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A Riordan Array Approach to Apostol Type-Sheffer Sequences

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Abstract. In this article, the generalized Apostol type-Sheffer sequences are introduced and their properties including the quasi-monomiality, determinant form and series and conjugate representations are derived via Riordan array techniques. The generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi-Sheffer sequences are considered as their special cases. Certain examples are framed in terms of the generalized Apostol Bernoulli-associated Laguerre sequences, generalized Apostol-Euler-Hermite sequences and generalized Apostol-Genocchi-Legendre sequences to give the applications of main results. The numerical results to calculate the zeros and approximate solutions of these sequences are given and their graphical representations are shown.

1. Introduction and preliminaries

Sheffer polynomial sequences arise in numerous problems of applied mathematics, theoretical physics, approximation theory and several other branches of the mathematical sciences. Sheffer polynomial sequences contain their associated sequences as well as the Appell sequences as two subclasses. Sheffer sequences are studied systematically by theory of modern umbral calculus (see, for example, [14], [15] and [13]; see also several related recent works including [3], [18], [4] and [19]). Some definitions and results to be used in this work are being recalled here from the work by Wang [23].

Let $\mathbb{K}$ be a field of characteristic zero and suppose that $\mathcal{F}$ is the set of all formal power series in the variable t over $\mathbb{K}$. An element of $\mathcal{F}$ has the following form:

$$f(t) = \sum_{k=0}^{\infty} a_k t^k,$$

where $a_k \in \mathbb{K}$ for all $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$. The order $o\big(f(t)\big)$ of a power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. The series $f(t)$ has a multiplicative inverse, denoted by $[f(t)]^{-1}$ or by $\frac{1}{f(t)}$, if and only if $o\big(f(t)\big) = 0$. In this case, $f(t)$ is called an *invertible series*. The series $f(t)$ has a compositional inverse, denoted by $\overline{f}(t)$ and satisfying the following condition:

$$f\big(\overline{f}(t)\big) = \overline{f}\big(f(t)\big) = t \iff o\big(f(t)\big) = 1.$$

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The series $f(t)$ with $o\big(f(t)\big) = 1$ is called a *delta series*.

Definition 1.1. Let $g(t)$ be an invertible series and $f(t)$ be a delta series of the following forms:

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!} \quad \big(f_0 = 0;\ f_1 \neq 0\big) \qquad \text{and} \qquad g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!} \quad (g_0 \neq 0). \tag{1}$$

Then the sequence $\big(s_n(x)\big)_{n\in\mathbb{N}_0}$ is called a *Sheffer sequence* for the pair $\big(g(t), f(t)\big)$ if and only if

$$\langle g(t)[f(t)]^k | s_n(x)\rangle = c_n \delta_{n,k},$$

for all $n, k \in \mathbb{N}_0$, $\delta_{n,k}$ being the Kronecker delta.

We recall the concept of the Riordan arrays, which was introduced by Shapiro *et al.* [17] and further studied by many authors (see, for example, [6] and [7]).

Definition 1.2. Let $g(t)$ be an invertible series and let $f(t)$ be a delta series. A *generalized Riordan array* with respect to the sequence $(c_n)_{n\in\mathbb{N}_0}$ is a pair $\big(g(t), f(t)\big)$, which defines an infinite lower triangular array $(a_{n,k})_{0\leqq k\leqq n<\infty}$ according to the following rule:

$$a_{n,k} = \left[\frac{t^n}{c_n}\right] g(t)\, \frac{[f(t)]^k}{c_k}, \tag{2}$$

where such quotients as $\frac{g(t)[f(t)]^k}{c_k}$ are called the *column generating functions* of the Riordan array. In particular, the *classical Riordan arrays* correspond to the case when $c_n = 1$ and the *exponential Riordan arrays* correspond to the case when $c_n = n!$.

For any fixed sequence $(c_n)_{n\in\mathbb{N}_0}$, the set of all Riordan arrays $\big(g(t), f(t)\big)$ is a group under matrix multiplication and is called a *Riordan group* with respect to the sequence $(c_n)_{n\in\mathbb{N}_0}$. The identity of this group is $(1, t)$ and the inverse of the array $\big(g(t), f(t)\big)$ is

$$\left(\frac{1}{g\big(\bar{f}(t)\big)}, \bar{f}(t)\right),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$. Moreover, for any fixed sequence $(c_n)_{n\in\mathbb{N}_0}$, the Riordan group and the Sheffer group are isomorphic [22].

It is shown by Wang [23] that the Sheffer sequences can be expressed as determinants. The determinant form for the Sheffer sequences is given by

Let $\big(s_n(x)\big)_{n\in\mathbb{N}_0}$ be a Sheffer sequence for the pair $\big(g(t), f(t)\big)$. Then we have

$$s_0(x) = \frac{1}{a_{0,0}} \tag{3}$$

$$s_n(x)=\frac{(-1)^n}{a_{0,0}\,a_{1,1}\,\cdots\,a_{n,n}}\begin{vmatrix}1 & x & x^2 & \cdots & x^{n-1} & x^n\\ a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{n-1,0} & a_{n,0}\\ 0 & a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1}\\ 0 & 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1}\end{vmatrix}$$

$$=\frac{(-1)^n}{a_{0,0}\,a_{1,1}\,\cdots\,a_{n,n}}\det\begin{pmatrix}X_{n+1}\\ S_{n\times(n+1)}\end{pmatrix}, \tag{4}$$

where

$$X_{n+1}=(1,x,x^2,\cdots,x^{n-1},x^n) \qquad \text{and} \qquad S_{n\times(n+1)}=(a_{j-1,i-1})_{1\leqq i\leqq n;\,1\leqq j\leqq n+1}$$

and $a_{n,k}$ is the (n,k) entry of the Riordan array $\big(g(t),f(t)\big)$.

The Sheffer sequences $\big(s_n(x)\big)_{n\in\mathbb{N}_0}$ defined by the equation (1.4) also satisfy the following condition:

$$s_n(x)=\sum_{k=0}^{n} b_{n,k}\,x^k, \tag{5}$$

where $b_{n,k}$ is the (n,k) entry of the Riordan array $\left(\frac{1}{g\left(\overline{f}(t)\right)},\overline{f}(t)\right)$ and $\big(s_n(x)\big)_{n\in\mathbb{N}_0}$ is a Sheffer sequence for the pair $\big(g(t),f(t)\big)$ (see [22]).

