Numerical Solution of Linear Stochastic Volterra Integral Equations via New Basis Functions

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Abstract. In this article, we use a new method based on orthogonal basis functions for the numerical solution of stochastic Volterra integral equations of the second kind (SVIE). By using this method, a SVIE can be reduced to a linear system of algebraic equations. Finally, to show the efficiency of the proposed method, we give two numerical examples.

1. Introduction

Due to increasing usage of stochastic differential equations or stochastic integral equations in applicable problems, the need to extend the numerical solution for this type of equation is felt. Some authors have suggested some numerical methods for solving stochastic differential equation [1–3]. M.Khodabin et al. have used interpolation solution generalized stochastic exponential population growth model [4, 5]. In [6], authors applied triangular functions for solving numerically stochastic Volterra integral equation. Asgari et al. suggested stochastic operational matrix based on Bernstein polynomials for obtaining numerical solution of nonlinear stochastic integral equation[7]. K.Maleknejad et al. used modified block pulse functions for solving stochastic Volterra integral equations [8].S. Bhattacharya et al. have obtained numerical solutions of Volterra integral equations by applying Bernstien polynomials [9]. Authors of [10] by using new basis functions have solved non stochastic and nonlinear Volterra-Fredholm integral equations numerically.
In this article, we apply the new basis functions method proposed in [10] for solving linear stochastic Volterra integral equation

$$h(x) = g(x) + \lambda_1 \int_0^x K_1(x,t)h(t)dt + \lambda_2 \int_0^x K_2(x,t)h(t)dB(t), \tag{1}$$

where the function $g \in L^2[0,1]$, and the kernels $K_1(.,.), K_2(.,.) \in L^2([0,1] \times [0,1])$ are known and $h(x)$ is stochastic processes defined on the probability space (Ω, Z, P), and $h(x)$ is unknown. $B(t)$ is a Brownian motion process and $\int_0^x K_2(x,t)h(x)dB(t)$ is the *Itô* integral. To this end we construct the operational matrix of integration, P and P_s, as follows:

$$\int_0^t \Psi(s)ds \simeq P\Psi(t),$$

$$\int_0^t \Psi(s)dB(s) \simeq P_s\Psi(t),$$

where, $\Psi(t) = [\psi_1(t), \psi_2(t), ..., \psi_n(t)]$ and clearly the matrices P and P_s can be calculated by using new orthogonal basis functions. The elements $\psi_1(t), \psi_2(t), ..., \psi_n(t)$ are the orthogonal basis functions defined on the certain interval [0,1]. The main advantage of this method is that this method reduces the integral equation (1) to a collection of algebraic equations by expanding $h(x)$, $g(x)$, $K_1(.,.)$ and $K_2(.,.)$ in (1) according to new orthogonal basis functions.

The rest of the paper is organized as follows: In Section 2, we introduce a new set of orthogonal basis functions. In Section 3, stochastic integrations operational matrix of new basis functions are introduced. Function approximation is introduced in Section 4. Error analysis was given in Section 5. The efficiency of the proposed method, is shown by some examples in Section 6. Finally, Section 7, gives our concluding remarks.

2. New basis functions

In an n-collection of new basis functions (NBFs) over interval [0,1], the ith left and right functions are introduced as [10]:

$$NF1_i(u) = \begin{cases} \frac{(i+1)^2 - (\frac{u}{h})^2}{2i+1}, & ih \le u < (i+1)h, \\ 0, & otherwise, \end{cases} \tag{2}$$

$$NF2_i(u) = \begin{cases} \frac{(\frac{u}{h})^2 - i^2}{2i+1}, & ih \le u < (i+1)h, \\ 0, & otherwise, \end{cases} \tag{3}$$

where $i = 0, 2, 3, ..., n-1$ is an arbitrary positive integer number, $h = \frac{1}{n}$, and $NF1_i(u)$ and $NF2_i(u)$ are the terms of ith of $NF1(u)$ and $NF2(u)$, respectively. Suppose that

