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# On the Conditional Edge Connectivity of Double-Orbit Graphs

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**Abstract.** The double-orbit graph is a generalization of vertex transitive graphs, which contains many classic network models. Conditional edge-connectivity is an important index to measure the fault-tolerance and reliability of the networks. In this paper, we characterize the super-$\lambda^{(2)}$ double-orbit graphs with two orbits of the same size. Moreover, we give a sufficient condition for regular double-orbit graphs to be $\lambda^{(3)}$-optimal, and characterize super-$\lambda^{(3)}$ regular double-orbit graphs.

## 1. Introduction

It is well-known that the network of a multiprocessor system can be represented by a graph, and the traditional connectivity is an important measure for networks, which can correctly reflect the fault-tolerance of systems with few processors. However, with the development of VLSI technology and software technology, multiprocessor systems with hundreds of thousands of processors have become available, and the traditional connectivity always underestimates the resilience of large networks. There is a discrepancy because the occurrence of events which would disrupt a large network after a few processor or link failures is highly unlikely, therefore, the disruption envisaged occurs in a worst-case scenario. To the design and maintenance purpose of multiprocessor systems, appropriate measure of reliability should be found. Harary [8] introduced the concept of conditional connectivity, and Fàbrega and Fiol [5, 6] proposed a new kind of conditional connectivity, called $k$-extra edge-connectivity defined as follows.

For a positive integer $k$, the *$k$-extra edge-connectivity* of $G$, denoted by $\lambda^{(k)}(G)$, is the minimum cardinality of a set of edges (named as *$k$-extra edge-cut*) of $G$, if any, whose deletion disconnects $G$, and every remaining component has at least $k$ vertices. Clearly, $\lambda^{(1)}(G) = \lambda(G)$ and $\lambda^{(2)}(G) = \lambda'(G)$ is the traditional *edge-connectivity* and the *restricted edge-connectivity* proposed by Esfahanian and Hakimi [4], respectively. We refer to [3,12,13,17,19,21,23,25] for the studies of restricted edge-connectivity. Known results about the existence of $\lambda^{(k)}(G)$ are referred to [2,4,26], and the graphs in which $\lambda^{(k)}(G)$ exists are said to be *$\lambda^{(k)}$-connected*. Zhang and Yuan [26] showed that for any graph $G$ with $|V(G)| \geq 2(\delta(G) + 1)$, $\lambda^{(k)}(G) \leq \xi_k(G) = \min\{|\omega(X)| : X \subset V(G), |X| = k \text{ and } G[X] \text{ is connected}\}$ holds for any $k \leq \delta(G) + 1$, except for a graph which consists of some copies of $K_{\delta(G)}$ and a vertex $u$ which is adjacent to all the vertices in those copies, where $\omega(X) = [X, \overline{X}]$ denotes the set of edges between $X$ and $\overline{X}$ in $G$, $G[X]$ is the subgraph of $G$ induced by $X$, $\overline{X} = V(G) \setminus X$, and $\delta(G)$ is the minimum degree of $G$. Thus, the graphs with $\lambda^{(k)}(G) = \xi_k(G)$ are called *$\lambda^{(k)}$-optimal*.

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It seems that the larger $\lambda^{(k)}(G)$ is, the more reliable the network is. Thus, $\lambda^{(k)}$-optimal graphs have received quit a lot of attention [9,11,12,16,17,23,24,27]. To have a more refined measurement for the fault-tolerance and reliability of the networks, a stronger concept is proposed. A graph $G$ is *super k-extra edge-connected* (*super-$\lambda^{(k)}$* for short), if every minimum $k$-extra edge-cut isolates at least one component of order $k$. Not only does this concept gives us the cardinality of $\lambda^{(k)}(G)$, but also it shows the construction of minimum $k$-extra edge-cut.

In the design of network topology, highly symmetric graphs are popular due to their desirable properties. It is proved that vertex transitive graph (digraph) has maximum edge-(arc-)connectivity [15,18,20]. And vertex transitive graph also has many good properties concerning with $\lambda^{(k)}(G)$ for $2 \le k \le 3$, for example [16,17,23,27]. It is then natural to consider the relationship between connectivity and the number of orbits and we refer to [11-14,19,24]. In fact, half vertex-transitive graphs [22], Bi-Cayley graphs [22] and mixed Cayley graphs [3] all have at most two orbits. This motivates us to study $\lambda^{(k)}(G)$ of double-orbit graphs.

Let $Aut(G)$ denote the automorphism group of $G$. For $x \in V(G)$, the set $\{g(x) : g \in Aut(G)\}$ is an orbit of $Aut(G)$; we abuse the terminology a little to call it an orbit of $G$. Let $W$ be a subgroup of the symmetric group over a set $S$. We say that $W$ acts transitively on a subset $T$ of $S$ if for any $h, l \in T$, there exists a permutation $\varphi \in W$ with $\varphi(h) = l$. Clearly, $Aut(G)$ acts transitively on each orbit of $Aut(G)$. A graph $G$ is called *double-orbit graph* if $G$ has exactly two orbits. We use $G = (G_1, G_2, (V_1, V_2))$ to denote a double-orbit graph, where $G_i$ is the subgraph of $G$ induced by the two orbits $V_i$, $i = 1, 2$, and $G(V_1, V_2)$ is the subgraph of $G$ induced by $E(G) \setminus E(G_1) \cup E(G_2)$.

The remainder of this paper is organized as follows. Section 2 introduces the notations and definitions used throughout the paper. Note that in 2018, Lin and Yang [14] studied super restricted edge-connectivity of double-orbit graphs. However, the proof in that paper contains a crucial flaw (mainly in the proof of [14, Lemma 2]) and the main result is incorrect. In Section 3, we make a correction to their result. In Section 4, we present a sufficient condition for a regular double-orbit graph to be $\lambda^{(3)}$-optimal. In Section 5, we characterize the super-$\lambda^{(3)}$ regular double-orbit graphs.

## 2. Preliminaries

For graph-theoretical terminology and notation not given here, we follow [1, 7]. Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of $G$ is the size of $|V(G)|$. We use $d_G(v)$ and $N_G(v)$ to denote the degree and neighbor set of vertex $v \in V(G)$ of $G$, respectively. And $N_G(A) = (\bigcup_{v\in A} N_G(v)) \setminus A$ for $A \subseteq V(G)$. We will usually omit subscript $G$ when no confusion can arise. $A$ is called an *independent set* of $G$ if no two vertices of $A$ are adjacent in $G$. The length of a shortest cycle of $G$ is called its *girth*, denoted by $g(G)$. For a bijection $\alpha \in Aut(G)$, we define $\alpha(X) = \{\alpha(x) : x \in X\}$ for $\emptyset \neq X \subset V(G)$. We use $K_{1,n-1}$, $P_n$, $C_n$, $K_n$ to denote the star, the path, the cycle and the complete graph of order $n$, respectively.

Let $X$ be a proper subset of $V(G)$. If $\omega(X)$ is a minimum restricted edge-cut of $G$, then $X$ is called a *$\lambda'$-fragment* of $G$. Clearly, if $X$ is a $\lambda'$-fragment, so is $\overline{X}$, and both $G[X]$ and $G[\overline{X}]$ are connected. We call a $\lambda'$-fragment $X$ *strict*, if $3 \le |X| \le |V(G)| - 3$. If $G$ contains strict $\lambda'$-fragments, then the ones with smallest cardinality are called *$\lambda'$-superatom*.

The concept of *$\lambda^{(3)}$-fragment* can be defined similarly to $\lambda'$-fragment. A $\lambda^{(3)}$-fragment with the least cardinality is called a *$\lambda^{(3)}$-atom*. We call a $\lambda^{(3)}$-fragment trivial if it contains exactly three vertices. A non-trivial $\lambda^{(3)}$-fragment with minimum cardinality is called a *$\lambda^{(3)}$-superatom*.

## 3. Super restricted edge-connectivity

In this section, we always assume that $|V_1| = |V_2|$, $G[V_i]$ is $k_i$-regular for $i = 1, 2$ and $G(V_1, V_2)$ is $d$-regular. For convenience, we write $\xi(G) = \xi_2(G)$.

**Lemma 3.1** ([24]). *Let $G = (G_1, G_2, (V_1, V_2))$ be a connected double-orbit graph with two orbits $V_1$ and $V_2$ such that $|V_1| = |V_2| \ge 3$ and $g(G) \ge 5$. Then $G$ is $\lambda'$-optimal.*

**Lemma 3.2.** *Let $G = (G_1, G_2, (V_1, V_2))$ be a connected double-orbit graph with two orbits $V_1$ and $V_2$ such that $|V_1| = |V_2| \geq 3$ and $k_1 \leq k_2$. For any $\lambda'$-superatom $X$ of $G$ and $A \subset X$ with $|A| \geq 3$, suppose $g(G) \geq 5$, $G[A]$ is not connected and $G[\overline{A}]$ is connected. Then $|\omega(A)| > \lambda'(G)$ holds if one of the following conditions is satisfied:*

(i) *$E(G_1) \neq \emptyset$ and $E(G_2) \neq \emptyset$;*

(ii) *$A \not\subseteq V_1$ when $E(G_1) = \emptyset$ or $E(G_2) = \emptyset$.*

*Proof.* If $G[A]$ contains a component $U$ with $|V(U)| \geq 3$, then by the fact that $G[X]$ and $G[\overline{A}]$ are connected, it is clear that $G[\overline{I}]$ is connected, where $I = V(U)$. So $\omega(I)$ is a restricted edge-cut with $|I| \geq 3$ and $|\omega(I)| \geq \lambda'(G)$. Because $X$ is a $\lambda'$-superatom and $I \subset X$, we obtain $|\omega(A)| \geq |\omega(I)| > \lambda'(G)$. Otherwise, we assume that every component in $G[A]$ is an isolated edge or an isolated vertex. If there is an isolated edge in $G[A]$, then by $|A| \geq 3$, we have $|\omega(A)| \geq \xi(G) + \lambda(G) > \lambda'(G)$. Thus, now assume that all components in $G[A]$ are isolated vertices.

