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Approximate Proper Efficiency for Multiobjective Optimization Problems

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Abstract. This paper is devoted to the study of a new kind of approximate proper efficiency in terms of proximal normal cone and co-radiant set for multiobjective optimization problem. We derive some properties of the new approximate proper efficiency and discuss the relations with the existing approximate concepts, such as approximate efficiency and approximate Benson proper efficiency. At last, we study the linear scalarizations for the new approximate proper efficiency under the generalized convexity assumption and give some examples to illustrate the main results.

1. Introduction

In the past few decades, the (weak, proper) efficient solutions of the multiobjective optimization problem were studied in many ways. When the existence conditions for these exact solutions were investigated, it is found that the compact conditions of constraint sets can not be removed. On the other hand, optimization models are solved frequently by using iterative algorithms or heuristic methods, and these procedures give approximations to the theoretical solution. For these reasons, many research focused on approximate concepts and study their characterizations and applications. The first concept of approximate solutions was introduced by Kutateladze [1] mainly to study the convex optimization problem. In the middle of the 1980s, Loridan [2] introduced the approximate efficient solutions for multiobjective optimization problem. And later, White [3] and Helbig [4] also gave the several concepts of approximate solutions for multiobjective optimization problem by using different tools. Notice that hereafter, the above approximate solutions for multiobjecitve optimization problems can be characterized by the co-radiant set, Gutiérrez et al [5] introduced a new kind of approximate solutions to unify several existing approximate solutions, and they also established nonlinear scalarization results for the unified approximate solutions. On the basis of these concepts, more characterizations, such as: the existence conditions, scalarizations, Lagrange multiplies rules and saddle points theorems were studied for the solutions of multiobjecitve optimization problems[6-15]. Especially, the optimality solutions can be characterized with the help of geometrical concepts, such as tangent cones and normal cones[12-13]. Lalitha et al [14] introduced a new proper efficiency by proximal normal cone, and relate it with Benson and Borwein proper efficiency. Moreover,

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they gave the linear scalarization results for the proximal proper efficiency under the generalized convexity assumption. Recently, Shahbeyk et al [15] defined the limiting proper minimal point for a nonconvex set with the limiting normal cone, and investigated its several properties.

Based on the proper efficiency in[14] and the approximate solution in [5,7], we want to give a new kind of approximate proper efficiency, and study the relation with existing approximate solutions as well as its scalarization results. This paper is organized as follows. In Section 2, some definitions and lemmas are given. The new approximate proper efficiency are defined in Section 3, the properties of $\varepsilon-$Proximal proper efficiency as well as the relationship with other approximate solutions are presented. Especially, the $\varepsilon-$Proximal proper efficient point can reduce to Proximal proper efficiency. Furthermore, under the certain conditions, we also illustrate the difference between $\varepsilon-$Proximal proper efficiency and some existing approximate proper efficient point. In section 4, the linear scalarizations for $\varepsilon-$Proximal proper efficiency are established under the locally starshapeness assumptions.

2. Preliminary

Let $\mathbb{R}^p$ be the p-dimensional Euclidean space and $\mathbb{R}^p_+$be its non-negative orthant. The int(C), cl(C) and conv(C) denote the interior, the closure and the convex hull of $C \subseteq \mathbb{R}^p$. If for all $c \in C$ and $\lambda \geqslant 0$, $\lambda c \in C$, then C is called cone. C is called pointed if $C \cap (-C) \subseteq \{0\}$. The generated cone of C is defined as

$$\text{cone}(C) := \{\lambda v : \lambda \geqslant 0, v \in C\}.$$

C is said to be a co-radiant set if $\alpha d \in C$ for all $d \in C, \alpha \geqslant 1$. Let $C(\varepsilon) = \varepsilon C, \forall \varepsilon > 0$ and $C(0) = \bigcup\limits_{\varepsilon>0} C(\varepsilon)$ (see [5]).

The positive dual cone and strict positive dual cone of C are defined as

$$C^+ = \{d \in \mathbb{R}^p | \langle d, c\rangle \geqslant 0, \forall c \in C\};$$
$$C^{s+} = \{d \in \mathbb{R}^p | \langle d, c\rangle > 0, \forall c \in C\backslash\{0\}\}.$$

The tangent cones play an important role in the field of optimization, we now introduce the definitions of two types of tangent cones and the corresponding normal cones in [16].

