18}\}$ as above. Thus $H_{v,u} = G_c[V(H_{15}) \cup \{v, u\}]$ consists of Γ_{11} and H_u. From Figure 5 and Figure 10, clearly, $H_{v,u}$ will be S_2^3, S_3^3 and S_4^3 if H_u takes Γ_{11}, Γ_{12} and H_{18}, respectively. However, S_2^3, S_3^3 and S_4^3 are all forbidden induced subgraphs of G_c.

If $H_v \cong \Gamma_{12}$, then $v_5 \rho v$ in Γ_{12} (see Figure 5). Similarly as above, G_c has a vertex $u \sim v_5$ but $u \nsim v$ such that $H_u = G_c[V(H_{15}) \cup \{u\}] \in \{\Gamma_{11}, \Gamma_{12}, H_{18}\}$. Additionally, $\{H_v, H_u\} \neq \{\Gamma_{11}, \Gamma_{12}\}$ as above. Now $H_{v,u} = G_c[V(H_{15}) \cup \{v, u\}]$ contain induced subgraphs which are isomorphic to H_v or H_u. Clearly, $H_{v,u}$ will be S_5^3 and S_6^3 if H_u takes Γ_{12} ($H_v = H_u = \Gamma_{12}$ corresponds S_5^3; $H_v, H_u \cong \Gamma_{12}$ corresponds S_6^3); $H_{v,u}$ will be S_7^3 if H_u takes H_{18}, respectively. However, S_5^3, S_6^3 and S_7^3 are all forbidden induced subgraphs.

The proof is complete. □

Theorem 4.12. *A graph $G \in \mathcal{G}_n^2([-1]^{n-5})$ contains an induced subgraph which is isomorphic to H_{15} if and only if its canonical graph $G_c \cong H_{15}, H_{18}$ or H_{19}.*

Proof. Assume that $G_c \cong H_{15}$, H_{18} or H_{19}. Since each of H_{18} and H_{19} has an induced subgraph which is isomorphic to H_{15}, G_c also has the induced subgraph which is isomorphic to H_{15}, and so has G.

Conversely, assume that G contains an induced subgraph which is isomorphic to H_{15}. By Lemma 2.3, G_c also has an induced subgraph isomorphic to H_{15}, and $G_c \cong H_{15}$ if $|V(G_c)| = 5$. If $|V(G_c)| \geq 6$ then $H_v = G_c[V(H_{15}) \cup \{v\}] \cong H_{18}$ for each $v \in V(G_c) \backslash V(H_{15})$ by Lemma 4.11. If G_c has exactly 6 vertices then $G_c \cong H_v \cong H_{18}$ as desired. Otherwise, G_c has another vertex $u \neq v$ such that $H_u = G_c[V(H_{15}) \cup \{u\}] \cong H_{18}$ again by Lemma 4.11. Thus, $H_{v,u} = G_c[V(H_{15}) \cup \{v, u\}]$ contains induced H_v, H_u which are isomorphic to H_{18}. Comparing H_{18}, clearly $N_{H_v}(v), N_{H_u}(u) = \{v_1, v_3, v_4\}$, $\{v_1, v_2, v_5\}$, $\{v_1, v_2, v_3\}$, or $\{v_1, v_4, v_5\}$. By the symmetry of H_{15}, we only need to distinguish the following cases.

Case 1. If $N_{H_v}(v) = \{v_1, v_2, v_5\}$ and $N_{H_u}(u) = \{v_1, v_3, v_4\}$, then

$$G_c[V(H_{15}) \cup \{v, u\}] \cong \begin{cases} F_{11}, & \text{if } v \sim u \\ F_{12}, & \text{if } v \nsim u \end{cases} \quad \text{(see } F_{11}, F_{12} \text{ in Figure 7)}$$

However, F_{11} and F_{12} are forbidden subgraphs of G_c, a contradiction.