If $E(G_1) \neq \emptyset$ and $E(G_2) \neq \emptyset$, then

$$|\omega(A)| \geq 3\delta(G) = 3(k_1 + d) > 2(k_1 + d) - 2 = \xi(G) = \lambda'(G).$$

If $E(G_1) = \emptyset$ or $E(G_2) = \emptyset$, by $k_1 \leq k_2$, we may assume that $E(G_1) = \emptyset$. Since $A \not\subseteq V_1$, we have $A \subseteq V_2$ or $A \cap V_i \neq \emptyset$ for $i = 1, 2$. If $A \subseteq V_2$, then

$$|\omega(A)| \geq 3(k_2 + d) > k_2 + 2d - 2 = \xi(G) = \lambda'(G).$$

If $A \cap V_i \neq \emptyset$, by $|A| \geq 3$, we obtain that

$$|\omega(A)| \geq 2d + (k_2 + d) > k_2 + 2d - 2 = \xi(G) = \lambda'(G).$$

Thus, the lemma follows. □

**Remark 3.3.** In the above lemma, if we replace the $\lambda'$-superatom $X$ by $\overline{X}$, then we can obtain that $|\omega(A)| \geq \lambda'(G)$ under the same condition.

Let $\mathcal{F}_1 = \{G : G$ is a connected double-orbit graph with $|V_1| = |V_2| \geq 3$ and $g(G) \geq 5$, $d = 1, k_1 = 0, k_2 \geq 2$ and $G_2$ is not super-$\lambda$ or $d = 2, k_1 = 0$ and $k_2 \geq 1\}$. In fact, we can show that, for any $G \in \mathcal{F}_1$, $G$ is not super-$\lambda'$. For the case $d = 1, k_1 = 0, k_2 \geq 2$ and $G_2$ is not super-$\lambda$, there exists some minimum edge-cut $S$ of $G_2$ such that every component of $G_2 - S$ contains at least 2 vertices. As $d = 1$ and $k_1 = 0$, $S$ is also a restricted edge-cut of $G$, which implies that $k_2 = \lambda'(G) \leq |S| = \lambda(G_2) = \delta(G_2) = k_2$. We see that $S$ is a minimum restricted edge-cut of $G$ such that $G - S$ contains no isolated edges, implying that $G$ is not super-$\lambda'$. For the case $d = 2, k_1 = 0$ and $k_2 \geq 1$, let $X \subseteq V(G)$ and $G[X] \cong K_{1,2}$, where $|X \cap V_1| = 2$ and $|X \cap V_2| = 1$. Clearly, $\lambda'(G) = \xi(G) = |\omega(X)| = k_2 + 2$. Since $d = 2$ and $g(G) \geq 5$, we have that $|\overline{X} \cap V_2| \geq 2$ and $|\overline{X}| \geq 3$. Suppose $G[\overline{X}]$ is not connected. If each component of $G[\overline{X}]$ is a single vertex, then $|\omega(\overline{X})| \geq 2(k_2 + d) + d > |\omega(X)|$, a contradiction. If $G[\overline{X}]$ contains a component of order at least 2, then $|\omega(\overline{X})| \geq \lambda'(G) + \lambda(G) > |\omega(X)|$, also a contradiction. Thus, $G[\overline{X}]$ is connected and $X$ is a $\lambda'$-superatom, implying that $G$ is not super-$\lambda'$.

**Lemma 3.4.** *Let $G = (G_1, G_2, (V_1, V_2)) \notin \mathcal{F}_1$ be a connected double-orbit graph with two orbits $V_1$ and $V_2$ such that $|V_1| = |V_2| \geq 3$ and $k_1 \leq k_2$. Suppose $g(G) \geq 5$ and $G$ is not super-$\lambda'$. If $X$ and $Y$ are two distinct $\lambda'$-superatoms with $\alpha(X) = Y$, where $\alpha \in Aut(G)$, then $X \cap Y = \emptyset$.*

*Proof.* Set $A = X \cap Y$, $B = X \cap \overline{Y}$, $C = \overline{X} \cap Y$ and $D = \overline{X} \cap \overline{Y}$. By contradiction, suppose $A \neq \emptyset$. Since $\|[A, \overline{X}]\| + \|[A, \overline{Y}]\| = \|[A, C]\| + \|[A, D]\| + \|[A, B]\| + \|[A, D]\|$, either $\|[A, \overline{X}]\| \geq \|[A, B]\|$ or $\|[A, \overline{Y}]\| \geq \|[A, C]\|$. In the following, without loss of generality, assume $\|[A, \overline{X}]\| \geq \|[A, B]\|$.

By the definition of $\lambda'$-superatom, $G[X]$, $G[\overline{X}]$, $G[Y]$ and $G[\overline{Y}]$ are all connected. Thus, $G[X \cup Y]$ and $G[\overline{X} \cup \overline{Y}]$ are connected since $A \neq \emptyset$ and $D \neq \emptyset$.

**Claim 1.** $|A| \leq 2$.

Suppose $|A| \geq 3$, we will derive a contradiction by two facts.

Fact 1. $|\omega(A)| \geq \lambda'(G)$.

If $G[A]$ is connected, then $|\omega(A)| \geq \lambda'(G)$ since $G[\overline{A}] = G[\overline{X} \cup \overline{Y}]$ is connected. If $G[A]$ is not connected, we claim that $A \nsubseteq V_1$ when $E(G_1) = \varnothing$ or $E(G_2) = \varnothing$.

Suppose to the contrary that $A \subseteq V_1$ when $E(G_1) = \varnothing$ or $E(G_2) = \varnothing$. Since $k_1 \leq k_2$, we may assume $E(G_1) = \varnothing$. By $\|[A, \overline{X}]\| \geq \|[A, B]\|$, there exists a vertex $x \in A$ such that

$$\|[x, B]\| \leq \|[x, C]\| + \|[x, D]\|.$$

Then

$$|\omega(X \setminus x)| = |\omega(X)| + \|[x, B]\| - \|[x, C]\| - \|[x, D]\| \leq |\omega(X)| = \lambda'(G).$$

Since $G[X]$ is connected and $A$ is an independent set, we have $\|[x, B]\| \geq 1$, which implies that $G[\overline{X} \cup \{x\}]$ is connected. We now show that every component in $G[X \setminus x]$ has at least 3 vertices. In fact, if there exists an isolated edge in $G[X \setminus x]$, by $|X \setminus x| = |X| - 1 = |A| + |B| - 1 \geq 3$, we have $|\omega(X \setminus x)| \geq \xi(G) + \lambda(G) > \lambda' G)$, a contradiction. If there is an isolated vertex $y$ in $G[X \setminus x]$, then $x$ is the only neighbor of $y$ in $G[X]$, and thus $d_{G[X]}(y) = 1$. Set $X' = X \setminus y$. Since $x \in A \subseteq V_1$, $E(G_1) = \varnothing$ and $y \in N_G(x)$, we have $y \in V_2$. If $d = 1$, then $xy$ is an isolated edge in $G[X]$, contradicting the fact that $X$ is a $\lambda'$-superatom. Hence, $d \geq 2$. By $d_G(y) = 1 + \|[y, \overline{X}]\| \geq d \geq 2$, we see that $G[\overline{X'}]$ is connected, which implies that

$$\lambda'(G) \leq |\omega(X')| = |\omega(X)| - \|[y, \overline{X}]\| + 1 \leq |\omega(X)| = \lambda'(G).$$

A contradiction arises, since $X' \subset X$ is a smaller strict $\lambda'$-fragment than $X$.

Now, for any component $U_1$ in $G[X \setminus x]$, set $I_1 = V(U_1)$, then $|I_1| \geq 3$ and

$$\lambda'(G) \leq |\omega(I_1)| \leq |\omega(X \setminus x)| \leq \lambda'(G),$$

which implies that $I_1$ is a smaller strict $\lambda'$-fragment contained in $X$, again a contradiction. By Lemma 3.2, Fact 1 follows.

Fact 2. $|\omega(D)| \geq \lambda'(G)$.

Note that $|D| = |V \setminus (X \cup Y)| = |V| - |X| - |Y| + |X \cap Y|$ and $|X| = |Y| \leq \frac{|V(G)|}{2}$, we have $|D| \geq |A| \geq 3$. In the following, we only need to show that $|\omega(D)| \geq \lambda'(G)$ if $G[D]$ is not connected.