Let Y be a set in $\mathbb{R}^p$ and $\bar{y} \in Y$. The tangent cone to Y at $\bar{y}$, denoted by $T(Y, \bar{y})$ is defined as

$$T(Y, \bar{y}) = \{d \in \mathbb{R}^p | \exists\, t_j \downarrow 0,\ d_j \to d \textit{ with } d_j \in Y \text{ such that } \bar{y} + t_j d_j \in Y\}.$$

The Clarke tangent cone to Y at $\bar{y}$, denoted by $T_c(Y, \bar{y})$ is defined as

$$T_c(Y, \bar{y}) = \{d \in \mathbb{R}^p | \forall\, t_j \downarrow 0,\ y_j \to \bar{y} \textit{ with } y_j \in Y,\ \exists\, d_j \to d \textit{ such that } y_j + t_j d_j \in Y\}.$$

Both tangent and Clarke tangent cones are closed and

$$T_c(Y, \bar{y}) \subseteq T(Y, \bar{y}) \subseteq clcone(Y - \bar{y}). \tag{2.1}$$

The notion of tangential regularity has been considered in [12]. Y is said to be tangentially regular at $\bar{y}$ if $T_c(Y, \bar{y}) = T(Y, \bar{y})$. It is obvious that if Y is convex , it is tangentially regular at any $\bar{y} \in Y$.

The normal cone to Y at $\bar{y}$, denoted by $N(Y, \bar{y})$, is the negative dual of $T(Y, \bar{y})$, that is

$$N(Y, \bar{y}) = \{d \in \mathbb{R}^p | \langle d, h\rangle \leqslant 0, \forall h \in T(Y, \bar{y})\},$$

and the Clarke normal cone to Y at $\bar{y}$, denoted by $N_c(Y, \bar{y})$, is the negative dual of $T_c(Y, \bar{y})$, that is

$$N_c(Y, \bar{y}) = \{d \in \mathbb{R}^p | \langle d, h\rangle \leqslant 0, \forall h \in T_c(Y, \bar{y})\}.$$

In [17], the proximal normal cone was introduced for a nonempty subset Y at a point $\bar{y} \in Y$. Let x be a point not lying in Y, and $\bar{y}$ be the projection of x onto Y, that is,

$$\|x - \bar{y}\| = \min_{y \in Y} \|x - y\|,$$

then the vector $x-\bar{y}$ is called proximal normal direction to Y at $\bar{y}$, any nonnegative multiples of such vectors are called proximal normal to Y at $\bar{y}$ and the set of all such vectors form the proximal normal cone to Y at $\bar{y}$, which is denoted by $N_p(Y,\bar{y})$. The proximal normal cone has the following properties.

Lemma 2.1[17] *Vector ξ belongs to $N_p(Y,\bar{y})$ if and only if there exists $\delta=\delta(\xi,\bar{y})\geqslant 0$ such that*

$$\langle \xi, y-\bar{y}\rangle \leqslant \delta\|y-\bar{y}\|^2 \ , \ \forall y\in Y.$$

In particular, if Y be a closed convex set, then

$$\langle \xi, y-\bar{y}\rangle \leqslant 0 \ , \ \forall y\in Y.$$

The following existing results will be needed to establish our main results.

Lemma 2.2[18]

(i) *Let C be a pointed convex cone in $\mathbb{R}^p$, then*

$$(-C)^+=-C^+,\ C^{s+}+C^+\subseteq C^{s+};$$

(ii) *If A and B are subsets of $\mathbb{R}^p$ such that $A\subseteq B$, then $B^+\subseteq A^+$;*

(iii) *Let $C_1, C_2, ..., C_m$ be a nonempty convex cones in $\mathbb{R}^p$, then*

$$(clC_1\cap clC_2\cap ...\cap C_m)^+=cl(C_1^++C_2^++...+C_m^+).$$

Lemma 2.3[5] *Let C be a pointed co-radiant set, then*

(i) $C(\varepsilon)$ *is a pointed co-radiant set for all $\varepsilon>0$.*

(ii) *If $0<\varepsilon_1<\varepsilon_2$, then $C(\varepsilon_2)\subseteq C(\varepsilon_1)$.*

(iii) $C(0)$ *is a pointed cone.*

3. Cone characterizations of approximate solutions

In this section, we introduce a new kind of approximate proper efficient concepts and consider their properties. Let Y be a closed set in $\mathbb{R}^p$, C be a closed pointed co-radiant set in $\mathbb{R}^p$ and D be a closed convex pointed cone in $\mathbb{R}^p$.