$$NF1(u) = [NF1_0(u), NF1_1(u), ..., NF1_{n-1}(u)]^T, \tag{4}$$

$$NF2(u) = [NF2_0(u), NF2_1(u), ..., NF2_{n-1}(u)]^T, \tag{5}$$

and

$$NF(u) = [NF1(u), NF2(u)]^T. \tag{6}$$

By using Eqs.(2) and (3), we conclude that

$$NF1_i(u) + NF2_i(u) = \psi_i(u),$$

where $\psi_i(u)$ are the ith block pulse functions.
Also

$$NF1(u)NF1^T(u) \simeq diag(NF1(u)), \tag{7}$$

$$NF2(u)NF2^T(u) \simeq diag(NF2(u)), \tag{8}$$

$$NF(u).NF^T(u).V \simeq \tilde{V}.NF(u), \tag{9}$$

where V is a $2n$-vector and $\tilde{V} = diag(V)$.
It can be clearly concluded that for an $n \times n$ matrix B:

$$NF^T(u).B.NF(u) \simeq \hat{B}^T.NF(u), \tag{10}$$

where $\hat{B}$ is a n-vector with elements equal to diagonal entries of matrix B.
It is easy to see that

$$\sum_{i=0}^{n-1} NF1_i(u) + \sum_{i=0}^{n-1} NF2_i(u) = 1, \quad 0 \le u < 1.$$

By some definitions and relations in [10] we can obtain the integration operational matrix as:

$$P = \frac{h}{6} \begin{bmatrix} 2 & 4 & \dots & 4 & 0 & 4 & \dots & 4 \\ 0 & 2 & \dots & . & 0 & . & \dots & \\ 0 & 0 & \dots & 4 & 0 & . & \dots & 4 \\ . & 0 & \dots & 2 & 0 & . & \dots & 0 \\ 1 & 2 & \dots & 2 & 0 & 2 & \dots & 2 \\ 0 & 1 & \dots & 2 & 0 & & \dots & 2 \\ 0 & 0 & \dots & 2 & 0 & 0 & \dots & 2 \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \end{bmatrix}_{2n \times 2n}.$$

3. Stochastic integration operational matrix

From [4], we have

$$P_s = \begin{bmatrix} a(1) & b(1) & b(1) & \dots & b(1) \\ 0 & a(2) & b(2) & \dots & b(2) \\ 0 & 0 & a(3) & \dots & b(3) \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ 0 & 0 & 0 & \dots & a(n) \end{bmatrix}_{n \times n}, \tag{11}$$

where

$$a(i) = B(\frac{(2i-1)h}{2}) - B((i-1)h) \quad , \quad b(i) = B(ih) - B((i-1)h),$$

and P_s is stochastic integration operational matrix of block pulse functions. By using (5) we can write

$$\int_0^x NF1_i(t)dB(t) + \int_0^x NF2_i(t)dB(t) = \int_0^x \psi_i(t)dB(t).$$

Furthermore, we can show

$$\int_0^x NF1_i(t)dB(t) = \frac{2}{3} \int_0^x \psi_i(t)dB(t),$$

and

$$\int_0^x NF2_i(t)dB(t) = \frac{1}{3} \int_0^x \psi_i(t)dB(t).$$

From (6), stochastic integration operational matrix biassed on new basis function, is introduced as

$$P_{NS} = \frac{1}{6} \begin{bmatrix} 4P_s & 2P_s \\ 2P_s & P_s \end{bmatrix}_{2n \times 2n}.$$

4. Function approximation

An arbitrary real bounded function $g(x) \in L^2[0, 1)$ can be approximated by:

$$g(x) \simeq \sum_{i=0}^{n-1} G1_i NF1_i(x) + \sum_{i=0}^{n-1} G2_i NF2_i(x) = G^T NF(x), \tag{12}$$

where G is a $2n$-vector given by $G = [G1^T, G2^T]$, and $NF(x) = [NF1(x), NF2(x)]^T$ are defined in Eq.(5). The coefficients in $G1$ and $G2$ can be computed by $G1_i = g(ih), G2_i = g((i+1)h)$.

Suppose that $K(x; t)$ be an arbitrary function defined on $L^2([0; 1) \times [0; 1))$. Clearly, it can be expanded by NBFs as the following form:

$$K(x; t) \simeq NF^T(x)KNF(t).$$

Also, $K = \begin{bmatrix} k11 & k12 \\ k21 & k22 \end{bmatrix}$, is $2n \times 2n$ coefficients matrix, where

$$[k11]_{mn} = K(mh, nh), [k12]_{mn} = K(mh, (n+1)h),$$

$$[k21]_{mn} = K((m+1)h, (n)h), [k22]_{mn} = K((m+1)h, (n+1)h).$$

Now, we proposed a numerical method based on new basis functions to solve linear stochastic Volterra integral equation:

$$h(x) = g(x) + \lambda_1 \int_0^x K_1(x, t)h(t)dt + \lambda_2 \int_0^x K_2(x, t)h(t)dB(t), \quad 0 \leq x < 1, \tag{13}$$

where, $g(x) \in L^2[0, 1)$ and kernels $K_1(x, t) \in L^2([0, 1) \times [0, 1))$, $K_2(x, t) \in L^2([0, 1) \times [0, 1))$ are known functions, $h(x)$ is the unknown function. By approximating functions $g(x), K_1(x, t)$, $K_2(x, t)$ and $h(x)$ in the matrix form, we have:

$$g(x) \simeq G^T NF(x), \tag{14}$$

$$K_1(x, t) \simeq NF^T(x)K_1NF(t), \tag{15}$$

$$K_2(x, t) \simeq NF^T(x)K_2NF(t), \tag{16}$$

$$h(x) \simeq H^T NF(x). \tag{17}$$

By substituting Eqs.(14)-(17) in Eq.(13), we obtain:

$$H^T NF(x) = G^T NF(x) + \lambda_1 \int_0^x NF^T(x)K_1NF(t)NF^T(t)Hdt +$$

$$\lambda_2 \int_0^x NF^T(x)K_2NF(t)NF^T(t)HdB(t),$$

$$H^T NF(x) = G^T NF(x) + \lambda_1 NF^T(x)K_1 \int_0^x NF(t)NF^T(t)Hdt +$$

$$\lambda_2 NF^T(x)K_2 \int_0^x NF(t)NF^T(t)HdB(t),$$

by using Eqs.(9)-(10) we have:

$$H^T NF(x) = G^T NF(x) + \lambda_1 NF^T(x)K_1 \int_0^x \tilde{H}NF(t)dt + \lambda_2 NF^T(x)K_2 \int_0^x \tilde{H}NF(t)dB(t),$$

by applying $\int_0^x NF(t)dt = P.NF(x)$, $\int_0^x NF(t)dB(t) = P_{NS}.NF(x)$ we have:

$$H^T NF(x) = G^T NF(x) + \lambda_1 NF^T(x)K_1\tilde{H}PNF(x) + \lambda_2 NF^T(x)K_2\tilde{H}P_{NS}NF(x), \tag{18}$$

let $B = K_1\tilde{H}P$, $B_s = K_2\tilde{H}P_{NS}$ we have:

$$H^T NF(x) = G^T NF(x) + \lambda_1 \tilde{B}^T NF(x) + \lambda_2 \tilde{B}_s^{\ T} NF(x),$$

where $\tilde{B}$ and $\tilde{B}_s$ are $2n$-vectors with elements equal to the diagonal entries of matrix B and B_s. So,

$$H = G + \lambda_1 \tilde{B} + \lambda_2 \tilde{B}_s.$$

This equation is a linear system of algebraic equations.

5. Error analysis

Theorem 1. Let f be an arbitrary real bounded function on $(0,1)$ and $|f'(t)| < M$ for all $t \in (0,1)$.
Put $e(t) = |f(t) - \hat{f}(t)|$ where, $\hat{f}(t)$ is approximation of $f(t)$ biassed on NBFs. Then

$$\|e(t)\|^2 < O(h^2),$$

where, $\|e(t)\|^2 = \int_0^1 |e(s)|^2 ds$.
Proof: By using properties of the NBFs, :

$$e(t) = |f(t) - \hat{f}(t)| = |f(t) - \sum_{i=0}^{m-1} (f(ih)NF1i + f((i+1)h)NF2i)|.$$

For $t \in (ih, (i+1)h)$, we conclude that

$$e(t) = |f(t) - \hat{f}(t)| = |f(t) - f(ih)NF1i - f((i+1)h)NF2i| = |f(t) - f(ih)NF1i - f((i+1)h)(1 - NF1i)|$$

$$\leq |f(t) - f((i+1)h)| + |f((i+1)h) - f(ih)|NF1i|.$$

By the mean value theorem, there exist $\eta, \eta_t \in (ih, (i+1)h)$ such that

$$e(t) \leq |f'(\eta)(t - (i+1)h| + |f'(\eta_t)hNF1i|$$

$$\leq |f'(\eta)h| + |f'(\eta_t)h| = (|f'(\eta)| + |f'(\eta_t)|)h \leq 2Mh.$$

So

$$\|e(t)\|^2 = \int_0^1 |e(s)|^2 ds \leq (2Mh)^2 \leq O(h^2).$$

In Eq.(1) let

$$l1^n(t) = \lambda_1 K_1(t, x_n(t))h(x_n), \quad q^n(t) = \lambda_2 K_2(t, x_n(t))h(x_n),$$