Suppose $D \subseteq V_1$ when $E(G_1) = \varnothing$. By the assumption $\|[A, \overline{X}]\| \geq \|[A, B]\|$, we have

$$|\omega(B)| = \|[B, A]\| + \|[B, C]\| + \|[B, D]\| \leq |\omega(X)| = \lambda'(G) = \xi(G) = k_2 + 2d - 2.$$

If $G[B]$ contains a component $U_2$ with $|V(U_2)| \geq 3$, then by a similar argument used in the proof of Fact 1, $V(U_2)$ is a strict $\lambda'$-fragment contained in $\lambda'$-superatom $X$, also a contradiction. Thus, every component in $G[B]$ is isomorphic to $K_1$ or $K_2$. By $|\omega(B)| \leq \lambda'(G) = k_2 + 2d - 2$, we see that one of the following cases occurs: (i) $G[B]$ is an edge belonging to $G(V_1, V_2)$, (Suppose it belongs to $G_2$, then $k_2 \neq 0$. But, in fact, by $k_2 + 2d - 2 \geq |\omega(B)| = 2(k_2 + d) - 2$, it follows that $k_2 = 0$, a contradiction.) or (ii) $B = \{y_1\}$ ($y_1 \in V_2$), or (iii) $B$ consists of some vertices in $V_1$.

Let $D = \{x_1, x_2, ..., x_s\}$, then $s \geq 3$. Since $\overline{Y} = B \cup D$, $G[\overline{Y}]$ is connected and $D \subseteq V_1$, case (iii) can not occur.

If (i) holds, let $B = \{x_0, y_0\}$ ($x_0 \in V_1, y_0 \in V_2$). Since $\alpha(X) = Y$, set $C = \{x'_0, y'_0\}$ and $x'_0 \in V_1, y'_0 \in V_2$. Since $G[\overline{X}]$ and $G[\overline{Y}]$ are connected, it follows that $x_i \in N_G(y_0)$, $x'_0 \cup x_i \in N_G(y'_0)$ for $1 \leq i \leq s$. Note that $g(G) \geq 5$, we have $s = 1$, a contradiction.

If (ii) holds, by $\alpha(X) = Y$, set $C = \{y'_1\}$ and $y'_1 \in V_2$. Similarly, $x_i \in N_G(y_1)$ and $x_i \in N_G(y'_1)$ for $1 \leq i \leq s$. Notice that $g(G) \geq 5$, $s = 1$, again a contradiction. Thus, $D \nsubseteq V_1$ when $E(G_1) = \varnothing$ or $E(G_2) = \varnothing$. Then by Remark 3.3, Fact 2 follows.

Combining Fact 1 with Fact 2, we obtain

$$2\lambda'(G) \leq |\omega(A)| + |\omega(D)| \leq |\omega(X)| + |\omega(Y)| = 2\lambda'(G).$$

It follows that $|\omega(A)| = \lambda'(G)$ and $G[A]$ is connected, which contradicts that $X$ is a $\lambda'$-superatom. Therefore, Claim 1 holds.

**Claim 2.** $|B| \leq 2$.

Suppose $|B| \geq 3$. Note that $|\omega(B)| \leq \lambda'(G) = \xi(G)$. If $G[B]$ is connected, then $|\omega(B)| = \lambda'(G)$ and $B$ is a strict $\lambda'$-fragment contained in $X$, a contradiction. If $G[B]$ is not connected, we show that $B \nsubseteq V_1$ when $E(G_1) = \emptyset$ or $E(G_2) = \emptyset$. By contradiction, let $k_1 = 0$ and $B = \{v_1, v_2, ..., v_l\} \subseteq V_1$, $l \geq 3$. By Claim 1, we have either $|A| = 1$ or $|A| = 2$.

If $|A| = 1$, set $A = \{u\}$. Since $G[X]$ is connected, then $u \in V_2$ and $v_i \in N_G(u)$ for $1 \leq i \leq l$. And so $d \geq l \geq 3$. We then have

$$\lambda'(G) = |\omega(X)| = \sum_{x\in X} d_G(x) - 2|E(G[X])| = k_2 + d + ld - 2l > k_2 + 2d - 2 = \xi(G),$$

a contradiction.

If $|A| = 2$, then since $G[X]$ is connected, we derive that $A$ satisfies one of the following conditions: (a) $G[A]$ is an edge belonging to $G(V_1, V_2)$, $\varepsilon = l + 1$ and $d \geq l + 1 \geq 4$, or (b) $G[A]$ is an edge belonging to $G_2$, $\varepsilon = l + 1$ (note that $g(G) \geq 5$), or (c) $G[A] = \{u_1, u_2\}$ and $u_1, u_2$ are independent vertices in $V_2$, $\varepsilon = l+1$, where $\varepsilon$ denotes the number of edges in $G[X]$. When (a) holds, $\lambda'(G) = |\omega(X)| = (l + 1)d + k_2 + d - 2(l + 1) > k_2 + 2d - 2 = \xi(G)$, a contradiction. If either (b) or (c) holds, $\lambda'(G) = |\omega(X)| = ld + 2(k_2 + d) - 2(l + 1)$. Since $G \notin \mathcal{F}_1$ and $G$ is not super-$\lambda'$, $d \geq 3$. Thus, $\lambda'(G) = |\omega(X)| > \xi(G)$, again a contradiction.

Thus, by Lemma 3.2, we have that $\lambda'(G) \geq |\omega(B)| > \lambda'(G)$, which is impossible. This completes the proof of Claim 2.

Now, we prove this lemma by considering the following two cases.

**Case 1.** $|A| = 1$.

By Claim 2, we have $|X| = 3$. Since $g(G) \geq 5$, we have $G[X] \cong P_3$. Note that $G \notin \mathcal{F}_1$ and $G$ is not super-$\lambda'$, then $\lambda'(G) = |\omega(X)| > \xi(G)$, which is impossible.

**Case 2.** $|A| = 2$.

In this case, $3 \leq |X| \leq 4$. If $|X| = 3$, as the proof of Case 1, we can get a contradiction. If $|X| = 4$, since $g(G) \geq 5$, we have $G[X] \cong P_4$ or $K_{1,3}$. Then, $\lambda'(G) = |\omega(X)| > \xi(G)$, again a contradiction. □

**Lemma 3.5** ([14]). *Let $G = (G_1, G_2, (V_1, V_2)) \notin \mathcal{F}_1$ be a connected double-orbit graph with two orbits $V_1$ and $V_2$ such that $|V_1| = |V_2| \geq 3$ and $k_1 \leq k_2$. Suppose $g(G) \geq 5$, $G$ is not super-$\lambda'$ and $A$ is a $\lambda'$-superatom of $G$. Then we have*

(i) *If $A \subseteq V_1$ (or $V_2$), then $V_1$ (or $V_2$) is a disjoint union of distinct $\lambda'$-superatoms and $G[A]$ is a vertex transitive graph;*

(ii) *If $A_i = A \cap V_i \neq \emptyset$ for $i = 1, 2$, then $V(G)$ is a disjoint union of distinct $\lambda'$-superatoms and $G[A]$ is a double-orbit graph, and $|A_1| = |A_2|$.*

**Lemma 3.6** ([14]). *Let $G = (G_1, G_2, (V_1, V_2))$ be a connected double-orbit graph with two orbits $V_1$ and $V_2$ such that $|V_1| = |V_2| \geq 3$. Suppose $g(G) \geq 4$ and $G$ is $\lambda'$-optimal but not super-$\lambda'$. Then for any $\lambda'$-superatom $A$ of $G$, we have*

$$|A| \geq \begin{cases} 2\delta(G) - 2 & if\ E(G_1) \neq \emptyset \text{ and } E(G_2) \neq \emptyset; \\ 2d + k_1 + k_2 - 2 & if\ E(G_1) = \emptyset \text{ or } E(G_2) = \emptyset. \end{cases}$$

**Lemma 3.7** ([10]). *If $G$ is a $k$-regular graph with girth $g$, then*

$$|V(G)| \geq n(k, g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{\frac{g-3}{2}} & if\ g \text{ is odd}; \\ 2(1 + k - 1 + \cdots + (k-1)^{\frac{g}{2}-1}) & if\ g \text{ is even}. \end{cases}$$

Now, we give a factor graph $G^*[A]$, shown in Fig. 1, to obtain a graph family $\mathcal{F}_2$, where $V(G^*[A]) = A = A_1 \cup A_2$ and $|A_1| = |A_2| = 4$, $A_1$ is an independent set, $G[A_2]$ consists of 2 copies of $K_2$, $G(A_1, A_2)$ is a 2-regular bipartite graph, and $\mathcal{F}_2$ denotes a family of 3-regular double-orbit graphs with $|V_1| = |V_2| \geq 3$, $g(G) \geq 5$,

$k_1 = k_2 = 1$ and $d = 2$. For any $G \in \mathcal{F}_2$ and some positive integer $m \geq 2$, $G$ is the disjoint union of $m$ copies of $G^*[A]$ together with a perfect matching between different copies of $A_1$. Such an example $G \in \mathcal{F}_2$, which consists of 3 copies of $G^*[A]$, is shown in Fig. 2.