We now introduce the following existing efficient and proper efficient points with respect to convex cone D(see [14, 18]).

Definition 3.1 *Let $\bar{y}\in Y$.*

(i) *$\bar{y}$ is called an efficient point of Y with respect to D, written $\bar{y}\in E[Y,D]$, if*

$$(Y-\bar{y})\cap -D=\{0\}.$$

(ii) *$\bar{y}$ is called a weakly efficient point of Y with respect to D, written $\bar{y}\in WE[Y,D]$, if*

$$(Y-\bar{y})\cap -intD=\emptyset.$$

(iii) *$\bar{y}$ is called Benson properly efficient point of Y with respect to D, written $\bar{y}\in Ben[Y,D]$, if*

$$\text{clcone}(Y+D-\bar{y})\cap -D=\{0\}.$$

(iv) *$\bar{y}$ is called Borwein properly efficient point of Y with respect to D, written $\bar{y}\in Bor[Y,D]$, if*

$$\text{T}(Y+D,\bar{y})\cap -D=\{0\}.$$

(v) *$\bar{y}$ is called a proximal properly efficient point of Y with respect to D, written $\bar{y} \in P_r[Y, D]$, if $\bar{y}$ is an efficient point of Y, and*

$$N_p(Y + D, \bar{y}) \cap (-D^{s+}) \neq \emptyset.$$

In recent years, Gutiérrez and Gao et al studied the following approximate efficiency and proper efficiency by using the co-radiant set, which extended and unified the existing approximate solutions(see [5,7]).

Definition 3.2 *Let $\varepsilon \geqslant 0$, $\bar{y} \in Y$.*

(i) *$\bar{y}$ is called an $\varepsilon-$efficient point of Y with respect to C, written $\bar{y} \in AE[Y, C(\varepsilon)]$, if*

$$(Y - \bar{y}) \cap (-C(\varepsilon)) \subseteq \{0\}.$$

(ii) *$\bar{y}$ is called an $\varepsilon-$Benson proper efficient point of Y with respect to C, written $\bar{y} \in Ben[Y, C(\varepsilon)]$, if*

$$\text{clcone}(Y + C(\varepsilon) - \bar{y}) \cap (-C(0)) \subseteq \{0\}.$$

Based on the proximal proper efficiency, Borwein proper efficiency and the above approximate solutions, we introduce the following approximate proper efficient points.

Definition 3.3. *Let $\varepsilon \geqslant 0$ and $\bar{y} \in Y \cap Y + C(\varepsilon)$.*

(i) *$\bar{y}$ is called an $\varepsilon-$Proximal properly efficient point of Y with respect to C, if*

$$N_p(Y + C(\varepsilon), \bar{y}) \cap (-C(0)^{s+}) \neq \emptyset.$$

(ii) *$\bar{y}$ is called an $\varepsilon-$Borwein properly efficient point of Y with respect to C, if*

$$T(Y + C(\varepsilon), \bar{y}) \cap (-C(0)) \subseteq \{0\}.$$

The set of all $\varepsilon-$Proximal properly efficient points and $\varepsilon-$Borwein proper efficient points of Y are denoted by $P_r[Y, C(\varepsilon)]$ and $Bor[Y, C(\varepsilon)]$, respectively.

Remark 3.1. (i) When $\varepsilon = 0$, Definition 3.3 reduces to $Pr[Y, D]$ and $Bor[Y, D]$.

(ii) In fact, for some problems, we can see that the approximate proximal proper efficient point may be exist, while the proximal efficient point does not exist.

Example 3.1. Let $p = 2, Y = \mathbb{R}^2_+ \cup \{(y_1, y_2)^{\mathrm{T}} | y_1 < 0, y_2 > 0\}$, $D = \mathbb{R}^2_+$, it is easy to check that the exact proximal properly efficient point does not exist. But, if we take $\varepsilon = \frac{1}{2}, C = \{(y_1, y_2)^{\mathrm{T}} | y_1 \geqslant 0, y_2 \geqslant 0, y_1 + y_2 \geqslant 1\}$, then $\bar{y} = (\frac{1}{2}, 0)^{\mathrm{T}}$ is an $\varepsilon-$Proximal efficient point of Y with respect to C.