and

$$\hat{l}1(t) = \hat{l}1_n(t), \quad \hat{q}(t) = \hat{l}2_n(t),$$

where $\hat{l}1(t)$ and $\hat{l}2(t)$ are defined by property of the NBFs.
Theorem 2. Let $h_n(t)$ is numerical solution of

$$h(x) = g(x) + \int_0^x \hat{l}1(t)dt + \int_0^x \hat{l}2(t)dB(t),$$

Then

$$\|h(t) - h_n(t)\|^2 \leq O(h^2),$$

where, $\|x\|^2 = E[x^2]$.
Proof.

$$h(t) - h_n(t) = \int_0^t (l1(s) - \hat{l}1(s))ds + \int_0^t (l2(s) - \hat{l}2(s))dB(s),$$

By using $(a + b)^2 \leq 2(a^2 + b^2)$, we have

$$\|h(t) - h_n(t)\|^2 \leq 2(\|\int_0^t (l1(s) - \hat{l}1(s))ds\|^2 + \|\int_0^t (l2(s) - \hat{l}2(s))dB(s)\|^2)$$

$$\leq 2(\|\int_0^t (l1(s) - \hat{l}1(s))\|^2 ds + \|\int_0^t (l2(s) - \hat{l}2(s)) dB(s)\|^2), \tag{19}$$

By using isometry property of the standard Brownian motion we have

$$\|h(t) - h_n(t)\|^2 \leq 2(\int_0^t \|l1(s) - \hat{l}1(s)\|^2 ds + \int_0^t \|l2(s) - \hat{l}2(s)\|^2) d(s)$$

$$= 2(\int_0^t \|l1(s) - l1^n(s) + l1^n(s) - \hat{l}1(s)\|^2 ds + \int_0^t \|l2(s) - l2^n(s) + l2^n(s) - \hat{l}2(s)\|^2) d(s)$$

$$\leq 2(2\int_0^t \|l1(s) - l1^n(s)\|^2 ds + 2\int_0^t \|l1^n(s) - \hat{l}1(s)\|^2 ds + 2\int_0^t \|l2(s) - l2^n(s)\|^2 ds + 2\int_0^t \|l2^n(s) - \hat{l}2(s)\|^2 ds)$$

$$\leq 4(\int_0^t \|l1(s) - l1^n(s)\|^2 ds + \int_0^t \|l1^n(s) - \hat{l}1(s)\|^2 ds + \int_0^t \|l2(s) - l2^n(s)\|^2 ds + \int_0^t \|l2^n(s) - \hat{l}2(s)\|^2 ds). \tag{20}$$

By using theorem 1 we have($k_1, k_2 > 0$)

$$\|l1^n(s) - \hat{l}1(s)\|^2 \leq k_1 h^2 \quad , \quad \|l1^n(s) - \hat{l}1(s)\|^2 \leq k_2 h^2 \tag{21}$$

By using Lipschitz condition we have

$$\int_0^t \|l1(s) - l1^n(s)\|^2 ds \leq l_1 \int_0^t \|h(s) - h_n(s)\|^2 ds, \tag{22}$$

$$\int_0^t \|l2(s) - l2^n(s)\|^2 ds \leq l_2 \int_0^t \|h(s) - h_n(s)\|^2 ds, \tag{23}$$

by substituting Eqs. (21),(22),(23) in Eq. (20) we get

$$\|h(t) - h_n(t)\|^2 \leq 4(k_1 h^2 + l_1 \int_0^t \|h(s) - h_n(s)\|^2 ds + k_2 h^2 + l_2 \int_0^t \|h(s) - h_n(s)\|^2 ds). \tag{24}$$

or

$$\theta(t) \leq \mu + \eta \int_0^t \theta(s) ds,$$

where,

$$\mu = 4(k_1 h^2 + k_2 h^2), \quad \eta = 4(l_1 + l_2), \quad \theta(t) = \|h(s) - h_n(s)\|^2$$

From Gronval inequality we obtain

$$\theta(t) \leq \mu(1 + \eta \int_0^t exp(\theta(t - s)) ds), \qquad t \in (0, 1)$$

so

$$\|h(t) - h_n(t)\|^2 \leq O(h^2).$$

6. Numerical examples

Here, we apply the proposed method in Section 4 for solving SVIE of the second kind. To illustrate the efficiency of the presented method, we compare the numerical solution with the exact solution.