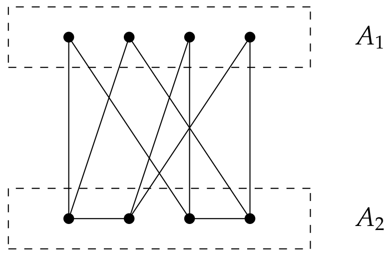


Fig.1. $G^*[A]$

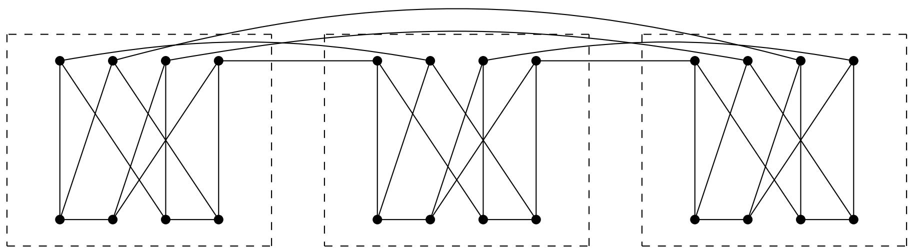

Fig.2. An example $G \in \mathcal{F}_2$

**Theorem 3.8.** *Let $G = (G_1, G_2, (V_1, V_2))$ be a connected double-orbit graph with two orbits $V_1$ and $V_2$ such that $|V_1| = |V_2| \geq 3$ and $k_1 \leq k_2$. If $g(G) \geq 5$, then $G$ is not super-$\lambda'$ if and only if $G \in \mathcal{F}_1 \cup \mathcal{F}_2$.*

*Proof.* If $G \in \mathcal{F}_1$, clearly $G$ is not super-$\lambda'$. If $G \in \mathcal{F}_2$, as $|\omega(A)| = 4 = \xi(G) = \lambda'(G)$, $G$ is not super-$\lambda'$. In the following, we prove the necessity by way of contradiction that $G$ is not super-$\lambda'$ and $G \notin \mathcal{F}_1 \cup \mathcal{F}_2$. Let $A$ be a $\lambda'$-superatom of $G$. We consider two cases.

**Case 1.** $E(G_1) \neq \emptyset$ and $E(G_2) \neq \emptyset$.

By Lemma 3.6, $|A| \geq 2\delta(G) - 2$. We claim that $A \not\subseteq V_1$ and $A \not\subseteq V_2$. If, to the contrary, $A \subseteq V_1$, then by Lemmas 3.1 and 3.6, $2\delta(G) - 2 = \lambda'(G) = |\omega(A)| = |A|d + |\omega_{G_1}(A)| \geq (2\delta(G) - 2)d + |\omega_{G_1}(A)|$. Thus we have $d = 1$ and $|\omega_{G_1}(A)| = 0$. Then, $|A| = 2\delta(G) - 2 = 2k_1$. On the other hand, by Lemma 3.7, $2k_1 = |A| \geq n(k_1, 5) = 1 + k_1^2$, thus $k_1 = 1$. Then $|A| = 2$, contradicting that $A$ is a $\lambda'$-superatom of $G$. Similarly, $A \not\subseteq V_2$. It follows that $A_i = A \cap V_i \neq \emptyset$ for $i = 1, 2$.

By Lemma 3.5, we have $|A_1| = |A_2| \geq \frac{2\delta(G)-2}{2} = \delta(G) - 1$. We proceed to prove by way of contradiction that exactly one of $A_1$ and $A_2$ has neighbors in $\overline{A}$. In fact, if to the contrary, each vertex of $A$ has a neighbor in $\overline{A}$, as $G$ is $\lambda'$-optimal, we have $2\delta(G) - 2 = |\omega(A)| \geq |A| = |A_1| + |A_2| \geq 2\delta(G) - 2$. Thus $|A_1| = |A_2| = \delta(G) - 1$, and each vertex of $A$ has exactly one neighbor in $\overline{A}$. Let $t = k_1 + d$. Then it is easy to see that $t = \delta(G) = |A_1| + 1 \geq 3$. By Lemma 3.7, we have $|A| \geq n(t-1, 5) = 1 + (t-1)^2 = t^2 - 2t + 2 > 2t - 2 = |A|$ for all $t \geq 3$, a contradiction. We consider two subcases.

**Subcase 1.1.** $N(A_1) \cap \overline{A} \neq \emptyset$ and $N(A_2) \cap \overline{A} = \emptyset$.

In this case, $N(A_1) \cap \overline{A} \subseteq V_1$. Since $|A_1| = |A_2| \geq \delta(G) - 1$, we see that each vertex of $A_1$ has at most 2 neighbors in $\overline{A}$.

**Subcase 1.1.1.** Each vertex of $A_1$ has exactly one neighbor in $\overline{A}$.

Then $\delta(G[A]) = t - 1$, $|A_1| = |\omega(A)| = \lambda'(G) = \xi(G) = 2t - 2$ and $|A| \geq n(t-1, 5) = t^2 - 2t + 2$. Hence, if $t \geq 5$, then $\lambda'(G) = |\omega(A)| = |A_1| = \frac{|A|}{2} \geq \frac{t^2-2t+2}{2} > 2t - 2 = \xi(G)$, a contradiction. Since $d \geq 1$ and $k_1 \geq 1$, we have $2 \leq t \leq 4$.

If $t = 2$, then $|A_1| = |A_2| = 2t - 2 = 2$, $k_1 = d = 1$ and $k_2 \geq 1$. Since $N(A_2) \cap \overline{A} = \emptyset$ and $|A_2| = 2$, we see that $k_2 \leq 1$. Thus, $k_2 = 1$. Then $G$ is a connected 2-regular graph and thus $G$ is isomorphic to a cycle, which contradicts that $G$ is not vertex transitive.

If $t = 3$, then $|A_1| = |A_2| = 4$. By Lemma 3.5 and the assumption that $N(A_2) \cap \overline{A} = \emptyset$, we see that $G[A_2]$ is a $k_2$-regular vertex transitive graph of order 4. Since $g(G) \geq 5$, then $k_2 = 1$ and $G[A_2]$ consists of two copies of $K_2$. Thus $k_1 = 1$, $d = 2$, $A_1$ is an independent set and $G(A_1, A_2)$ is 2-regular. For any two distinct vertices $x, y$ in $A_1$, the two neighbors of $x$ in $A_2$ lie in different copies of $K_2$ and $|N_{G[A_2]}(x) \cap N_{G[A_2]}(y)| \leq 1$. Thus $G[A] \cong G^*[A]$ and $G \in \mathcal{F}_2$, a contradiction.

If $t = 4$, then $|A_1| = |A_2| = 6$. Since $G[A_2]$ is a $k_2$-regular graph of order 6 and $g(G[A_2]) \geq 5$, we have $1 \leq k_2 \leq 2$. If $k_2 = 1$, then $k_1 = 1$ and $d = 3$. But then $g(G[A]) \leq 4$, contradicting that $g(G) \geq 5$. If $k_2 = 2$, $G[A_2] \cong C_6$ and we can also obtain that $g(G[A]) \leq 4$, again a contradiction.

**Subcase 1.1.2.** Each vertex of $A_1$ has exactly two neighbors in $\overline{A}$.

Then $\delta(G[A]) = t - 2$, $2|A_1| = |\omega(A)| = \lambda'(G) = \xi(G) = 2t - 2$ and $|A| \geq n(t - 2, 5) = t^2 - 4t + 5$. Thus, if $t \geq 5$, then $\lambda'(G) = |\omega(A)| = 2|A_1| = |A| \geq t^2 - 4t + 5 > 2t - 2 = \xi(G)$, a contradiction. Clearly, $k_1 \geq 2$. Hence, $3 \leq t \leq 4$.

If $t = 3$, then $|A_1| = |A_2| = t - 1 = 2$, $k_1 = 2$ and $d = 1$. But $k_2 \geq k_1 = 2$, which is impossible, since $G[A_2]$ is a $k_2$-regular graph of order 2.

If $t = 4$, $|A_1| = |A_2| = 3$. Thus, $G[A_2] \cong C_3$, contradicting that $g(G) \geq 5$.

**Subcase 1.2.** $N(A_1) \cap \overline{A} = \emptyset$ and $N(A_2) \cap \overline{A} \neq \emptyset$.

In this case, we may assume $k_2 > k_1$. Similarly as above, each vertex of $A_2$ has at most two neighbors in $\overline{A}$. If each vertex of $A_2$ has exactly one neighbor in $\overline{A}$, then $\delta(G[A]) = t$ and $|A| \geq n(t, 5) = 1 + t^2$. But then $\lambda'(G) = |\omega(A)| = |A_2| = \frac{|A|}{2} \geq \frac{1+t^2}{2} > 2t - 2 = \xi(G)$, a contradiction. If each vertex of $A_2$ has exactly two neighbors in $\overline{A}$, then $\delta(G[A]) \geq t - 1$ and $|A| \geq n(t - 1, 5) = t^2 - 2t + 2$. When $t \geq 3$, we have $\lambda'(G) = |\omega(A)| = 2|A_2| = |A| \geq t^2 - 2t + 2 > 2t - 2 = \xi(G)$, a contradiction. When $t = 2$, $|A_1| = |A_2| = t - 1 = 1$, contradicting that $|A| \geq 3$.

**Case 2.** $E(G_1) = \emptyset$ or $E(G_2) = \emptyset$.

As $k_1 \leq k_2$, we may assume $E(G_1) = \emptyset$. Thus, $k_1 = 0$. By Lemma 3.6, we have $|A| \geq 2d + k_2 - 2$. Since $G[A]$ is connected, $A \nsubseteq V_1$. If $A \subseteq V_2$, by an argument similar to that of Case 1, we get a contradiction. Thus, $A_i = A \cap V_i \neq \emptyset$ for $i = 1, 2$.