For the pair $\big(g(t),f(t)\big)$, the exponential generating function and conjugate representation for the Sheffer sequences $s_n(x)$ are given by

$$\frac{1}{g\left(\overline{f}(t)\right)}\,\epsilon_x\left(\overline{f}(t)\right)=\sum_{n=0}^{\infty}s_n(x)\,\frac{t^n}{c_n}, \qquad \text{where} \qquad \epsilon_x(t)=\sum_{k=0}^{\infty}\frac{x^kt^k}{c_k} \tag{6}$$

and

$$s_n(x)=\sum_{k=0}^{n}\frac{\left\langle (g(\overline{f}(t)))^{-1}(\overline{f}(t))^k\mid x^n\right\rangle}{c_k}x^k. \tag{7}$$

In particular, the Sheffer sequence for the pair $\big(1,f(t)\big)$ is called the *associated Sheffer sequence* and the Sheffer sequence for the pair $(g(t),t)$ is called the *Appell sequence* for $g(t)$ (see, for details, [14]; see also [20]).

The Sheffer sequences $\{s_n(x)\}_{n\in\mathbb{N}_0}$ are shown to be quasi-monomial, for details see [2]. The Sheffer sequences are also studied from algebraic point of view, see for example [4]. According to the monomiality principle, there exist two operators Φ^+ and Φ^- playing, respectively, the roles of multiplicative and derivative operators for a polynomial set $\{s_n(x)\}_{n\in\mathbb{N}_0}$, that is, Φ^+ and Φ^- satisfy the following identities for all $n\in\mathbb{N}_0$:

$$\Phi^+\{s_n(x)\}=s_{n+1}(x) \qquad \text{and} \qquad \Phi^-\{s_n(x)\}=n\,s_{n-1}(x). \tag{8}$$

The polynomial set $\{s_n(x)\}_{n\in\mathbb{N}_0}$ is then called quasi-monomial. If Φ^+ and Φ^- have derivative-type realizations, then the polynomial set $\{s_n(x)\}_{n\in\mathbb{N}_0}$ satisfy the following differential equation:

$$\Phi^+\Phi^-\{s_n(x)\} = n\, s_n(x). \tag{9}$$

Several important results for the Apostol-Bernoulli, -Euler and -Genocchi polynomials are derived in [8, 21]. We give the following generalized unified form of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials (see also [12]):

For $\alpha,\ \beta \in \mathbb{C}$, $a,\ b \in \mathbb{R}\setminus\{0\}$ and $k \in \mathbb{N}_0$, the *generalized unified Apostol type polynomials* $\mathcal{P}^{(\alpha)}_{n,\beta}(x;k,a,b)$ are defined by the following generating function:

$$\left(\frac{2^{1-k}t^k}{\beta^b\ \epsilon(t) - a^b}\right)^{\alpha} \epsilon_x(t) = \sum_{n=0}^{\infty} \mathcal{P}^{(\alpha)}_{n,\beta}(x;k,a,b)\frac{t^n}{c_n}. \tag{10}$$

In fact, the following special cases hold:

$$\mathcal{P}^{(\alpha)}_{n,\lambda}(x;1,1,1) := \mathfrak{B}^{(\alpha)}_n(x;\lambda), \tag{11}$$

$$\mathcal{P}^{(\alpha)}_{n,\lambda}(x;0,-1,1) := \mathfrak{E}^{(\alpha)}_n(x;\lambda), \tag{12}$$

$$\mathcal{P}^{(\alpha)}_{n,\frac{\lambda}{2}}\left(x;1,-\frac{1}{2},1\right) := \mathcal{G}^{(\alpha)}_n(x;\lambda), \tag{13}$$

where $\mathfrak{B}^{(\alpha)}_n(x;\lambda)$, $\mathfrak{E}^{(\alpha)}_n(x;\lambda)$ and $\mathcal{G}^{(\alpha)}_n(x;\lambda)$ are the generalized forms of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials (see [9–11]).

We also note that

$$\mathfrak{B}^{(\alpha)}_n(x;1) := B^{(\alpha)}_n(x),\ \ \mathfrak{E}^{(\alpha)}_n(x;1) := E^{(\alpha)}_n(x) \quad \text{and} \quad \mathcal{G}^{(\alpha)}_n(x;1) := G^{(\alpha)}_n(x), \tag{14}$$

where

$$B^{(\alpha)}_n(x),\quad E^{(\alpha)}_n(x) \qquad \text{and} \qquad G^{(\alpha)}_n(x)$$

are the generalized Bernoulli, Euler and Genocchi polynomials (see [5, 16]). Similarly, we have

$$B^{(1)}_n(x) := B_n(x),\quad E^{(1)}_n(x) := E_n(x) \qquad \text{and} \qquad G^{(1)}_n(x) := G_n(x), \tag{15}$$

where $B_n(x)$, $E_n(x)$ and $G_n(x)$ are the Bernoulli, Euler and Genocchi polynomials (see [5, 16]).

In this article, the generalized Apostol type-Sheffer sequences are introduced and their quasi-monomial properties, determinant forms and series and conjugate representation are established via Riordan arrays. As the special cases of these sequences, the generalized Apostol Bernoulli, Euler and Genocchi-Sheffer sequences are deduced. The examples of some members belonging to the Sheffer sequences are framed to give the applications of main results. The numerical results to calculate the zeros and approximate solutions of these sequences are also given.

2. Generalized Apostol type-Sheffer sequences

The generalized Apostol type-Sheffer sequences are introduced as the combination of generalized Apostol type polynomials and Sheffer sequences via Riordan arrays. For this, we give the following definition:

Definition 2.1. *For the pair $\left(g(t), f(t)\right)$ and for all $x,\ \alpha,\ \beta \in \mathbb{C}$; $a,\ b \in \mathbb{R}\setminus\{0\}$ and $k \in \mathbb{N}_0$, the generating function for the generalized Apostol type-Sheffer sequences ${}_{\mathcal{P}}s_{n,\beta}^{(\alpha)}(x;k,a,b)$ is given by*

$$\frac{1}{g\left(\overline{f}(t)\right)}\left(\frac{2^{1-k}\left(\overline{f}(t)\right)^{k}}{\beta^{b}\epsilon(\overline{f}(t))-a^{b}}\right)^{\alpha}\epsilon_x\left(\overline{f}(t)\right)=\sum_{n=0}^{\infty}{}_{\mathcal{P}}s_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{c_n}. \tag{16}$$

The following special cases of ${}_{\mathcal{P}}s_{n,\beta}^{(\alpha)}(x;k,a,b)$ can be deduced as:

Case1: Put $\beta\to\lambda$ and $k=a=b=1$ and using relation

$${}_{\mathcal{P}}s_{n,\lambda}^{(\alpha)}(x;1,1,1)={}_{\mathfrak{B}}s_{n}^{(\alpha)}(x;\lambda) \tag{17}$$

in ${}_{\mathcal{P}}s_{n,\beta}^{(\alpha)}(x;k,a,b)$, we find the generalized Apostol-Bernoulli-Sheffer sequences ${}_{\mathfrak{B}}s_{n}^{(\alpha)}(x;\lambda)$ defined by

$$\frac{1}{g\left(\overline{f}(t)\right)}\left(\frac{\overline{f}(t)}{\lambda\ \epsilon(\overline{f}(t))-1}\right)^{\alpha}\epsilon_x\left(\overline{f}(t)\right)=\sum_{n=0}^{\infty}{}_{\mathfrak{B}}s_{n}^{(\alpha)}(x;\lambda)\frac{t^n}{c_n}. \tag{18}$$

Case2: Put $\beta\to\lambda$ and $k=a+1=b-1=0$ and using relation

$${}_{\mathcal{P}}s_{n,\lambda}^{(\alpha)}(x;0,-1,1)={}_{\mathfrak{E}}s_{n}^{(\alpha)}(x;\lambda) \tag{19}$$

in ${}_{\mathcal{P}}s_{n,\beta}^{(\alpha)}(x;k,a,b)$, we find the generalized Apostol-Euler-Sheffer sequences ${}_{\mathfrak{E}}s_{n}^{(\alpha)}(x;\lambda)$ defined by

$$\frac{1}{g\left(\overline{f}(t)\right)}\left(\frac{2}{\lambda\ \epsilon(\overline{f}(t))+1}\right)^{\alpha}\epsilon_x\left(\overline{f}(t)\right)=\sum_{n=0}^{\infty}{}_{\mathfrak{E}}s_{n}^{(\alpha)}(x;\lambda)\frac{t^n}{c_n}. \tag{20}$$

Case3: Put $\beta\to\frac{\lambda}{2}$, $k=1$, $a=-\frac{1}{2}$ and $b=1$ and using relation

$${}_{\mathcal{P}}s_{n,\frac{\lambda}{2}}^{(\alpha)}\left(x;1,-\frac{1}{2},1\right)={}_{\mathcal{G}}s_{n}^{(\alpha)}(x;\lambda) \tag{21}$$

in ${}_{\mathcal{P}}s_{n,\beta}^{(\alpha)}(x;k,a,b)$, we find the generalized Apostol-Genocchi-Sheffer sequences ${}_{\mathcal{G}}s_{n}^{(\alpha)}(x;\lambda)$ defined by

$$\frac{1}{g\left(\overline{f}(t)\right)}\left(\frac{2\overline{f}(t)}{\lambda\ \epsilon(\overline{f}(t))+1}\right)^{\alpha}\epsilon_x\left(\overline{f}(t)\right)=\sum_{n=0}^{\infty}{}_{\mathcal{G}}s_{n}^{(\alpha)}(x;\lambda)\frac{t^n}{c_n}. \tag{22}$$

We note that

$${}_{\mathfrak{B}}s_{n}^{(\alpha)}(x;1):={}_{B}s_{n}^{(\alpha)}(x),\quad {}_{\mathfrak{E}}s_{n}^{(\alpha)}(x;1):={}_{E}s_{n}^{(\alpha)}(x)\quad\text{and}\quad {}_{\mathcal{G}}s_{n}^{(\alpha)}(x;1):={}_{G}s_{n}^{(\alpha)}(x), \tag{23}$$

where

$${}_{B}s_{n}^{(\alpha)}(x),\quad {}_{E}s_{n}^{(\alpha)}(x)\qquad\text{and}\qquad {}_{G}s_{n}^{(\alpha)}(x)$$

are the generalized Bernoulli-Sheffer, the generalized Euler-Sheffer and the generalized Genocchi-Sheffer sequences.

Similarly, we have

$${}_{B}s_{n}^{(1)}(x):={}_{B}s_{n}(x),\quad {}_{E}s_{n}^{(1)}(x):={}_{E}s_{n}(x)\quad\text{and}\quad {}_{G}s_{n}^{(1)}(x):={}_{G}s_{n}(x), \tag{24}$$

where

$${}_{B}s_{n}(x),\quad {}_{E}s_{n}(x)\qquad\text{and}\qquad {}_{G}s_{n}(x)$$

are the Bernoulli-Sheffer, the Euler-Sheffer and the Genocchi-Sheffer sequences.

In order to show that the generalized Apostol type-Sheffer sequences are quasi-monomial, we prove the following results:

Theorem 2.2. *The multiplicative and derivative operators for the generalized Apostol type-Sheffer sequences are given by*

$$\Phi^{+}_{\mathcal{P}_S} = \left(\left(x - \frac{g'(D_x)}{g(D_x)} + \frac{\alpha k}{D_x} - \frac{\alpha\beta^b\epsilon(D_x)}{(\beta^b\epsilon(D_x) - a^b)}\right)\frac{1}{f'(D_x)}\right), \tag{25}$$

$$\Phi^{-}_{\mathcal{P}_S} = f(D_x). \tag{26}$$

Proof. On differentiation of generating equation (2.1) with respect to t and after simplification of the resultant equation, we obtain

$$\sum_{n=0}^{\infty} {}_{\mathcal{P}}s^{(\alpha)}_{n+1,\beta}(x;k,a,b)\frac{t^n}{c_n} = \left(\frac{2^{1-k}\left(\bar{f}(t)\right)^k}{\beta^b\ \epsilon(\bar{f}(t)) - a^b}\right)^{\alpha}\frac{1}{g\left(\bar{f}(t)\right)}\epsilon_x(\bar{f}(t)) \tag{27}$$

$$\left[x - \frac{g'\left(\bar{f}(t)\right)}{g\left(\bar{f}(t)\right)} - \frac{\alpha k}{\bar{f}(t)} - \frac{\alpha\beta^b\ \epsilon(\bar{f}(t))}{(\beta^b\ \epsilon(\bar{f}(t)) - a^b)}\right]\overline{f'}(\bar{f}(t)), \tag{28}$$

which on using generating equation (2.1) and the following identity:

$$f(D_x)\left\{\frac{1}{g\left(\bar{f}(t)\right)}\left(\frac{2^{1-k}\left(\bar{f}(t)\right)^k}{\beta^b\ \epsilon(\bar{f}(t)) - a^b}\right)^{\alpha}\epsilon_x(\bar{f}(t))\right\} = t\left\{\frac{1}{g\left(\bar{f}(t)\right)}\left(\frac{2^{1-k}\left(\bar{f}(t)\right)^k}{\beta^b\ \epsilon(\bar{f}(t)) - a^b}\right)^{\alpha}\epsilon_x(\bar{f}(t))\right\} \tag{29}$$

becomes

$$\sum_{n=0}^{\infty} {}_{\mathcal{P}}s^{(\alpha)}_{n+1,\beta}(x;k,a,b)\frac{t^n}{c_n} = \sum_{n=0}^{\infty}\left\{\left[x - \frac{g'\left(D_x\right)}{g\left(D_x\right)} - \frac{\alpha k}{D_x} - \frac{\alpha\beta^b\epsilon(D_x)}{(\beta^b\epsilon(D_x) - a^b)}\right]\overline{f'}(D_x)\right\}{}_{\mathcal{P}}s^{(\alpha)}_{n,\beta}(x;k,a,b)\frac{t^n}{c_n}. \tag{30}$$

Equating the coefficients of the same powers of t together with the use of the monomiality equation (1.8), we get the assertion (2.10).

Also, by using the equation (2.1) in the identity (2.14) and after some simplification, we find that

$$f(D_x)\sum_{n=0}^{\infty} {}_{\mathcal{P}}s^{(\alpha)}_{n,\beta}(x;k,a,b)\frac{t^n}{c_n} = \sum_{n=1}^{\infty} {}_{\mathcal{P}}s^{(\alpha)}_{n-1,\beta}(x;k,a,b)\frac{t^n}{c_{n-1}}. \tag{31}$$

Finally, by equating the coefficients of the same powers of t together with the use of the monomiality equation (1.8), we get the assertion (2.11). Our proof of Theorem 2.1 is thus completed. □

Theorem 2.3. *The generalized Apostol type-Sheffer sequences satisfy the following differential equation:*

$$\left(\left(x - \frac{g'(D_x)}{g(D_x)} + \frac{\alpha k}{D_x} - \frac{\alpha\beta^b\epsilon(D_x)}{(\beta^b\epsilon(D_x) - a^b)}\right)\frac{f(D_x)}{f'(D_x)} - n\right){}_{\mathcal{P}}s^{(\alpha)}_{n,\beta}(x;k,a,b) = 0. \tag{32}$$

Proof. Use of equations (2.10) and (2.11) in equation (1.9) yields assertion (2.17). □

To express the generalized Apostol type-Sheffer sequences in a determinant form, we prove the following theorem.

Theorem 2.4. *The generalized Apostol type-Sheffer sequences of degree n are given by*

$$_{\mathcal{P}}s^{(\alpha)}_{0,\beta}(x;k,a,b) = \frac{1}{a_{0,0}} \tag{33}$$

$$_{\mathcal{P}}s^{(\alpha)}_{n,\beta}(x;k,a,b) = \frac{(-1)^n}{a_{0,0}\ \cdots\ a_{n,n}} \begin{vmatrix} 1 & \mathcal{P}^{(\alpha)}_{1,\beta}(x;k,a,b) & \mathcal{P}^{(\alpha)}_{2,\beta}(x;k,a,b) & \cdots & \mathcal{P}^{(\alpha)}_{n-1,\beta}(x;k,a,b) & \mathcal{P}^{(\alpha)}_{n,\beta}(x;k,a,b) \\ a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{n-1,0} & a_{n,0} \\ 0 & a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix} \tag{34}$$

$$= \frac{(-1)^n}{a_{0,0}\ a_{1,1}\ \cdots\ a_{n,n}} \det\begin{pmatrix} \mathcal{P}^{(\alpha)}_{n+1,\beta}(x;k,a,b) \\ \mathcal{M}_{n\times(n+1)} \end{pmatrix}, \tag{35}$$

where

$$\mathcal{P}^{(\alpha)}_{n+1,\beta}(x;k,a,b) = \left(1, \mathcal{P}^{(\alpha)}_{1,\beta}(x;k,a,b), \mathcal{P}^{(\alpha)}_{2,\beta}(x;k,a,b), \cdots, \mathcal{P}^{(\alpha)}_{n,\beta}(x;k,a,b)\right)$$

and

$$\mathcal{M}_{n\times(n+1)} = (a_{j-1,i-1})_{1\leqq i\leqq n,\ 1\leqq j\leqq n+1},$$

$a_{n,k}$ *being the* (n,k) *entry of the Riordan array* $\left(g(t), f(t)\right)$.

Proof. Upon replacing the powers x^n $(n = 0,1,2\ldots)$ by the polynomials $\mathcal{P}^{(\alpha)}_{n,\beta}(x;k,a,b)$ $(n = 0,1,2\ldots)$ in the right-hand side and x by $\mathcal{P}^{(\alpha)}_{1,\beta}(x;k,a,b)$ in left-hand side of equations (1.3) and (1.4) and on appropriately using the following relation:

$$_{\mathcal{P}}s^{(\alpha)}_{n,\beta}(x;k,a,b) = s_n(\mathcal{P}^{(\alpha)}_{1,\beta}(x;k,a,b)) \tag{36}$$

in the left-hand side of the resultant equation yields assertions (2.18) and (2.19). □

Theorem 2.5. *The series and conjugate representations for the generalized Apostol type-Sheffer sequences are given by*

$$_{\mathcal{P}}s^{(\alpha)}_{n,\beta}(x;k,a,b) = \sum_{k=0}^{n} b_{n,k}\, \mathcal{P}^{(\alpha)}_{n,\beta}(x;k,a,b), \tag{37}$$

$$_{\mathcal{P}}s^{(\alpha)}_{n,\beta}(x;k,a,b) = \sum_{k=0}^{n} \frac{\left\langle (g(\overline{f}(t)))^{-1}(\overline{f}(t))^k \mid \mathcal{P}^{(\alpha)}_{n,\beta}(x;k,a,b)\right\rangle}{c_k} \mathcal{P}^{(\alpha)}_{k,\beta}(x;k,a,b), \tag{38}$$

where $b_{n,k}$ *is the is the* (n,k) *entry of the Riordan array* $\left(\frac{1}{g(\overline{f}(t))}, \overline{f}(t)\right)$ *and* $(_{\mathcal{P}}s^{(\alpha)}_{n,\beta}(x;k,a,b))_{n\in\mathbb{N}}$ *is Sheffer for the pair* $\left((g(t), f(t))\right)$.