Example 1. Consider the following stochastic Volterra integral equation of the second kind:

$$u(t) = \frac{1}{12} + \int_0^t cos(s)u(s)ds + \int_0^t sin(s)u(s)dB(s), \tag{25}$$

with the exact solution $u(t) = 1/12 \exp(-t/4 + sin(t) + sin(2t)/8) + \int_0^t sin(s)dB(s))$, $0 \le t < 1$.
The numerical results are shown in Table 1.

Table 1:The error mean($\bar{X}_E$), mean of analytical solution $\bar{X}_A$, mean of numerical solution $\bar{X}_N$ and 95%confidence interval for error mean values of x (n=32)

x	$\bar{X}_E$	$\bar{X}_A$	$\bar{X}_N$	95%confidence interval for$\bar{X}_E$	
				Lower	Upper
0	4.938e-04	0.0833651	0.0846491	4.939e-04	4.939e-04
0.1	8.524e-04	0.0921918	0.0926241	8.524e-04	8.525e-04
0.2	2.076e-03	0.1014824	0.1010376	2.074e-03	2.078e-03
0.3	2.090e-03	0.1117954	0.1096639	2.085e-03	2.095e-03
0.4	3.026e-03	0.1230248	0.1181889	3.012e-03	3.041e-03
0.5	6.452e-04	0.1336782	0.1315328	6.078e-04	6.826e-04
0.6	6.363e-04	0.1477232	0.1387214	5.802e-04	6.924e-04
0.7	5.380e-04	0.1629618	0.1444476	4.958e-04	5.803e-04
0.8	3.410e-04	0.1750085	0.1482378	2.761e-04	4.059e-04
0.9	2.746e-03	0.1889018	0.1497249	2.659e-03	2.833e-03

Example 2. Consider the following linear stochastic Volterra integral equation,

$$u(t) = \frac{1}{3} + \int_0^t ln(s+1)u(s)ds + \int_0^t \sqrt{ln(s+1)}u(s)dB(s) \qquad s, t \in [0, 0.5), \tag{26}$$

with the exact solution $u(t) = \frac{1}{3}e^{-\frac{1}{2}t+\frac{1}{2}ln(t+1)t+\frac{1}{2}ln(t+1)+\int_0^t \sqrt{ln(s+1)}dB(s)}$, for $0 \le t < 0.5$.
The numerical results are shown in Table 2 .

Table 2: The error mean($\bar{X}_E$), mean of analytical solution $\bar{X}_A$, mean of numerical solution $\bar{X}_N$ and 95%confidence interval for error mean values of x (n=32)

x	$\bar{X}_E$	$\bar{X}_A$	$\bar{X}_N$	95%confidence interval for $\bar{X}_E$	
				Lower	Upper
0	2.78015e-05	0.333334	0.333333	2.78014e-05	2.78011e-05
0.1	1.72133e-05	0.336691	0.334142	6.996e-06	2.7430e-05
0.2	3.52856e-05	0.345342	0.336481	-5.481e-06	7.6052e-05
0.3	7.26905e-05	0.359038	0.340248	-3.848e-06	1.4922e-04
0.4	1.10451e-04	0.377406	0.365839	-1.699e-05	2.3789e-05
0.5	1.70366e-04	0.408059	0.381858	-5.009e-05	3.9083e-04
0.6	2.41803e-04	0.437881	0.459660	-5.899e-05	5.4260e-04
0.7	3.16882e-04	0.473096	0.468789	-9.168e-05	7.2545e-04
0.8	3.97393e-04	0.514373	0.479251	-1.354e-04	9.3015e-04
0.9	4.92036e-04	0.562033	0.491097	-1.854e-04	1.1695e-04

7. Conclusions

In this article, we used a new orthogonal basis functions for solving SVIE of the second kind. By using this method, we reduce Eq.(1) to the linear stochastic system. Also, numerical examples are show the accuracy of presented method. We can extend this idea to SVIE by fractional Brownian motion.

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