**Subcase 2.1.** Each vertex of $A$ has a neighbor in $\overline{A}$.

Since $2d + k_2 - 2 = \xi(G) = \lambda'(G) = |\omega(A)| \geq |A| \geq 2d + k_2 - 2$, we have $|A_1| = |A_2| = \frac{|A|}{2} = d + \frac{k_2}{2} - 1$, and each vertex of $A$ has exactly one neighbor in $\overline{A}$. Thus, $k_2$ is even, $G(A_1, A_2)$ is $(d - 1)$-regular and $G[A_2]$ is $k_2$-regular. On the other hand, by $d + \frac{k_2}{2} - 1 = |A_2| \geq n(k_2, 5) = 1 + k_2^2$, then $d \geq k_2^2 - \frac{k_2}{2} + 2$. If $k_2 \geq 2$, then $d \geq 5$. For any vertex $v \in A_2$, $|N(v) \cap A_1| = d - 1$, $N(u_i) \cap N(u_j) \cap A_2 = \{v\}$, and $N(u_i) \cap N(v) \cap A_2 = \emptyset$, where $u_i \neq u_j \in N(v) \cap A_1$. Hence, we have $|A| = 2|A_2| \geq 2[(d-1)(d-2)+k_2+1] > 2d+k_2-2 = |A|$, a contradiction. If $k_2 = 0$, then $|A| = 2d - 2$. Thus, $G[A]$ is a $(d - 1)$-regular graph and $|A| \geq n(d - 1, 5) = 1 + (d - 1)^2 > 2d - 2 = |A|$ when $d \geq 3$, a contradiction. If $d = 2$, then $G$ is isomorphic to a cycle, also a contradiction. If $d = 1$, $G \cong K_2$, contradicting that $|V_1| = |V_2| \geq 3$.

If $A_1$ has neighbors in $\overline{A}$, then since $E(G_1) = \emptyset$, $A_2$ also has neighbors in $\overline{A}$. In view of Subcase 2.1, this is impossible.

**Subcase 2.2.** Only $A_2$ has neighbors in $\overline{A}$.

By Lemma 3.6, we see that each vertex of $A_2$ has at most 2 neighbors in $\overline{A}$. And these neighbors of $A_2$ lie in $V_2 \setminus A$. We consider the following two cases.

**Subcase 2.2.1.** Each vertex of $A_2$ has exactly one neighbor in $V_2 \setminus A$.

In this case, $2d + k_2 - 2 = \lambda'(G) = |\omega(A)| = |A_2|$. By a similar argument as above, we obtain that $|A_2| \geq k_2 + d(d - 1)$. If $d \geq 3$, $|A_2| \geq k_2 + d(d - 1) > 2d + k_2 - 2 = |A_2|$, a contradiction. Since $k_1 = 0$ and $G$ is a connected double-orbit graph which is not super-$\lambda'$, when $1 \leq d \leq 2$, $G \in \mathcal{F}_1$, a contradiction.

**Subcase 2.2.2.** Each vertex of $A_2$ has exactly two neighbors in $V_2 \setminus A$.

In this case, $k_2 \geq 2$ and $2|A_2| = |\omega(A)| = 2d + k_2 - 2$. Clearly, $|A_2| \geq k_2 + d(d - 1) - 1$. If $d \geq 2$, $|A| = 2|A_2| \geq 2[k_2 + d(d - 1) - 1] > 2d + k_2 - 2 = |A|$, a contradiction. When $d = 1$, $G \in \mathcal{F}_1$, also a contradiction. □

## 4. 3-Extra edge-connectivity

In this section, we will investigate 3-extra edge-connectivity for $k$-regular double-orbit graphs.

**Lemma 4.1** ([2]). *A $\lambda^{(3)}$-connected graph $G$ is $\lambda^{(3)}$-optimal if and only if the cardinality of any $\lambda^{(3)}$-atom is three.*

**Theorem 4.2** ([12]). *Let $G$ be a $k$-regular connected double-orbit graph with $k \geq 3$ and girth $g(G) \geq 6$. Then $G$ is $\lambda'$-optimal.*

**Lemma 4.3.** *Let $G$ be a $k$-regular connected double-orbit graph with $k \geq 3$ and girth $g(G) \geq 6$. If $G$ is not $\lambda^{(3)}$-optimal, then any two distinct $\lambda^{(3)}$-atoms are disjoint.*

*Proof.* We prove this lemma by contradiction. Assume that $X$ and $Y$ are two distinct $\lambda^{(3)}$-atoms of $G$ with non-empty intersection $A$, using the same notation as Lemma 3.4. Since $G$ is not $\lambda^{(3)}$-optimal, then $\lambda^{(3)}(G) < \xi_3(G) = 3k - 4$ and $|X|, |Y| \geq 4$. By Theorem 3.2, we have that $\lambda'(G) = \xi(G) = 2k - 2$ and thus $G$ is $\lambda$-optimal, i.e. $\lambda(G) = \delta(G)$. Clearly, $G[X \cup Y]$ and $G[\overline{X} \cup \overline{Y}]$ are connected.

**Claim 1.** $|A| \leq 2$.

Suppose $|A| \geq 3$. If $G[A]$ is connected, then $|\omega(A)| \geq \lambda^{(3)}(G)$. If $G[A]$ is not connected, we will show $|\omega(A)| > \lambda^{(3)}(G)$. In fact, if $G[A]$ contains exactly two components, then one of them has more than one vertex, and thus $|\omega(A)| \geq \lambda'(G) + \lambda(G) = \xi(G) + \delta(G) > 3k - 4 = \xi_3(G) > \lambda^{(3)}(G)$. Otherwise, we may assume that $G[A]$ contains at least three components, then $|\omega(A)| \geq 3\lambda(G) > \lambda^{(3)}(G)$.

Note that $|D| \geq |A| \geq 3$. By a similar argument as above, we have $|\omega(D)| \geq \lambda^{(3)}(G)$. By submodular inequality, we obtain that

$$2\lambda^{(3)}(G) \leq |\omega(A)| + |\omega(D)| \leq |\omega(X)| + |\omega(Y)| = 2\lambda^{(3)}(G).$$

It follows that $|\omega(A)| = \lambda^{(3)}(G)$ and $G[A]$ is connected, which contradicts that $X$ is a $\lambda^{(3)}$-atom. Therefore, Claim 1 holds.

**Claim 2.** $|A| = 2$.

Suppose $|A| = 1$. Since $|X|, |Y| \geq 4$, we have $|B|, |C| \geq 3$. By an argument similar to the proof of Claim 1, we can always derive a contradiction that $B$ is a $\lambda^{(3)}$-fragment contained in X.

**Claim 3.** $|X| = 4$.

Suppose $|X| > 4$, then $|B| = |X| - |A| \geq 3$ and $|C| = |Y| - |A| \geq 3$. By a similar argument to that of the proof of Claim 1, we can obtain a contradiction.

Since $|X| = 4$ and $G[X]$ is a connected subgraph with $g(G[X]) \geq 6$, we have $G[X] \cong P_4$ or $K_{1,3}$, which gives that $4k - 6 = |\omega(X)| = \lambda^{(3)}(G) < \xi_3(G) = 3k - 4$ and $k \leq 1$, again a contradiction. □

Similar to the proof of Lemma 3.5, and by Lemma 4.3, we have the following result.

**Lemma 4.4.** *Let $G$ be a $k$-regular connected double-orbit graph with two orbits $V_1$ and $V_2$, $k \geq 3$ and $g(G) \geq 6$. Suppose that $G$ is not $\lambda^{(3)}$-optimal and $A$ is a $\lambda^{(3)}$-atom of $G$. Then we have*

(i) *If $A \subseteq V_1$ (or $V_2$), then $V_1$ (or $V_2$) is a disjoint union of isomorphic $\lambda^{(3)}$-atoms and $G[A]$ is a vertex transitive graph;*

(ii) *If $A_i = A \cap V_i \neq \emptyset$ for $i = 1, 2$, then $V(G)$ is a disjoint union of distinct $\lambda^{(3)}$-atoms, $G[A]$ is a double-orbit graph and $G[A_i]$ is vertex transitive for $i = 1, 2$.*

By the above lemma, in this section we may assume that $G[A_i]$ is a $k_i$-regular graph for $i = 1, 2$, $|[x, A_2]| = r_1$ and $|[y, A_1]| = r_2$ for $x \in A_1$ and $y \in A_2$.

**Lemma 4.5.** *Let $G$ be a $k$-regular connected double-orbit graph with two orbits $V_1$ and $V_2$, $k \geq 3$ and $g(G) \geq 4$. Suppose that $G$ is not $\lambda^{(3)}$-optimal and $A$ is a $\lambda^{(3)}$-atom of $G$, then $|A| \geq 2k - 3$.*

*Proof.* Since $G$ is not $\lambda^{(3)}$-optimal and $g(G) \geq 4$, we have $|\omega(A)| = \lambda^{(3)}(G) < \xi_3(G) = 3k-4$ and $|A| \geq 4$. Then by Turán's Theorem,

$$\frac{|A|^2}{2} \geq \sum_{x\in A} d_{G[A]}(x) = \sum_{x\in A} d_G(x) - |\omega(A)| > k|A| - (3k-4),$$

which implies that $(\frac{|A|}{2} - k + \frac{3}{2})(|A|-3) + \frac{1}{2} > 0$ and thus $|A| \geq 2k-3$. □

**Lemma 4.6.** *Let $G$ be a $k$-regular connected double-orbit graph with two orbits $V_1$ and $V_2$, $k \geq 3$ and $g(G) \geq 6$. Suppose that $G$ is not $\lambda^{(3)}$-optimal and $A$ is a $\lambda^{(3)}$-atom of $G$, then $|A_i| = |A \cap V_i| \geq 3$ for $i = 1, 2$.*

*Proof.* Without loss of generality, we assume that $|A_1| \leq |A_2|$. By contradiction, suppose $|A_1| \leq 2$. We consider three cases.