Proof. By replacing the powers x^n $(n = 1,2\ldots)$ by the polynomials $\mathcal{P}^{(\alpha)}_{n,\beta}(x;k,a,b)$ $(n = 1,2\ldots)$ in both sides of equation (1.5) and (1.7) with use of relation (2.21), we get assertions (2.22) and (2.23). □

3. Examples

By choosing particular members of the Sheffer family, we explore some new special members belonging to the generalized Apostol type-Sheffer family. The corresponding properties are also obtained.

Example 3.1. *Taking*

$$g(t)=\frac{1}{(1-t)^{\gamma+1}} \qquad \text{and} \qquad f(t)=\bar{f}(t)=\frac{t}{t-1}$$

of the associated Laguerre sequences $L_n^{(\gamma)}(x)$ [14] in generating function (2.1), we find that the resultant generalized Apostol-Bernoulli-associated Laguerre sequences ${}_{\mathfrak{B}}L_n^{(\alpha,\gamma)}(x;\lambda)$ are defined by

$$(1-t)^{\gamma+1}\left(\frac{t}{(t-1)(\lambda\,\epsilon(\frac{t}{t-1})-1)}\right)^{\alpha}\epsilon_x\Big(\frac{t}{t-1}\Big)=\sum_{n=0}^{\infty}{}_{\mathfrak{B}}L_n^{(\alpha,\gamma)}(x;\lambda)\frac{t^n}{c_n}. \tag{39}$$

The multiplicative and derivative operators and differential equation for the generalized Apostol-Bernoulli-associated Laguerre sequences are given by

$$\Phi^{+}_{\mathfrak{B}L}=\left(-\left(x+\frac{(1-D_x)}{\gamma+1}+\frac{\alpha}{D_x}-\frac{\alpha\lambda\,\epsilon(D_x)}{(\lambda\,\epsilon(D_x)-1)}\right)(D_x-1)^2\right),\qquad \Phi^{-}_{\mathfrak{B}L}=-\frac{1}{(D_x-1)^2}, \tag{40}$$

$$\left(x+\frac{(1-D_x)}{\gamma+1}+\frac{\alpha}{D_x}-\frac{\alpha\lambda\epsilon(D_x)}{(\lambda\epsilon(D_x)-1)}-n\right){}_{\mathfrak{B}}L_n^{(\alpha,\gamma)}(x;\lambda)=0. \tag{41}$$

Taking

$$a_{n,k}=(-1)^k\frac{n!}{k!}\binom{n+\gamma}{n-k}$$

and then using relations (1.11) in the right-hand side and (2.2) with $s_n(x)=L_n^{(\gamma)}(x)$ in the left-hand side of equations (2.18) and (2.19), the following determinant form for the ${}_{\mathfrak{B}}L_n^{(\alpha,\gamma)}(x;\lambda)$ is obtained as:

$${}_{\mathfrak{B}}L_0^{(\alpha,\gamma)}(x;\lambda)=1,$$

$${}_{\mathfrak{B}}L_n^{(\alpha,\gamma)}(x;\lambda)=(-1)^{\frac{n(n+3)}{2}}\begin{vmatrix} 1 & \mathfrak{B}_1^{(\alpha)}(x;\lambda) & \mathfrak{B}_2^{(\alpha)}(x;\lambda) & \cdots & \mathfrak{B}_{n-1}^{(\alpha)}(x;\lambda) & \mathfrak{B}_n^{(\alpha)}(x;\lambda)\\ 1 & (\gamma+1) & (\gamma+2)_2 & \cdots & (\gamma+n-1)_{n-1} & (\gamma+n)_n\\ 0 & -1 & -2(\gamma+2) & \cdots & -(n-1)(\gamma+n-1)_{n-2} & -n(\gamma+n)_{n-1}\\ 0 & 0 & 1 & \cdots & \frac{(n-1)(n-2)}{2}(\gamma+n-1)_{n-3} & \frac{n(n-1)}{2}(\gamma+n)_{n-2}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & (-1)^{n-1} & (-1)^{n-1}n(n+\gamma) \end{vmatrix}. \tag{42}$$

The series and conjugate representations for the sequences ${}_{\mathfrak{B}}L_n^{(\alpha,\gamma)}(x;\lambda)$ are given by

$${}_{\mathfrak{B}}L_n^{(\alpha,\gamma)}(x;\lambda)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n+\gamma}{n-k}\mathfrak{B}_k^{(\alpha)}(x;\lambda), \tag{43}$$

$${}_{\mathfrak{B}}L_n^{(\alpha,\gamma)}(x;\lambda)=\sum_{k=0}^{n}\frac{\left\langle(1-t)^{-\gamma-1}\left(\frac{t}{t-1}\right)^k\mid\mathfrak{B}_n^{(\alpha)}(x;\lambda)\right\rangle}{c_k}\mathfrak{B}_k^{(\alpha)}(x;\lambda). \tag{44}$$

Example 3.2. *Taking*

$$g(t) = e^{\nu t^2/2} \qquad \text{and} \qquad f(t) = \bar{f}(t) = t$$

of the Hermite polynomials $H_n^{(\nu)}(x)$ of variance ν [14] in generating function (2.1), we find that the resultant generalized Apostol-Euler-Hermite sequences ${}_{\mathfrak{E}}H_n^{(\alpha,\nu)}(x;\lambda)$ are defined by

$$e^{-\nu t^2/2}\left(\frac{2}{\lambda\,\epsilon(t)+1}\right)^{\alpha}\epsilon_x(t) = \sum_{n=0}^{\infty} {}_{\mathfrak{E}}H_n^{(\alpha,\nu)}(x;\lambda)\frac{t^n}{c_n}. \tag{45}$$

The multiplicative and derivative operators and differential equation for the generalized Apostol-Euler-Hermite sequences ${}_{\mathfrak{E}}H_n^{(\alpha,\nu)}(x;\lambda)$ of variance ν are given by

$$\Phi^{+}_{\mathfrak{E}H} = \left(x - \nu D_x - \frac{\alpha\lambda\epsilon(D_x)}{(\lambda\epsilon(D_x)+1)}\right), \qquad \Phi^{-}_{\mathfrak{E}H} = D_x, \tag{46}$$

$$\left(xD_x - \nu D_x^2 - \frac{\alpha\lambda\epsilon(D_x)}{(\lambda\epsilon(D_x)+1)} - n\right){}_{\mathfrak{E}}H_n^{(\alpha,\nu)}(x;\lambda) = 0. \tag{47}$$