Some properties of $P_r[Y, C(\varepsilon)]$ and $Bor[Y, C(\varepsilon)]$ can be established in the following Theorem.

Theorem 3.1.

(i) *$P_r[Y, C(0)] \subseteq P_r[Y, C(\varepsilon)]$, for any $\varepsilon > 0$.*

(ii) *If $0 < \varepsilon_1 < \varepsilon_2$, then $P_r[Y, C(\varepsilon_1)] \subseteq P_r[Y, C(\varepsilon_2)]$.*

(iii) *$Bor[Y, C(0)] \subseteq Bor[Y, C(\varepsilon)]$, for any $\varepsilon > 0$.*

(iv) *If $0 < \varepsilon_1 < \varepsilon_2$, then $Bor[Y, C(\varepsilon_1)] \subseteq Bor[Y, C(\varepsilon_2)]$.*

Proof. (i) Since $C(\varepsilon) \subseteq C(0), \forall \varepsilon > 0$,

$$Y + C(\varepsilon) \subseteq Y + C(0).$$

If $\bar{y} \in P_r[Y, C(0)]$, then there exists $h \in -C(0)^{s+}$ such that $h \in N_p(Y + C(0), \bar{y})$. According to Lemma 2.1, there exists $\delta \geqslant 0$ such that

$$\langle h, y - \bar{y} \rangle \leqslant \delta \|y - \bar{y}\|^2, \quad \forall y \in Y + C(0).$$

Hence, for any $y \in Y + C(\varepsilon)$, we also have $\langle h, y - \bar{y} \rangle \leqslant \delta \|y - \bar{y}\|^2$. This implies $\bar{y} \in P_r[Y, C(\varepsilon)]$. Therefore $P_r[Y, C(0)] \subset P_r[Y, C(\varepsilon)], \forall \varepsilon > 0$.

(ii) Take any $\varepsilon_1, \varepsilon_2$ with $0 < \varepsilon_1 < \varepsilon_2$, if $\bar{y} \in P_r[Y, C(\varepsilon_1)]$, then by Lemma 2.3 we know that $C(\varepsilon_2) \subseteq C(\varepsilon_1)$. Therefore

$$Y + C(\varepsilon_2) \subseteq Y + C(\varepsilon_1).$$

Hence, we have that $P_r[Y, C(\varepsilon_1)] \subseteq P_r[Y, C(\varepsilon_2)]$.

(iii) Since $C(\varepsilon) \subseteq C(0), \forall \varepsilon > 0$,

$$T(Y + C(\varepsilon), \bar{y}) \cap (-C(0)) \subseteq T(Y + C(0), \bar{y}) \cap (-C(0)).$$

It follows that $Bor[Y, C(0)] \subseteq Bor[Y, C(\varepsilon)], \forall \varepsilon > 0$.

(iv) The proof is similar to (ii), and can be omitted. □

Generally speaking, the approximate proper efficient point is included in the approximate efficient point, but the $\varepsilon-$Proximal efficiency is not satisfied, which is illustrated by an example below.

Example 3.2. Let $p = 2$, $C = \{(y_1, y_2)^{\mathrm{T}} | y_1 \geqslant 1, y_2 \geqslant 1\}$, $Y = \{(y_1, y_2)^{\mathrm{T}} | y_1 + y_2 = 1, y_1 \geqslant 0, y_2 \geqslant 0\} \cup \{(y_1, y_2)^{\mathrm{T}} | 0 \leqslant y_1 \leqslant 1, y_2 = 0\} \cup \{(y_1, y_2)^{T} | 0 \leqslant y_2 \leqslant 1, y_1 = 0\}$. Take $\varepsilon = \frac{1}{2}, \bar{y} = (\frac{1}{2}, \frac{1}{2})^{\mathrm{T}}$, then $N_p(Y + C(\varepsilon), \bar{y}) \cap (-C(0)^{s+}) \neq \emptyset$, that is $\bar{y} \in P_r[Y, C(\varepsilon)]$. Since $(Y - \bar{y}) \cap (-C(\varepsilon)) \nsubseteq \{0\}$, we have $\bar{y} = (\frac{1}{2}, \frac{1}{2})^{\mathrm{T}} \notin AE[Y, C(\varepsilon)]$(see Figure 1).