**Case 1.** $|A_1| = 0$ ($A \subseteq V_2$).

By Lemmas 4.4 and 4.5, if each vertex of $A$ has at least two neighbors in $G[\overline{A}]$, then

$$|\omega(A)| \geq 2|A| \geq 2(2k-3) > 3k-4 = \xi_3(G) > \lambda^{(3)}(G),$$

a contradiction. Thus, we see that each vertex of $A$ has exactly one neighbor in $G[\overline{A}]$ and $G[A]$ is $(k-1)$-regular. Then

$$\lambda^{(3)}(G) = |\omega(A)| = |A| \geq n(k-1, 6) = 2(k^2 - 3k + 3) > 3k-4 = \xi_3(G),$$

again a contradiction.

**Case 2.** $|A_1| = 1$.

Then $k_1 = 0$, $r_2 = 1$ and $r_1 = |A_2| = |A| - |A_1| \geq 3$. Clearly, if $k_2 \geq 1$, then $g(G[A]) = 3$. Hence, $k_2 = 0$ and

$$\lambda^{(3)}(G) = |\omega(A)| = (k - r_1) + |A_2|(k - r_2) = r_1(k-2) + k > 3k-4 = \xi_3(G),$$

a contradiction.

**Case 3.** $|A_1| = 2$.

Subcase 3.1. $k_1 = 1$.

If $r_2 \geq 2$, then $g(G[A]) = 3$, contradicting that $g(G) \geq 6$. Thus, $r_2 = 1$ and $|A_2| = r_1|A_1| = 2r_1$. Clearly, if $k_2 \geq 1$, then $g(G[A]) \leq 4$, a contradiction. Hence, $k_2 = 0$ and then

$$\lambda^{(3)}(G) = |\omega(A)| = 2(k - k_1 - r_1) + |A_2|(k - r_2) = 2r_1(k-2) + 2(k-1) > 3k-4 = \xi_3(G),$$

a contradiction.

Subcase 3.2. $k_1 = 0$.

When $r_1 = 1$, by $|A_1| = r_2|A_2| \leq |A_2|$, we have $r_2 = 1$ and $|A_1| = |A_2| = 2$. Since $G[A]$ is connected, $k_2 = 1$. Thus,

$$\lambda^{(3)}(G) = |\omega(A)| = 2(k - r_1) + 2(k - k_2 - r_2) > 3k-4 = \xi_3(G),$$

a contradiction.

When $r_1 \geq 2$, we have $\frac{2r_1}{r_2} = |A_2| \geq 2r_1 - 1$, and so $r_2 = 1$ and $|A_2| = 2r_1$. If $k_2 \geq 2$, then $g(G[A]) \leq 4$, a contradiction. Thus, $0 \leq k_2 \leq 1$ and

$$\lambda^{(3)}(G) = |\omega(A)| = 2(k - r_1) + |A_2|(k - k_2 - r_2) = 2r_1(k - k_2 - 2) + 2k > 3k-4 = \xi_3(G),$$

which is impossible. □

**Theorem 4.7.** *Let $G$ be a $k$-regular connected double-orbit graph with two orbits $V_1$, $V_2$, $k \geq 3$ and $g(G) \geq 7$. Then $G$ is $\lambda^{(3)}$-optimal.*

*Proof.* We prove this theorem by way of contradiction that $G$ is not $\lambda^{(3)}$-optimal. Let $A$ be a $\lambda^{(3)}$-atom of $G$. Then $|\omega(A)| = \lambda^{(3)}(G) < \xi_3(G) = 3k-4$. By Lemma 4.6, $A_i = A \cap V_i \neq \emptyset$ for $i = 1,2$ and $|A_i| \geq 3$. We distinguish two cases.

**Case 1.** $N(A_1) \cap \overline{A} \neq \emptyset$ and $N(A_2) \cap \overline{A} \neq \emptyset$.

In this case, $\lambda^{(3)}(G) = |\omega(A)| = |A_1|(k-k_1-r_1) + |A_2|(k-k_2-r_2)$. Assume that $k-k_1-r_1 \leq k-k_2-r_2$. We claim that $k-k_1-r_1 = 1$. In fact, if otherwise $k-k_1-r_1 \geq 2$, by Lemma 4.5, $|\omega(A)| \geq 2|A_1| + 2|A_2| = 2|A| \geq 2(2k-3) > \xi_3(G)$, which is impossible.

If $k-k_2-r_2 \geq 2$, then since $g(G) \geq 7$, by counting the neighbors of $A_1$ in $A_2$, we have $|A_2| \geq r_1(1+k_1)$. Therefore,

$$|\omega(A)| \geq |A_1| + 2|A_2| = |A| + |A_2| \geq (2k-3) + r_1(1+k_1) = -r_1^2 + kr_1 + (2k-3) \geq 3k-4 > \lambda^{(3)}(G),$$

a contradiction.

If $k-k_2-r_2 = 1$, then $G[A]$ is $(k-1)$-regular. Hence,

$$|\omega(A)| = |A_1| + |A_2| = |A| \geq n(k-1,7) = k^3 - 4k^2 + 6k - 2 > 3k-4 > \lambda^{(3)}(G),$$

again a contradiction.

**Case 2.** Either $N(A_1) \cap \overline{A} \neq \emptyset$ or $N(A_2) \cap \overline{A} \neq \emptyset$.

Without loss of generality, we assume that only vertices in $A_1$ have neighbors in $\overline{A}$, then $\lambda^{(3)}(G) = |\omega(A)| = |A_1|(k-k_1-r_1)$ and $k_2 + r_2 = k$. We claim that $2 \leq r_2 \leq k-1$.

Suppose $r_2 = k$, then $k_2 = 0$. By Lemma 4.6 and $g(G) \geq 7$, $|A_1| \geq 3r_2 - 2 = 3k-2$. Thus $|\omega(A)| \geq |A_1| \geq 3k-2 > \xi_3(G) > \lambda^{(3)}(G)$, a contradiction. Thus $r_2 \leq k-1$.

Suppose $r_2 = 1$, then $G[A_2]$ is $(k-1)$-regular and $|A_2| \geq n(k-1,7)$. Since $k-k_1-r_1 \geq 1$, then $r_1 \leq k-1$ and $|A_1| \leq 3k-5$. When $k \geq 4$, $|A_2| \geq n(k-1,7) > (k-1)(3k-5) \geq r_1|A_1| = |A_2|$, which is impossible. When $k = 3$, $k_2 = 2$ and $G[A_2]$ is a cycle of order $|A_2|$. If $r_1 = 1$, then $|A_2| = |A_1| \leq 3k-5 = 4$, contradicting that $g(G) \geq 7$. If $r_1 = 2$, then $|A_2| = 2|A_1| \leq 8$. Since $g(G) \geq 7$ and $|A_2|$ is even, then $|A_1| = 4$, $|A_2| = 8$ and $g(G[A]) \leq 6$, which is impossible. This completes the proof of the claim.

In the following, we consider four subcases.

Subcase 2.1. $k-k_1-r_1 = 1$.

In this case, $\delta(G[A]) = k-1$ and $|A| \geq n(k-1,7)$. On the other hand, we have $|A| = |A_1| + |A_2| = \frac{r_1+r_2}{r_2}|A_1| \leq \frac{(k-1)+(k-1)}{2}(3k-5) = (k-1)(3k-5)$. When $k \geq 4$, $|A| \geq n(k-1,7) > (k-1)(3k-5) \geq |A|$, a contradiction. When $k = 3$, $|A_1| \leq 3k-5 = 4$, and by the claim $2 \leq r_2 \leq k-1$ and $k_2 + r_2 = k$, we have $r_2 = 2$ and $k_2 = 1$. If $r_1 = 1$, then $|A_1| = 2|A_2| \leq 4$ and $A_2 \leq 2$, contradicting Lemma 4.6. If $r_1 = 2$, then $k_1 = 0$ and $|A_1| = |A_2| \leq 4$. So, $|A_2| = 4$ and $g(G[A]) \leq 6$, again a contradiction.

Subcase 2.2. $k-k_1-r_1 = 2$.

Then $|\omega(A)| = 2|A_1| \leq 3k-5$ and $|A_1| \leq \frac{3k-5}{2}$. If $k \geq 5$, $|A| \geq n(k-2,7) = k^3 - 7k^2 + 17k - 13 > \frac{(k-2)+(k-1)}{2}\frac{3k-5}{2} \geq \frac{r_1+r_2}{r_2}|A_1| = |A|$, a contradiction. If $k = 3$, then $|A_1| \leq 2$, contradicting Lemma 4.6. If $k = 4$, then $1 \leq r_1 \leq k-2 = 2$ and $2 \leq r_2 \leq k-1 = 3$. By Lemma 4.6, $3 \leq |A_1| \leq \frac{3k-5}{2} = \frac{7}{2}$, thus $|A_1| = 3$. When $r_2 = 3$, then $|A_2| = \frac{r_1|A_1|}{r_2} \leq 2$, contradicting Lemma 4.6. When $r_2 = 2$, $k_2 = 2$ and $G[A_2]$ is a cycle of order $|A_2|$. But $|A_2| = \frac{r_1|A_1|}{r_2} \leq 3$, contradicting that $g(G) \geq 7$.

Subcase 2.3. $k-k_1-r_1 = 3$.