Taking

$$a_{n,k} = \binom{n}{k} g_{n-k},$$

where

$$g_k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \dfrac{k!\left(\frac{\nu}{2}\right)^{k/2}}{\left(\frac{k}{2}\right)!} & \text{if } k \text{ is even,} \end{cases} \tag{48}$$

and then using relations (1.12) in the right-hand side and (2.4) with $s_n(x) = H_n^{(\nu)}(x)$ in the left-hand side of equations (2.18) and (2.19), the following determinant form for the sequences ${}_{\mathfrak{E}}H_n^{(\alpha,\nu)}(x;\lambda)$ is obtained as:

$$\begin{aligned} &{}_{\mathfrak{E}}H_0^{(\alpha,\nu)}(x;\lambda) = 1, \\ &{}_{\mathfrak{E}}H_n^{(\alpha,\nu)}(x;\lambda) = (-1)^n \begin{vmatrix} 1 & \mathfrak{E}_1^{(\alpha)}(x;\lambda) & \mathfrak{E}_2^{(\alpha)}(x;\lambda) & \cdots & \mathfrak{E}_{n-1}^{(\alpha)}(x;\lambda) & \mathfrak{E}_n^{(\alpha)}(x;\lambda) \\ 1 & 0 & 1 & \cdots & \frac{(n-1)!\left(\frac{\nu}{2}\right)^{n-1/2}}{\left(\frac{n-1}{2}\right)!} & \frac{n!\left(\frac{\nu}{2}\right)^{n/2}}{\left(\frac{n}{2}\right)!} \\ 0 & 1 & 0 & \cdots & \binom{n-1}{1}\frac{(n-2)!\left(\frac{\nu}{2}\right)^{n-2/2}}{\left(\frac{n-2}{2}\right)!} & \binom{n}{1}\frac{(n-1)!\left(\frac{\nu}{2}\right)^{n-1/2}}{\left(\frac{n-1}{2}\right)!} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{(n-3)!\left(\frac{\nu}{2}\right)^{n-3/2}}{\left(\frac{n-3}{2}\right)!} & \binom{n}{2}\frac{(n-2)!\left(\frac{\nu}{2}\right)^{n-2/2}}{\left(\frac{n-2}{2}\right)!} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}. \end{aligned} \tag{49}$$

The series and conjugate representations for the sequences ${}_{\mathfrak{E}}H_n^{(\alpha,\nu)}(x;\lambda)$ are given by

$${}_{\mathfrak{E}}H_n^{(\alpha,\nu)}(x;\lambda) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k\nu^k n!\,\mathfrak{E}_{n-2k}^{(\alpha)}(x;\lambda)}{2^k\,k!\,(n-2k)!}, \tag{50}$$

$$
{}_{\mathfrak{E}}H_n^{(\alpha,\nu)}(x;\lambda) = \sum_{k=0}^{n} \frac{\left\langle e^{\frac{\nu t^2}{2}} t^k \mid \mathfrak{E}_n^{(\alpha)}(x;\lambda)\right\rangle}{c_k} \mathfrak{E}_k^{(\alpha)}(x;\lambda). \tag{51}
$$

Example 3.3. *Taking*

$$
g(t) = \left(\frac{2}{1+\sqrt{1-t^2}}\right)^{1/2}, \qquad f(t) = \frac{-t}{1+\sqrt{1-t^2}} \qquad \text{and} \qquad \bar{f}(t) = \frac{-2t}{1+t^2},
$$

of the Legendre polynomials $P_n(x)$ [1] in generating function (2.1), we find that the resultant generalized Apostol-Genocchi-Legendre sequences ${}_{\mathcal{G}}P_n^{(\alpha)}(x;\lambda)$ are defined by

$$
\frac{1}{\sqrt{1+t^2}}\left(\frac{-4t}{(1+t^2)(\lambda\, \epsilon\left(\frac{-2t}{1+t^2}\right)+1)}\right)^{\alpha} \epsilon_x\left(\frac{-2t}{1+t^2}\right) = \sum_{n=0}^{\infty} {}_{\mathcal{G}}P_n^{(\alpha)}(x;\lambda)\frac{t^n}{c_n}. \tag{52}
$$

The multiplicative and derivative operators and differential equation for the generalized Apostol-Genocchi-Legendre sequences ${}_{\mathcal{G}}P_n^{(\alpha)}(x;\lambda)$ are given by

$$
\Phi^+_{\mathcal{G}P} = \left(\left(x - \frac{(1+\sqrt{1-D_x^2})^{3/2}}{4\sqrt{-D_x}(1-D_x^2)} + \frac{\alpha}{D_x} - \frac{\alpha\lambda\epsilon(D_x)}{2(\lambda\epsilon(D_x)+1)}\right)\frac{(\sqrt{1-D_x^2}-2D_x^2+1)}{D_x^2\sqrt{1-D_x^2}}\right), \quad \Phi^-_{\mathcal{G}P} = \frac{-2D_x}{1+D_x^2} \tag{53}
$$

$$
\left(\left(x - \frac{(1+\sqrt{1-D_x^2})^{3/2}}{4\sqrt{-D_x}(1-D_x^2)} + \frac{\alpha}{D_x} - \frac{\alpha\lambda\epsilon(D_x)}{2(\lambda\epsilon(D_x)+1)}\right)\frac{(-2D_x\sqrt{1-D_x^2}+4D_x^3-2D_x)}{(D_x^2+D_x^4)\sqrt{1-D_x^2}} - n\right){}_{\mathcal{G}}P_n^{(\alpha)}(x;\lambda) = 0. \tag{54}
$$