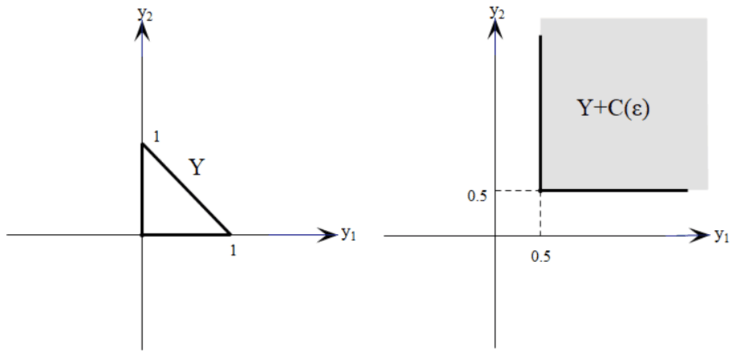

Figure 1: The image spaces of Example 3.2

The following theorems give the relationship between the new two kinds of approximate proper efficient points and the other approximate efficient points.

Theorem 3.2. Let $\varepsilon \geqslant 0$, $\delta \in [0, 1)$, then

$$\bigcap_{\delta \leqslant \alpha < 1} P_r[Y, C(\alpha\varepsilon)] \subseteq AE[Y, C(\varepsilon)].$$

Proof. Let $\bar{y} \in \bigcap_{\delta \leqslant \alpha < 1} P_r[Y, C(\alpha\varepsilon)]$, then there exists $h \in -C(0)^{s+}$ such that $h \in N_p(Y + C(\alpha\varepsilon), \bar{y})$. It follows from Lemma 2.1, for all $\alpha \in [\delta, 1)$, there exists $\delta_1 \geqslant 0$ such that

$$\langle h, y - \bar{y} \rangle \leqslant \delta_1 \|y - \bar{y}\|^2, \quad \forall y \in Y + C(\alpha\varepsilon). \tag{3.1}$$

Now we proof that $\bar{y} \in AE[Y, C(\varepsilon)]$. Suppose to the contrary that $\bar{y} \notin AE[Y, C(\varepsilon)]$, that is, there exists $q \neq 0$ such that $q \in (Y - \bar{y}) \cap (-C(\varepsilon))$. This implies that there exists $p \in C$ such that $q = -\varepsilon p$. We have $\alpha q = -\varepsilon\alpha p \in -C(\alpha\varepsilon)$. Therefore,

$$q + \bar{y} - \alpha q = \bar{y} + (1 - \alpha)q \in Y + C(\alpha\varepsilon).$$

Combining with (3.1), we have

$$\langle h, (1-\alpha)q\rangle \leqslant \delta_1(1-\alpha)^2\|q\|^2.$$

Since $\alpha \in [\delta, 1)$,

$$\langle h, q\rangle \leqslant \delta_1(1-\alpha)\|q\|^2.$$

Taking $\alpha \to 1$ in the above inequality, we have that $\langle h, q\rangle \leqslant 0$.

On the other hand, since $q \in -C(0)\setminus\{0\}$ and $h \in -C(0)^{s+}$, $\langle h, q\rangle > 0$, which leads to a contradiction. Hence, $\bar{y} \in AE[Y, C(\varepsilon)]$. □

Theorem 3.3. Let $\varepsilon \geqslant 0$, then $P_r[Y, C(\varepsilon)] \subseteq Bor[Y, C(\varepsilon)]$.

Proof. Let $\bar{y} \in P_r[Y, C(\varepsilon)]$, then there exists $h \in -C(0)^{s+}$ such that $h \in N_p(Y + C(\varepsilon), \bar{y})$. It follows from Lemma 2.1, there exists $\delta \geqslant 0$ such that

$$\langle h, y-\bar{y}\rangle \leqslant \delta\|y-\bar{y}\|^2, \quad \forall y \in Y + C(\varepsilon). \tag{3.2}$$

Next we proof $\bar{y} \in Bor[Y, C(\varepsilon)]$. Suppose that $\bar{y} \notin Bor[Y, C(\varepsilon)]$, then there exists $d \neq 0$ such that $d \in T(Y + C(\varepsilon), \bar{y}) \cap (-C(0))$. Since $d \in T(Y + C(\varepsilon), \bar{y})$, there exists $t_j \downarrow 0, d_j \to d$ with $d_j \in Y + C(\varepsilon)$ such that

$$\bar{y} + t_j d_j \in Y + C(\varepsilon).$$

Which together with (3.2) yields

$$\langle h, t_j d_j\rangle \leqslant \delta\|t_j d_j\|^2.$$

That is,

$$\langle h, d_j\rangle \leqslant \delta t_j\|d_j\|^2.$$

Taking $j \to +\infty$ in the above inequality, we have $\langle h, d\rangle \leqslant 0$. Since $d \in -C(0)\setminus\{0\}$ and $h \in -C(0)^{s+}$, $\langle h, d\rangle > 0$, which leads to a contradiction. Hence, $\bar{y} \in Bor[Y, C(\varepsilon)]$. □