Since $1 \leq r_1 \leq k-3$, we have $k \geq 4$. If $k \geq 6$, then $|A| \geq n(k-3,7) = k^3 - 10k^2 + 34k - 38 > \frac{(k-3)+(k-1)}{2}\frac{3k-5}{3} \geq |A|$, a contradiction. If $k = 4$, then $|A_1| \leq \frac{3k-5}{3} < 3$, contradicting Lemma 4.6. If $k = 5$, then $3 \leq |A_1| \leq \frac{3k-5}{3} = \frac{10}{3}$, thus $|A_1| = 3$. When $r_1 = 1$, then $|A_2| = \frac{|A_1|}{r_2} < 3$, contradicting Lemma 4.6. When $r_1 = 2$, by $|A_2| = \frac{2|A_1|}{r_2} \geq 3$ and the claim $r_2 \geq 2$, we have $r_2 = 2$, $k_2 = 3$ and $|A_2| = 3$, which is impossible for $G[A_2]$ is a $k_2$-regular graph.

Subcase 2.4. $k-k_1-r_1 \geq 4$.

In this case, $k \geq 5$ and $2 \leq r_2 \leq |A_1| \leq \frac{3k-5}{4}$. Thus $k_2 \geq \frac{k+5}{4}$ and $|A_2| \geq n(\frac{k+5}{4},7) > \frac{k-4}{2}\frac{3k-5}{4} \geq \frac{r_1}{r_2}|A_1| = |A_2|$, a contradiction. □

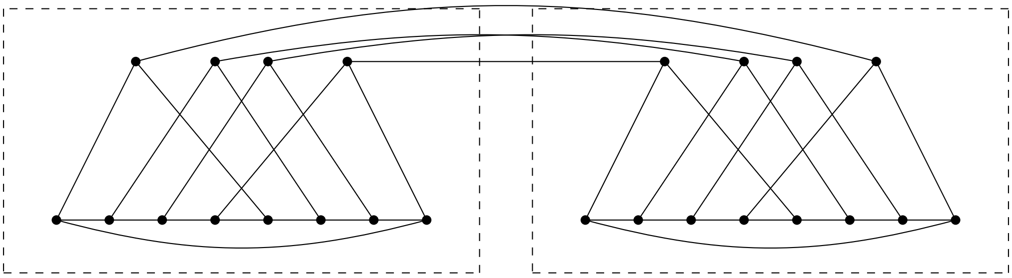

Fig.3. A 3-regular graph $G \in \mathcal{F}_3$

**Remark 4.8.** Now, we show that the girth bound given in Theorem 4.7 is best possible.

We construct one family of graphs with $g(G) \geq 6$, which is not $\lambda^{(3)}$-optimal, to illustrate that the girth bound is best possible. Using the same notation as Theorem 4.7, $A_1$ is an independent set with $2k-2 \leq |A_1| \leq 3k-5$, $G[A_2]$ is a $(k-1)$-regular subgraph, every vertex of $A_1$ has $k-1$ neighbors in $A_2$ and every vertex of $A_2$ has 1 neighbors in $A_1$. Let $\mathcal{F}_3$ be a family of $k$-regular double-orbit graphs such that for any $G \in \mathcal{F}_3$, $G$ consists of $m$ copies of $G[A]$ by adding a perfect matching between different copies of $A_1$. A 3-regular example $G \in \mathcal{F}_3$ is shown in Fig. 3.

Since a $\lambda'$-connected graph is super-$\lambda'$ if and only if either $G$ is not $\lambda^{(3)}$-connected or $\lambda^{(3)}(G) > \xi(G)$, we deduce the following result by Theorem 4.7.

**Corollary 4.9.** *Let G be a k-regular connected double-orbit graph with $k \geq 3$ and $g(G) \geq 7$. Then G is super-$\lambda'$.*

## 5. Super 3-extra edge-connectivity

**Lemma 5.1.** *Let G be a k-regular connected double-orbit graph with $k \geq 3$ and girth $g(G) \geq 7$. If G is not super-$\lambda^{(3)}$, then any two distinct $\lambda^{(3)}$-superatoms are disjoint.*

*Proof.* Assume that $X$ and $Y$ are two distinct $\lambda^{(3)}$-superatoms of $G$. Use the same notation as Lemma 3.4 and we suppose to the contrary that $A \neq \emptyset$. Since $G$ is not super-$\lambda^{(3)}$, then $|X|, |Y| \geq 4$. And by Theorems 4.2 and 4.7, $\lambda^{(3)}(G) = \xi_3(G) = 3k-4$, $\lambda'(G) = \xi(G) = 2k-2$ and thus $\lambda(G) = \delta(G)$. Clearly, $G[X \cup Y]$ and $G[\overline{X} \cup \overline{Y}]$ are connected.

**Claim 1.** $|A| \leq 3$.

Suppose $|A| \geq 4$. If $G[A]$ is connected, then $|\omega(A)| \geq \lambda^{(3)}(G)$. If $G[A]$ is not connected, we will show $|\omega(A)| > \lambda^{(3)}(G)$.

If $G[A]$ contains a component, say $U_3$, such that $|V(U_3)| \geq 3$, then $|\omega(A)| \geq \lambda^{(3)}(G) + \lambda(G) > \lambda^{(3)}(G)$. Otherwise, we may assume that every component in $G[A]$ is an isolated edge or an isolated vertex. If there is an isolated edge in $G[A]$, then we have $|\omega(A)| > \xi(G) + \lambda(G) > \lambda^{(3)}(G)$. Thus, now assume that all components in $G[A]$ are isolated vertices, then $|\omega(A)| \geq 4\lambda(G) > \lambda^{(3)}(G)$.

Note that $|D| \geq |A| \geq 4$. By a similar argument as above, we have $|\omega(D)| \geq \lambda^{(3)}(G)$. By submodular inequality, it follows that $A$ is a non-trivial $\lambda^{(3)}$-fragment contained in $X$, a contradiction.

**Claim 2.** $|B| \leq 3$.

Suppose $|B| \geq 4$, then $G[\overline{B}], G[\overline{C}]$ is connected. By an argument similar to the proof of Claim 1, we can always derive a contradiction that $B$ is a non-trivial $\lambda^{(3)}$-fragment contained in $X$, which is impossible.

We prove this lemma by considering three cases.

**Case 1.** $|A| = 1$.

Since $|X| = |A| + |B| = 1 + |B| \geq 4$, then $|B| = 3$ by Claim 2. Thus $G[X]$ is a connected subgraph of order 4 with $g(G[X]) \geq 7$, and we see that $G[X]$ is a tree of order 4, which implies that $4k-6 = |\omega(X)| = \lambda^{(3)}(G) = 3k-4$ and $k = 2$, contradicting $k \geq 3$.

**Case 2.** $|A| = 2$.

Since $|X| = 2 + |B| \geq 4$, then $2 \leq |B| \leq 3$ by Claim 2. Thus we see that $G[X]$ is a tree of order 4 or 5, which implies that $|X|k - 2|E(G[X])| = |\omega(X)| = \lambda^{(3)}(G) = 3k - 4$ and $k = 2$, a contradiction.

**Case 3.** $|A| = 3$.

In this case $G[X]$ is a tree of order 4, 5 or 6, which implies that $k = 2$, again a contradiction. □

**Lemma 5.2.** *Let $G$ be a $k$-regular connected double-orbit graph with two orbits $V_1$ and $V_2$, $k \geq 3$ and $g(G) \geq 7$. Suppose that $G$ is not super-$\lambda^{(3)}$ and $A$ is a $\lambda^{(3)}$-superatom of $G$. Then we have*

(i) *If $A \subseteq V_1$ (or $V_2$), then $V_1$ (or $V_2$) is a disjoint union of isomorphic $\lambda^{(3)}$-superatoms and $G[A]$ is a vertex transitive graph;*

(ii) *If $A_i = A \cap V_i \neq \emptyset$ for $i = 1, 2$, then $V(G)$ is a disjoint union of distinct $\lambda^{(3)}$-superatoms, $G[A]$ is a double-orbit graph and $G[A_i]$ is vertex transitive for $i = 1, 2$.*

In this section, we use the same notation as Section 4. By an argument similar to the proof of Lemma 4.5, we have the following lemma.

**Lemma 5.3.** *Let $G$ be a $k$-regular connected double-orbit graph with two orbits $V_1$ and $V_2$, $k \geq 3$ and $g(G) \geq 4$. Suppose that $G$ is $\lambda^{(3)}$-optimal but not super-$\lambda^{(3)}$ and $A$ is a $\lambda^{(3)}$-superatom of $G$, then $|A| \geq 2k - 4$.*

**Lemma 5.4.** *Let $G$ be a $k$-regular connected double-orbit graph with two orbits $V_1$ and $V_2$, $k \geq 3$ and $g(G) \geq 7$. Suppose that $G$ is not super-$\lambda^{(3)}$ and $A$ is a $\lambda^{(3)}$-superatom of $G$, then $|A_i| = |A \cap V_i| \geq 3$ for $i = 1, 2$.*

*Proof.* The outline of the proof is similar to that of Lemma 4.6 except for the case $|A_1| = 0$. In the following, we assume $|A_1| = 0$.