Taking

$$
a_{n,k} = \left\{\begin{array}{ll} 0 & (n-k \text{ is odd}) \\ \frac{c_n(-1)^k\,(2k+1)}{c_k\,2^n\,(2n+1)}\left[\binom{\frac{1}{2}+n}{\frac{n-k}{2}}\right] & (n-k \text{ is even}) \end{array}\right\}, \tag{55}
$$

where

$$
c_n = \frac{1}{\binom{-1/2}{n}} \qquad \text{and} \qquad \binom{-n}{k} = (-1)^k\binom{n+k-1}{k}
$$

and then using relations (1.13) in the right-hand side and (2.6) with $s_n(x) = P_n(x)$ in the left-hand side of equations (2.18) and (2.19), the following determinant form for the ${}_{\mathcal{G}}P_n^{(\alpha)}(x;\lambda)$ is obtained as:

$$
{}_{\mathcal{G}}P_0^{(\alpha)}(x;\lambda) = 1,
$$

$$
{}_{\mathcal{G}}P_n^{(\alpha)}(x;\lambda) = (-1)^{\frac{n(n+3)}{2}} 2^{\frac{n(n+1)}{2}} \begin{vmatrix} 1 & \mathcal{G}_1^{(\alpha)}(x;\lambda) & \mathcal{G}_2^{(\alpha)}(x;\lambda) & \cdots & \mathcal{G}_{n-1}^{(\alpha)}(x;\lambda) & \mathcal{G}_n^{(\alpha)}(x;\lambda) \\ 1 & 0 & \frac{1}{3} & \cdots & 0 & \frac{n!\frac{1}{2}\left(n-\frac{1}{2}\right)!}{2^n\left(\frac{n}{2}\right)!\frac{1}{2}\frac{3}{2}\cdots\left(n-\frac{1}{2}\right)\left(\frac{n+1}{2}\right)!} \\ 0 & -\frac{1}{2} & 0 & \cdots & & 0 \\ 0 & 0 & \frac{1}{4} & \cdots & 0 & \frac{n(n-1)\ldots 3\frac{5}{2}\left(n-\frac{1}{2}\right)!}{2^n\left(\frac{n-2}{2}\right)!\frac{5}{2}\cdots\left(n-\frac{1}{2}\right)\left(\frac{n+3}{2}\right)!} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \left(\frac{-1}{2}\right)^{n-1} & 0 \end{vmatrix}. \tag{56}
$$

The series and conjugate representations for the sequences ${}_{\mathcal{G}}P_n^{(\alpha)}(x;\lambda)$ *are given by*

$$ {}_{\mathcal{G}}P_n^{(\alpha)}(x;\lambda)=\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(\frac{1}{2})_{n-k}2^{n-2k}\mathcal{G}_{n-2k}^{(\alpha)}(x)}{k!\,(n-2k)!}, \tag{57}$$

where $(x)_n:=\prod_{k=0}^{n-1}(x-k)=x(x-1)(x-2)\ldots(x-n+1)$ *being the falling factorial and*

$$ {}_{\mathcal{G}}P_n^{(\alpha)}(x;\lambda)=\sum_{k=0}^{n}\frac{\left\langle(1+t^2)^{-1}\left(\frac{-2t}{1+t^2}\right)^k \mid \mathcal{G}_n^{(\alpha)}(x;\lambda)\right\rangle}{c_k}\mathcal{G}_k^{(\alpha)}(x;\lambda). \tag{58}$$

In the next section, we give the numerical results to compute the zeros and approximate solutions for the generalized hybrid Sheffer sequences.

4. Numerical Results

Recently, the computing environment is making more and more rapid progress. By using numerical investigations and computer experiments, we find the real zeros and observe the phenomenon of distribution of the real zeros of some hybrid Sheffer sequences for certain values of index n, which seems to be an interesting approach.

By taking $\alpha=\lambda=\gamma=\nu=1$ in the generalized Apostol-Bernoulli-associated Laguerre sequences, generalized Apostol-Euler-Hermite sequences and generalized Apostol-Genocchi-Legendre sequences, we find

$$ {}_{B}L_n(x):={}_{\mathfrak{B}}L_n^{(1,1)}(x;1);\qquad {}_{E}H_n(x):={}_{\mathfrak{E}}H_n^{(1,1)}(x;1)\qquad\text{and}\qquad {}_{G}P_n(x):={}_{\mathcal{G}}P_n^{(1)}(x;1),$$

where

$$ {}_{B}L_n(x),\qquad {}_{E}H_n(x)\qquad\text{and}\qquad {}_{G}P_n(x)$$

are the Bernoulli-Laguerre, Euler-Hermite and Genocchi-Legendre sequences.

To investigate the zeros of above sequences, we need the expressions of first few Bernoulli polynomials $B_n(x)$, Euler polynomials $E_n(x)$ and Genocchi polynomials $G_n(x)$. These are given in Table 1:

Table 1. Expressions of first six $B_n(x)$, $E_n(x)$ and $G_n(x)$.

n	0	1	2	3	4	5
$B_n(x)$	1	$x-\frac{1}{2}$	$x^2-x+\frac{1}{6}$	$x^3-\frac{3}{2}x^2+\frac{x}{2}$	$x^4-2x^3+x^2-\frac{1}{30}$	$x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{x}{6}$
$E_n(x)$	1	$x-\frac{1}{2}$	x^2-x	$x^3-\frac{3}{2}x^2+\frac{1}{6}$	$x^4-2x^3+\frac{2}{3}x$	$x^5-\frac{5}{2}x^4+\frac{5}{3}x^2-\frac{1}{2}$
$G_n(x)$	0	1	$2x-1$	$3x^2-3x$	$4x^3-6x^2+1$	$5x^4-10x^3+5x$

From Table 1(III), it is to be noted that the degree of $G_n(x)$ *is* $n-1$*. Therefore,* $G_n(x)$ *is considered in the class of polynomial sequences which are not Apostol type in the strong sense.*

By making use of expressions given in Table 1 with $\alpha=\lambda=\gamma=\nu=1$ in equations (3.5), (3.12) and (3.19), we find the expressions of Bernoulli-Laguerre, Euler-Hermite and Genocchi-Legendre sequences for $n=0,\ 1,\ 2,\ 3,\ 4,\ 5$. These are given in Table 2:

Table 2. Expressions of first six ${}_BL_n(x)$, ${}_EH_n(x)$ and ${}_GP_n(x)$.

n	0	1	2	3	4	5
${}_BL_n(x)$	1	$-x+\frac{3}{2}$	$\frac{x^2}{2}-\frac{7x}{2}+\frac{19}{12}$	$-\frac{x^3}{6}+\frac{9}{4}x^2-\frac{97x}{12}+\frac{22}{3}$	$\frac{x^4}{24}-\frac{11x^3}{12}+\frac{151x^2}{24}$ $-\frac{185x}{12}+\frac{7799}{720}$	$-\frac{x^5}{120}+\frac{13}{48}x^4-\frac{217}{72}x^3$ $+\frac{56x^2}{4}-\frac{18899x}{720}+\frac{1819}{120}$
${}_EH_n(x)$	1	$x-\frac{1}{2}$	x^2-x-1	$x^3-\frac{3}{2}x^2-3x+\frac{10}{6}$	$x^4-2x^3-6x^2+\frac{20}{3}x+3$	$x^5-\frac{5}{2}x^4-10x^3$ $+\frac{50}{3}x^2+15x+\frac{32}{6}$
${}_GP_n(x)$	0	1	$-x+\frac{1}{2}$	$\frac{3x^2}{2}-\frac{3x}{2}-1$	$-\frac{5x^3}{2}+\frac{15x^2}{4}-\frac{3x}{2}+\frac{1}{8}$	$\frac{35x^4}{8}-\frac{70x^3}{8}+\frac{15x^2}{4}+\frac{5x}{8}+\frac{3}{8}$

The manual computation of zeros is difficult, so we use "*Matlab software*" to compute the zeros of ${}_BL_n(x)$, ${}_EH_n(x)$ and ${}_GP_n(x)$. The zeros of these sequences are given in Table 3.