Case 2. If $N_{H_v}(v) = \{v_1, v_2, v_5\} = N_{H_u}(u)$, then $u \sim v$, since otherwise $H_{v,u} = G_c[V(H_{15}) \cup \{v, u\}] \cong F_{13}$ (see Figure 7), but $\mu_4(F_{13}) \neq -1$. Thus $u \rho v$ in $H_{v,u}$, and so $H_{v,u}$ is a proper subgraph of G_c. There exists $w \in V(G_c)$ such that $w \sim v$ but $w \nsim u$. Again by Lemma 4.11, $H_w = G_c[V(H_{15}) \cup \{w\}] \cong H_{18}$. Similarly, $N_{H_w}(w) = \{v_1, v_3, v_4\}$, $\{v_1, v_2, v_5\}$, $\{v_1, v_2, v_3\}$, or $\{v_1, v_4, v_5\}$. Now we consider $H_{w,v} = G_c[V(H_{15}) \cup \{w, v\}]$. Regarding $w = u$ we know that $N_{H_w}(w) = \{v_1, v_3, v_4\}$ should be eliminated because of the reason in Case 1. If $N_{H_w}(w) = \{v_1, v_2, v_3\}$ or $\{v_1, v_4, v_5\}$ then $H_{v,w} \cong F_{14}$ (see Figure 7), but $\mu_4(F_{14}) \neq -1$. At last, $N_{H_w}(w) = \{v_1, v_2, v_5\} = N_{H_v}(v) = N_{H_u}(u)$. It means $w \sim u$ by arguments above. It contradicts the selection of $w \nsim u$.

Case 3. If $N_{H_v}(v) = \{v_1, v_2, v_5\}$ and $N_{H_u}(u) = \{v_1, v_2, v_3\}$, then

$$H_{v,u} = G_c[V(H_{15}) \cup \{v, u\}] \cong \begin{cases} F_{14}, & \text{if } v \sim u \\ H_{19}, & \text{if } v \nsim u \end{cases} \quad \text{(see Figure 3)}$$

Since F_{14} is a forbidden subgraph, we have finished the argument if $H_{v,u} \cong G_c$. Otherwise, $H_{v,u}$ is a proper subgraph of G_c. There exists a vertex $w \neq v, u$ such that $H_w = G_c[V(H_{15}) \cup \{w\}] \cong H_{18}$ by Lemma 4.11. Similarly, $N_{H_w}(w) = \{v_1, v_2, v_5\}, \{v_1, v_2, v_3\}$, $\{v_1, v_3, v_4\}$ or $\{v_1, v_4, v_5\}$. However, the case of $N_{H_v}(v) =$

$\{v_1, v_2, v_5\} = N_{H_w}(w)$ (similarly, $N_{H_u}(u) = \{v_1, v_2, v_3\} = N_{H_w}(w)$) should be eliminated as in Case 2; the case of $N_{H_v}(v) = \{v_1, v_2, v_5\}$ and $N_{H_w}(w) = \{v_1, v_3, v_4\}$ (similarly, $N_{H_u}(u) = \{v_1, v_2, v_3\}$ and $N_{H_w}(w) = \{v_1, v_4, v_5\}$) should be eliminated as in Case 1. It is a contradiction.

Case 4. $N_{H_v}(v) = \{v_1, v_2, v_5\}$ and $N_{H_u}(u) = \{v_1, v_4, v_5\}$. The two graphs corresponding to $H_{v,u} = G_c[V(H_{15}) \cup \{v, u\}]$ will be isomorphic in the Cases of 3 and 4. Thus the Case 3 is equivalent to the Case 4.

The proof is complete. □

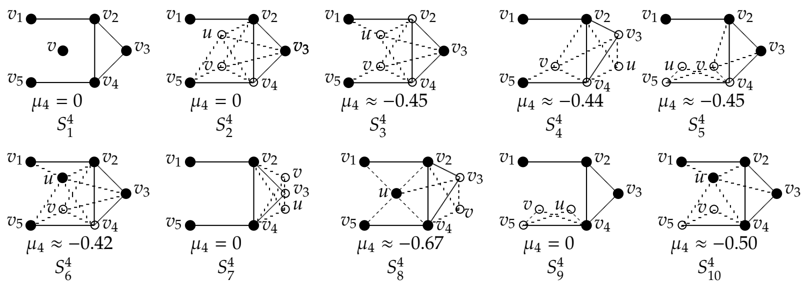

Figure 11: Forbidden subgraphs $\mu_4 \neq -1$.

Lemma 4.13. *Let $G_c \in \mathcal{G}_n^2([-1]^{n-5})$ contain an induced subgraph which is isomorphic to H_{16} and $H_v = G_c[V(H_{16}) \cup \{v\}]$ for $v \in V(G_c) \backslash V(H_{16})$. Then $H_v \cong H_{22}$.*

Proof. Obviously, the graph H_v has six vertices and $\mu_4(H_v) = -1$ by Lemma 4.2. Additionally, H_v will be connected, since otherwise $H_v \cong S_1^4$ (see Figure 11) but $\mu_4(S_1^4) = 0$. By using the Table A3 in [11] (also can using software SageMath 8.0 under the restriction of $\mu_4(G_c) = -1$), we find that $\Gamma_{13}, \Gamma_{14}, \Gamma_{15}$ and H_{22} are only four connected graphs on 6 vertices whose fourth largest eigenvalue equal -1 and each of them contains an induced subgraph isomorphic to H_{16}. Thus we have $H_v \in \{\Gamma_{13}, \Gamma_{14}, \Gamma_{15}, H_{22}\}$. It suffices to eliminate the graphs: Γ_{13}–Γ_{15}.