By Lemmas 5.2 and 5.3, if each vertex of $A$ has at least two neighbors in $G[\overline{A}]$, then $|\omega(A)| \geq 2|A| \geq 2(2k - 4) = 4k - 8$. If $k \geq 5$, then $|\omega(A)| \geq 4k - 8 > 3k - 4 = \lambda^{(3)}(G)$, contradicting Theorem 4.7. If $k = 3$, by $0 \leq k_2 \leq k - 2 = 1$, then $5 = 3k - 4 = \lambda^{(3)}(G) = |\omega(A)| = (k - k_2)|A| \geq 2|A| \geq 8$, which is impossible. If $k = 4$, $\lambda^{(3)}(G) = 8$. When $0 \leq k_2 \leq 1$, $8 = |\omega(A)| = (k - k_2)|A| \geq 12$, a contradiction. When $k_2 = 2$, $G[A] \cong C_t$ and $t \geq 7$. But, $8 = |\omega(A)| = 2t \geq 14$, which is impossible. Thus, we see that each vertex of $A$ has exactly one neighbor in $G[\overline{A}]$, $G[A]$ is $(k - 1)$-regular, and

$$\lambda^{(3)}(G) = |\omega(A)| = |A| \geq n(k - 1, 7) > 3k - 4 = \xi_3(G),$$

a contradiction. □

Let $\mathcal{F}_4$ be a family of 4-regular connected double-orbit graphs with two orbits $V_1$, $V_2$ and girth at least 7. For any $G \in \mathcal{F}_4$, $G$ is the disjoint union of $m$ copies of $G[A]$ by adding a perfect matching between different copies of $A_1$, where $A_1$ is an independent set of order 8, $G[A_2]$ is a 3-regular vertex transitive graph of order 24, $G(A_1, A_2)$ is the disjoint union of 8 copies of $K_{1,3}$ and $A_i = A \cap V_i$ for $i = 1, 2$.

**Theorem 5.5.** *Let $G$ be a $k$-regular connected double-orbit graph with two orbits $V_1$, $V_2$, $k \geq 3$ and $g(G) \geq 7$. Then $G$ is super-$\lambda^{(3)}$ except the graphs in $\mathcal{F}_4$.*

*Proof.* We prove this theorem by way of contradiction that $G$ is not super-$\lambda^{(3)}$ and $G \notin \mathcal{F}_4$. By Theorem 4.7, $\lambda^{(3)}(G) = \xi_3(G) = 3k - 4$. Let $A$ be a $\lambda^{(3)}$-superatom of $G$. Then by Lemma 5.4, $A_i = A \cap V_i \neq \emptyset$ for $i = 1, 2$ and $|A_i| \geq 3$. We distinguish two cases.

**Case 1.** $N(A_1) \cap \overline{A} \neq \emptyset$ and $N(A_2) \cap \overline{A} \neq \emptyset$.

In this case, $\lambda^{(3)}(G) = |\omega(A)| = |A_1|(k - k_1 - r_1) + |A_2|(k - k_2 - r_2)$. Assume that $k - k_1 - r_1 \leq k - k_2 - r_2$, and we have $k - k_1 - r_1 = 1$. In fact, if $k - k_1 - r_1 \geq 2$, by Lemma 5.3, when $k \geq 5$, $|\omega(A)| \geq 2|A| \geq 2(2k - 4) > 3k - 4 = \lambda^{(3)}(G)$, a contradiction. When $3 \leq k \leq 4$, by lemma 5.4, $|\omega(A)| \geq 2|A_1| + 2|A_2| \geq 12 > \lambda^{(3)}(G)$, again a contradiction.

If $k - k_2 - r_2 \geq 2$, we claim that $2 \leq r_1 \leq k - 2$. If $r_1 = 1$, then $k_1 = k - 2$. Thus, $G[A_1]$ is $(k - 2)$-regular and $|A_1| \geq (k - 2, 7) = k^3 - 7k^2 + 17k - 13$. Then $|\omega(A)| \geq |A_1| + 2|A_2| \geq (k^3 - 7k^2 + 17k - 13) + 6 > 3k - 4 = \lambda^{(3)}(G)$, a contradiction. If $r_1 = k-1$, then $k_1 = 0$. By $g(G) \geq 7$ and $|A_1| \geq 3$, then $|A_2| \geq 3r_1 - 2$ and $|\omega(A)| \geq |A_1| + 2|A_2| \geq 3 + 2(3r_1 - 2) > 3k - 4 = \lambda^{(3)}(G)$, which is impossible. So, the claim follows.

By the claim, we have $k \geq 4$. When $k = 4$, $8 = \lambda^{(3)}(G) = |\omega(A)| \geq 9$, which is impossible. When $k \geq 5$, $|\omega(A)| \geq |A_1| + 2|A_2| = |A| + |A_2| \geq (2k-4) + r_1(1 + k_1) = -r_1^2 + kr_1 + (2k-4) > 3k - 4 = \lambda^{(3)}(G)$, again a contradiction.

If $k - k_2 - r_2 = 1$, then $G[A]$ is $(k-1)$-regular. Hence, $|\omega(A)| = |A_1| + |A_2| = |A| \geq n(k-1, 7) > 3k - 4 = \lambda^{(3)}(G)$, also a contradiction.

**Case 2.** Either $N(A_1) \cap \overline{A} \neq \emptyset$ or $N(A_2) \cap \overline{A} \neq \emptyset$.

Without loss of generality, we assume that only vertices in $A_1$ has neighbors in $\overline{A}$, then $\lambda^{(3)}(G) = |\omega(A)| = |A_1|(k - k_1 - r_1)$ and $k_2 + r_2 = k$. We claim that $3 \leq r_2 \leq k-1$.

Suppose $r_2 = k$, then $k_2 = 0$ and $|\omega(A)| \geq |A_1| \geq 3k - 2 > 3k - 4 = \lambda^{(3)}(G)$, a contradiction.

Suppose $r_2 = 1$, then $k_2 = k-1$. When $k \geq 5$, $|A_2| \geq n(k-1, 7) > (k-1)(3k-4) \geq r_1|A_1| = |A_2|$, a contradiction. When $k = 3$, $k_2 = 2$ and $G[A_2]$ is a cycle. If $r_1 = 1$, then $|A_2| = |A_1| \leq 5$, contradicting that $g(G[A_2]) \geq 7$. If $r_1 = 2$, then $|A_2| = 2|A_1| = 10$, contradicting $g(G[A]) \geq 7$. When $k = 4$, $k_2 = 3$. If $1 \leq r_1 \leq 2$, then $16 \geq r_1|A_1| = |A_2| \geq n(3, 7) = 22$, which is impossible. If $r_1 = 3$, then $|A_2| = 3|A_1| = 24$ and it is easy to see that $G \in \mathcal{F}_4$, a contradiction.

Suppose $r_2 = 2$, then $k_2 = k-2$. When $k \geq 6$, $|A_2| \geq n(k-2, 7) > \frac{k-1}{2}(3k-4) \geq \frac{r_1}{r_2}|A_1| = |A_2|$, a contradiction. When $k = 3$, $k_2 = 1$ and $1 \leq r_1 \leq k-1 = 2$. If $r_1 = 1$, then $|A_2| = \frac{1}{2}|A_1| < 3$, contradicting Lemma 5.4. If $r_1 = 2$, then $|A_1| = |A_2| \leq 5$. By $k_2 = 1$, we have $|A_1| = |A_2| = 4$, contradicting $g(G[A]) \geq 7$. When $k = 4$, $k_2 = 2$ and $G[A_2]$ is a cycle of order $|A_2|$. If $r_1 = 1$, then $|A_2| = \frac{1}{2}|A_1| \leq 4$, contradicting $g(G[A_2]) \geq 7$. If $r_1 = 2$, then $|A_1| = |A_2| \leq 8$. By $g(G) \geq 7$, we have $|A_1| = |A_2| = 8$, contradicting $g(G[A]) \geq 7$. If $r_1 = 3$, then $k_1 = 0$, $|A_1| = 8$ and $|A_2| = 12$. Since $\delta(G[A]) = 3$, we have $|A| \geq n(3, 7) = 22 > 20 = |A_1| + |A_2| = |A|$, which is impossible. When $k = 5$, $|A_2| \geq n(3, 7) = 22 \geq \frac{r_1}{r_2}|A_1| = |A_2|$, thus $|A_2| = 22$. Since $11 = 3k - 4 \geq |A_1| = \frac{r_2}{r_1}|A_2|$, we have $r_1 = 4$ and $|A_1| = 11$. By $\delta(G[A]) = 4$, $|A| \geq n(4, 7) = 53 > 33 = |A|$, which is impossible. Hence, the claim follows.

We complete the proof by considering four subcases.

Subcase 2.1. $k - k_1 - r_1 = 1$.

By the claim, $k \geq 4$, then $|A| \geq n(k-1, 7) > \frac{(k-1)+(k-1)}{3}(3k-4) \geq \frac{r_1+r_2}{r_2}|A_1| = |A|$, a contradiction.

Subcase 2.2. $k - k_1 - r_1 = 2$.

In this case, $k \geq 4$, $|A| \geq n(k-2, 7) > \frac{(k-2)+(k-1)}{3}\frac{3k-4}{2} \geq |A|$, again a contradiction.

Subcase 2.4. $k - k_1 - r_1 = 3$.

By $3k - 4 = \lambda^{(3)}(G) = |\omega(A)| = 3|A_1|$, we have $|A_1| = \frac{3k-4}{3} = k - \frac{4}{3}$, contradicting that $|A_1|$ is an integer.

Subcase 2.3. $k - k_1 - r_1 \geq 4$.

In this case, $k \geq 5$ and $3 \leq r_2 \leq |A_1| \leq \frac{3k-4}{4}$. Thus $k_2 \geq \frac{k+4}{4}$ and $|A_2| \geq n(\frac{k+4}{4}, 7) > \frac{k-4}{3}\frac{3k-4}{4} \geq |A_2|$, a contradiction. □

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