Table 3. Zeros of ${}_BL_n(x)$, ${}_EH_n(x)$ and ${}_GP_n(x)$

Degree n	${}_BL_n(x)$	${}_EH_n(x)$	${}_GP_n(x)$
1	1.5000	0.5000	-
2	6.5139, 0.4861	-0.6180, 1.6180	0.5000
3	8.2896, 3.8214, 1.3890	2.4475, -1.4253, 0.4778	1.4574, -0.4574
4	11.4879, 6.2582, 3.0799, 1.1741	3.1379, -2.0836, 1.2989, -0.3533	0.8873, 0.5000, 0.1127
5	14.7958, 8.9308, 5.1342, 2.6132, 1.0260	-2.7385, 3.6502, 2.3150 -0.3634 + 0.3137i, -0.3634 - 0.3137i	1.1125 + 0.2318i, 1.1125 - 0.2318i -0.1125 + 0.2318i, -0.1125 - 0.2318i

In order to make the above discussions more clear, we draw the graphs showing shapes with scattered zeros of the sequences ${}_BL_n(x)$, ${}_EH_n(x)$ and ${}_GP_n(x)$ by making use of corresponding expressions from Table 2.

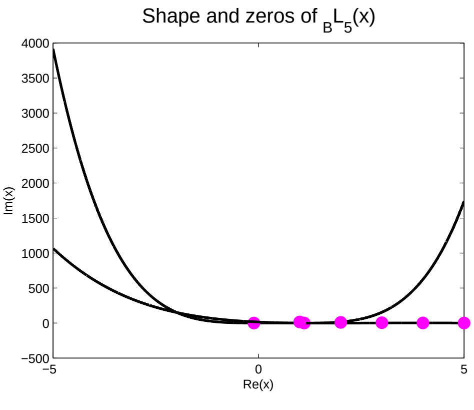

Figure 1

Note 1. It is to be noted that in Figure 1 there is no any complex zero and all five real zeros are located on real line $Im(x) = 0$.

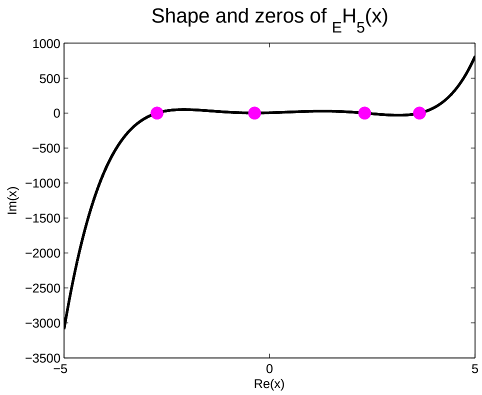

Figure 2

Note 2. It is to be noted that in Figure 2 out of total two complex zeros only one with positive imaginary part is visible, due to the absence of negative imaginary axis in these graphs.

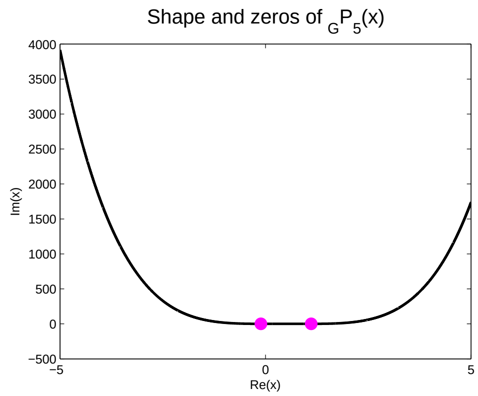

Figure 3

Note 3. It is to be noted that in Figure 3 all the four are complex zeros and out of this only two with positive imaginary part is visible, due to the absence of negative imaginary axis in these graphs.

We note that the real zeros of the sequences $_BL_n(x)$, $_EH_n(x)$ and $_GP_n(x)$ give the numerical results for the approximate solutions of these sequences. These solutions are given in Table 4.

Table 4. Approximate solutions of $_BL_n(x) = 0$, $_EH_n(x) = 0$ and $_GP_n(x) = 0$, $x \in \mathbb{R}$

Degree n	Real zeros of $_BL_n(x)$	Real zeros of $_EH_n(x)$	Real zeros of $_GP_n(x)$
1	1.5000	0.5000	×
2	6.5139, 0.4861	-0.6180, 1.6180	0.5000
3	8.2896, 3.8214, 1.3890	2.4475, -1.4253, 0.4778	1.4574, -0.4574
4	11.4879, 6.2582, 3.0799, 1.1741	3.1379, -2.0836, 1.2989, -0.3533	0.8873, 0.5000, 0.1127
5	14.7958, 8.9308, 5.1342, 2.6132, 1.0260	-2.7385, 3.6502, 2.3150	×

Remark 4.1. From Table 3, the following general relation is observed: denoting by m the number of Complex zeros of ($_BL_n(x)$, $_EH_n(x)$, $_GP_n(x)$), the number of Real zeros of ($_BL_n(x)$, $_EH_n(x)$, $_GP_n(x)$), i.e. the zeros lying on the real plane Im $(x) = 0$ is given by $n - m$, where n is the degree of polynomial.

5. Conclusion

A hybrid family of generalized Apostol type-Sheffer sequences is introduced and their properties comprising the quasi-monomiality, determinant form and series and conjugate representations are investigated by making use of Riordan arrays. Several examples are framed in terms of the members of the Sheffer sequences. The numerical results to calculate the zeros and approximate solutions of these sequences are given and their graphical representations are shown. With a view to further generalize the hybrid families associated with Sheffer sequences to their q-analogues and studying their properties via q-Riordan arrays is a subject of new research work.

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