If $H_v \cong \Gamma_{13}$, then $v_4 \rho v$ in Γ_{13} (see Figure 5). Thus Γ_{13} is a proper subgraph of G_c, and we may assume that there exists $u \sim v_4$ but $u \nsim v$ such that $H_u = G_c[V(H_{16}) \cup \{u\}] \in \{\Gamma_{13}, \Gamma_{14}, \Gamma_{15}, H_{22}\}$ as above. Now $H_{v,u} = G_c[V(H_{16}) \cup \{v, u\}]$ consists of induced subgraphs isomorphic to Γ_{13} and H_u. From Figure 11, obviously, $H_{v,u}$ will be S_2^4 or S_3^4 if H_u takes Γ_{13} (where $H_v = H_u = \Gamma_{13}$ corresponds S_2^4; $H_v, H_u \cong \Gamma_{13}$ corresponds S_3^4), and $H_{v,u}$ will be S_4^4, S_5^4 and S_6^4 if H_u takes Γ_{14}, Γ_{15} and H_{22}, respectively. However, $S_2^4, S_3^4, S_4^4, S_5^4$ and S_6^4 are all forbidden induced subgraphs of G_c.

If $H_v \cong \Gamma_{14}$, then $v_3 \rho v$ in Γ_{14} (see Figure 5). Similarly as above, G_c has another vertex $u \sim v_3$ but $u \nsim v$ such that $H_u = G_c[V(H_{16}) \cup \{u\}] \in \{\Gamma_{13}, \Gamma_{14}, \Gamma_{15}, H_{22}\}$. Additionally, $\{H_v, H_u\} \neq \{\Gamma_{13}, \Gamma_{14}\}$ as above. Now $H_{v,u} = G_c[V(H_{16}) \cup \{v, u\}]$ contains induced subgraphs isomorphic to Γ_{14} and H_u. Since $u \nsim v_3$ in Γ_{15}, $H_u \ncong \Gamma_{15}$. Clearly, $H_{v,u}$ will be S_7^4 and S_8^4 if H_u takes Γ_{14} and H_{22}, respectively. However, S_7^4 and S_8^4 are all forbidden induced subgraphs of G_c.

If $H_v \cong \Gamma_{15}$, then $v_5 \rho v$ in Γ_{15} (see Figure 5). Similarly, G_c has another vertex $u \sim v_5$ but $u \nsim v$ such that $H_u = G_c[V(H_{16}) \cup \{u\}] \in \{\Gamma_{15}, H_{22}\}$ (Γ_{13}, Γ_{14} will be abandoned as above). Thus $H_{v,u} = G_c[V(H_{16}) \cup \{v, u\}]$ will be S_9^4 and S_{10}^4 if H_u takes H_{15} and H_{22}, respectively. However, S_9^4 and S_{10}^4 are all forbidden induced subgraphs of G_c.

The proof is complete. □

Theorem 4.14. *A graph $G \in \mathcal{G}_n^2([-1]^{n-5})$ contains an induced subgraph which is isomorphic to H_{16} if and only if its canonical graph $G_c \cong H_{16}$ or H_{22} .*

Proof. Assume that $G_c \cong H_{16}$ or H_{22}. Since H_{22} has an induced subgraph isomorphic to H_{16}, G_c has the induced subgraph isomorphic to H_{16}, and so has G.

Conversely, assume that G contains an induced subgraph which is isomorphic to H_{16}. By Lemma 2.3, G_c has induced subgraph isomorphic to H_{16}, and $G_c \cong H_{16}$ if $|V(G_c)| = 5$. If $|V(G_c)| \geq 6$ then $H_v = G_c[V(H_{16}) \cup \{v\}] \cong H_{22}$ for each $v \in V(G_c)\backslash V(H_{16})$ by Lemma 4.13. If G_c has exactly 6 vertices then $G_c \cong H_v \cong H_{22}$ as desired. Otherwise, G_c has another vertex $u \neq v$ such that $H_u = G_c[V(H_{16}) \cup \{u\}] \cong H_{22}$ again by Lemma 4.11. Thus $H_{v,u} = G_c[V(H_{16}) \cup \{v, u\}]$ contains induced subgraphs H_v and H_u. From Figure 3, we see that $N_{H_v}(v) = V(H_{16}) = N_{H_u}(u)$. If $v \nsim u$ then $H_{v,u} \cong F_{15}$ (see Figure 7), but $\mu_4(F_{15}) \neq -1$. Thus $v \sim u$ and $v\rho u$ in $H_{v,u}$. Since G_c is a primitive, there exists another vertex $w \neq u, v$. Again, $H_w = G_c[V(H_{16}) \cup \{w\}] \cong H_{22}$. Now $N_{H_w}(w) = V(H_{16}) = N_{H_v}(v) = N_{H_u}(u)$. We have $w \sim u$ by arguments above, however $w \nsim u$ by our choice. It implies that such u and w do not exist.