Remark 3.2. (i) *If $\varepsilon = 0$, Theorem 3.3 can reduce to Theorem 3.1 in [14].*
(ii) *If $Y + C(\varepsilon)$ is a convex set, it is easy to prove that*

$$P_r[Y, C(\varepsilon)] = Bor[Y, C(\varepsilon)] = Ben[Y, C(\varepsilon)].$$

The following two examples illustrate that there is no reciprocal inclusion relations between $\varepsilon-$Proximal efficient points and $\varepsilon-$Benson proper efficient points without convexity assumptions.

Example 3.3. Let $p = 2$, $C = \{(y_1, y_2)^T | y_1 \geqslant 1, y_2 \geqslant 1\}$, $Y = \{(y_1, y_2)^T | 2y_2 \geqslant -y_1 \cup \{(y_1, y_2)^T | y_2 \geqslant -2y_1\}$. Take $\varepsilon = \frac{1}{2}, \bar{y} = (\frac{1}{2}, \frac{1}{2})^T$, then $clcone(Y + C(\varepsilon) - \bar{y}) = Y$ and $Y \cap -C(0) \subseteq \{0\}$. that is $\bar{y} \in Ben[Y, C(\varepsilon)]$, and $\bar{y} \in Bor[Y, C(\varepsilon)]$. But $N_p(Y + C(\varepsilon), \bar{y}) \cap (-C(0)^{s+}) = \emptyset$, that is $\bar{y} \notin P_r[Y, C(\varepsilon)]$(see Figure 2).

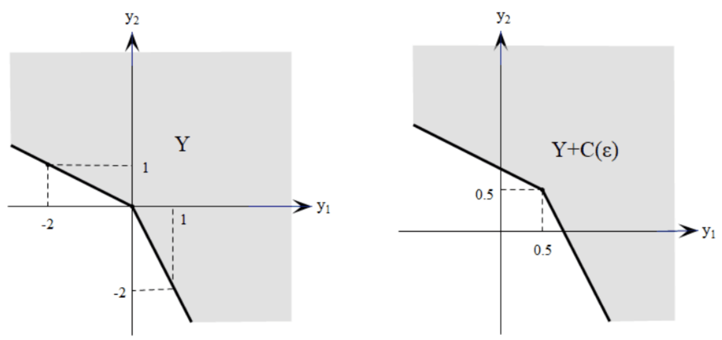

Figure 2: The image spaces of Example 3.3

Example 3.4. Let $p = 2$, $C = \{(y_1, y_2)^T | y_1 + y_2 \geqslant 1, y_1 \geqslant 0, y_2 \geqslant 0\}$, $Y = \{(y_1, y_2)^T | y_1 < 0, y_2 = 1\} \cup \mathbb{R}^2_+$. Take $\varepsilon = \frac{1}{2}, \bar{y} = (0, \frac{1}{2})^T$, then $clcone(Y + C(\varepsilon) - \bar{y}) \cap -C(0) = \{(y_1, y_2)^T | y_1 \leqslant -\frac{1}{2}, y_2 = 0\}$, that is $\bar{y} \notin Ben[Y, C(\varepsilon)]$. Since $N_p(Y + C(\varepsilon), \bar{y}) \cap (-C(0)^{s+}) \neq \emptyset$, $\bar{y} \in P_r[Y, C(\varepsilon)]$ (see Figure 3).