The proof is complete. □

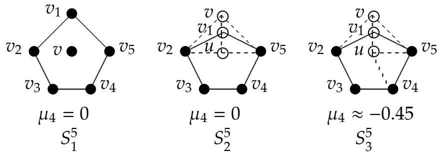

Figure 12: Forbidden subgraphs $\mu_4 \neq -1$.

Theorem 4.15. *A graph $G \in \mathcal{G}_n^2([-1]^{n-5})$ contains an induced subgraph which is isomorphic to H_{17} if and only if its canonical graph $G_c \cong H_{17}$.*

Proof. The sufficiency is obvious. For the necessity, let G contain an induced subgraph isomorphic to H_{17}. By Lemma 2.3, G_c has an induced subgraph isomorphic to H_{17}, and $G_c \cong H_{17}$ if $|V(G_c)| = 5$. If $|V(G_c)| \geq 6$, then $H_v = G_c[V(H_{17}) \cup \{v\}]$ for each $v \in V(G_c)\backslash V(H_{17})$, and thus $\mu_4(H_v) = -1$ by Lemma 4.2. Additionally, H_v will be connected, since otherwise $H_v \cong S_1^5$ (see Figure 12) but $\mu_4(S_1^5) = 0$. By using the Table A3 in [11] (also can using software SageMath 8.0 under the restriction of $\mu_4(G_c) = -1$), we find that Γ_{16}, shown in Figure 5, is the only connected graph of 6 vertices whose fourth largest eigenvalue equals -1 and contains an induced subgraphs isomorphic to H_{17}. Thus we have $H_v \cong \Gamma_{16}$. Obviously, Γ_{16} is imprimitive (in fact, $v_1 \rho v$ in Γ_{16} (see Figure 5)). However, since G_c is primitive, H_v should be a proper subgraph of G_c. There exists $u \in V(G_c)\backslash V(H_v)$ such that $H_u = G_c[V(H_{17}) \cup \{u\}] \cong \Gamma_{16}$ by the arguments above. Now the subgraph $H_{v,u} = G_c[V(H_{17}) \cup \{v, u\}]$ contains two induced subgraphs H_u, H_v which are all isomorphic to Γ_{16}. Furthermore, $H_{v,u}$ will be S_2^5 or S_3^5 if H_u takes Γ_{16} (in fact, $H_v = H_u \cong \Gamma_{16}$ corresponds S_2^5; $H_v, H_u \cong \Gamma_{16}$ corresponds S_3^5). However, S_2^5 and S_3^5 are the forbidden induced subgraphs of G_c.

The proof is complete. □

Finally, we obtain our main result below.

Theorem 4.16. *A graph $G \in \mathcal{G}_n([-1]^{n-5})$ if and only if its canonical graph G_c is isomorphic to H_i, for $1 \geq i \geq 23$(see H_1–H_{23} in Figure 2 and Figure 3).*

Proof. By definition we know that $\mathcal{G}_n([-1]^{n-5}) = \mathcal{G}_n^1([-1]^{n-5}) \cup \mathcal{G}_n^2([-1]^{n-5})$.

The Theorem 4.1 completely characterize $\mathcal{G}_n^1([-1]^{n-5})$, i.e., $G \in \mathcal{G}_n^1([-1]^{n-5})$ if and only if its canonical graph G_c is isomorphic to one of H_1–H_{11}.

By Lemma 4.4 we know that H_{12}–H_{17} are exactly six minimal graphs in $\mathcal{G}_n^2([-1]^{n-5})$, i.e, G must contain at least one induced subgraph which is isomorphic to one of H_{12}–H_{17} if $G \in \mathcal{G}_n^2([-1]^{n-5})$. Thus, by Theorems 4.7–4.15, we know that G contains an induced subgraph isomorphic to one of H_{12}–H_{17} if and only if its canonical graph is isomorphic to one of H_{12}–H_{23}.

The proof is complete. □

Acknowledgement We would like to thank the referees and the editors for very careful reading and for helpful comments, which helped us to improve the manuscript.

References

[1] B. Borovićanin, I. Gutman, Nullity of Graphs, in D. Cvetković, I. Gutman (Eds.), Applications of Graph Spectra, Mathematical Institute, Belgrade, 2009, pp. 107–122.
[2] W. G. Bridges, R. A. Mena, Multiplicative cones – a family of three eigenvalue graphs, Aequ. Math. 22 (1981) 208–214.
[3] D. de Caen, E. R. van Dam, E. Spence, A nonregular analogue of conference graphs, J. Combin. Theory Ser. A 88 (1999) 194–204.
[4] M. Cámara, W. H. Haemers, Spectral characterizations of almost complete graphs, Discrete Appl. Math. 176 (2014) 19–23.
[5] G. J. Chang, L. H. Huang, H. G. Yeh, A characterization of graphs with rank 4, Linear Algebra Appl. 434 (2011) 1793–1798.
[6] G. J. Chang, L. H. Huang, H. G. Yeh, A characterization of graphs with rank 5, Linear Algebra Appl. 436 (2012) 4241–4250.
[7] X. M. Cheng, A. L. Gavrilyuk, G. R. W. Greaves, J.H. Koolen, Biregular graphs with three eigenvalues, Europ. J. Combin. 56 (2016) 57–80.
[8] B. Cheng, B. Liu, On the nullity of graphs, Electron. J. Linear Algebra 16 (2007) 60–67.
[9] S. M. Cioabă, W. H. Haemers, J. R. Vermette, W. Wong, The graphs with all but two eigenvalues equal to ± 1, J. Algebraic Combin. 41(3) (2015) 887–897.
[10] S. M. Cioabă, W. H. Haemers, J. R. Vermette, The graphs with all but two eigenvalues equal to -2 or 0, Des. Codes Cryptogr. 84(1–2) (2017) 153–163.
[11] D. Cvetković, P. Rowlinson, S. Simić, An introduction to the theory of graph spectra, Cambridge University Press, Cambridge, 2010.
[12] E. R. van Dam, Regular graphs with four eigenvalues, Linear Algebra Appl. 226–228 (1995) 139–163.
[13] E. R. van Dam, Nonregular graphs with three eigenvalues, J. Combin. Theory Ser. B 73 (1998) 101–118.
[14] E. R. van Dam, J.H. Koolen, Z.J. Xia, Graphs with many valencies and few eigenvalues, Electron. J. Linear Algebra 28 (2015) 12–24.
[15] E. R. van Dam, E. Spence, Small regular graphs with four eigenvalues, Discrete Math. 189 (1998) 233–257.
[16] E. R. van Dam, E. Spence, Combinatorial designs with two singular values I: uniform multiplicative designs, J. Comb. Theory Ser. A 107 (2004) 127–142.
[17] E. R. van Dam, E. Spence, Combinatorial designs with two singular values II. Partial geometric designs, Linear Algebra Appl. 396 (2005) 303–316.
[18] M. Doob, Graphs with a small number of distinct eigenvalues, Ann. New York Acad. Sci. 175 (1970) 104–110.
[19] X. Y. Huang, Q. X. Huang, On regular graphs with four distinct eigenvalues, Linear Algebra Appl. 512 (2017) 219–233.
[20] L. S. de Lima, A. Mohammadian, C.S. Oliveira, The non-bipartite graphs with all but two eigenvalues in $[-1, 1]$, Linear Multilinear Algebra 65(3) (2017) 526–544.
[21] M. Muzychuk, M. Klin, On graphs with three eigenvalues, Discrete Math. 189 (1998) 191–207.
[22] M. R. Oboudi, On the third largest eigenvalue of graphs, Linear Algebra Appl. 503 (2016) 164–179.
[23] M. Petrović, On graphs with exactly one eigenvalue less than -1, J. Combin. Theory Ser. B 52 (1991) 102–112.
[24] P. Rowlinson, On graphs with just three distinct eigenvalues, Linear Algebra Appl. 507 (2016) 462–473.
[25] I. Sciriha, On the construction of graphs of nullity one, Discrete Math. 181(1-3) (1998) 193–211.
[26] I. Sciriha, A characterization of singular graphs, Electron. J. Linear Algebra 16 (2007) 451–462.
[27] S. S. Shrikhande, Bhagwandas, Duals of incomplete block designs, J. Indian. Stat. Assoc. 3 (1965) 30–37.
[28] W. Rudin, Real and Complex Analysis, (3rd edition), McGraw-Hill, New York, 1986.
[29] J. A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1967) 145–174.
[30] P. Erdös, S. Shelah, Separability properties of almost-disjoint families of sets, Israel Journal of Mathematics 12 (1972) 207–214.