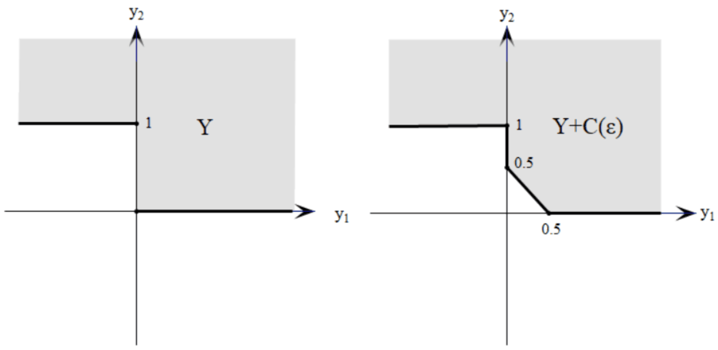

Figure 3: The image spaces of Example 3.4

Consider the following multiobjective optimization problem

$$(\text{MOP})\begin{cases}\min & f(x)\\ \text{s.t.} & x\in S\end{cases}$$

Where $f : X \to Z, S$ be a nonempty subset of X .

Based on the Definition 3.2 and Definition 3.3, we can get the following approximate proper efficient solutions for (MOP).

Definition 3.4. *Let $\varepsilon \geq 0, \bar{x} \in S$.*

(i) *$\bar{x} \in S$ is said to be an $\varepsilon-$Benson proper efficient solutions of* (MOP), *if*

$$\text{clcone}(f(S) + C(\varepsilon) - f(\bar{x})) \cap (-C(0)) \subseteq \{0\}.$$

(ii) *$\bar{x} \in S$ is said to be an $\varepsilon-$Borwein proper efficient solutions of* (MOP), *if*

$$T(f(S) + C(\varepsilon), f(\bar{x})) \cap (-C(0)) \subseteq \{0\}.$$

(iii) *$\bar{x} \in S$ is said to be an $\varepsilon-$Proximal efficient solutions of* (MOP), *if*

$$N_p(f(S) + C(\varepsilon), f(\bar{x})) \cap (-C(0)^{s+}) \neq \emptyset.$$

(iv) *Let $\epsilon \in K$. $\bar{x} \in S$ is said to be an $\epsilon-$Benson efficient solutions of* (MOP), *if*

$$\text{clcone}(f(S) + \epsilon + K - f(\bar{x})) \cap (-K) = \{0\}.$$

We denote the above approximate proper efficient solutions by $Ben(f, S, C, \varepsilon)$, $Bor(f, S, C, \varepsilon)$, $Pr(f, S, C, \varepsilon)$, $Be(f, S, K, \epsilon)$, respectively.

If $X = \mathbb{R}^n, Z = \mathbb{R}^p, f : S \to \mathbb{R}^p$. The approximate proper efficient solutions based on the Geoffrion proper efficient solutions are as follows.

Definition 3.5[19]. *Let $\epsilon \in \mathbb{R}^p_+$. A feasible point $\bar{x} \in S$ is called an $\epsilon-$Geoffrion proper efficient solution if there is no $x \in S$ such that $f_i(x) \leq f_i(\bar{x}) - \epsilon_i, 1 \leq i \leq p$ and there exists a real positive number $M > 0$ such that for each i we have*

$$\frac{f_i(\bar{x}) - f_i(x) - \epsilon_i}{f_j(x) - f_j(x) + \epsilon_j} \leqq M,$$

for some $j \in \{1, 2, ..., p\}$ such that $f_j(\bar{x}) - \epsilon_j < f_j(x)$ whenever $x \in S$ and $f_i(x) < f_i(\bar{x}) - \epsilon_i$. We denoted this by $Ge(f, S, \mathbb{R}^p_+, \epsilon)$.

Definition 3.6[20]. *Let Z be a normed linear space and $\epsilon \in K$. A feasible point $\bar{x} \in S$ is called an $\epsilon-$Super proper efficient solution of (MOP) if there exist $\lambda \in (0, 1)$ and $M > 0$ such that*

$$\text{clcone}(f(S) + \lambda\epsilon - f(\bar{x})) \cap (B - K) \subset M \cdot B,$$

where B is the closed unit ball in Z. We denote this by $\bar{x} \in Se(f, S, K, \epsilon)$.

Definition 3.7[20]. *Let Z be a normed linear space, $K \subset Y$ be a proper pointed convex cone and $\epsilon \in K$. A feasible point $\bar{x} \in S$ is called an $\epsilon-$Henig global efficient proper efficient solution of (MOP) if there exists a proper convex cone K' with $K \backslash \{0\} \subset corK'$ such that*

$$(f(S) - f(\bar{x}) + \epsilon) \cap (-K' \backslash \{0\}) = \emptyset,$$

where

$$\text{core}(A) := \{y \in Y : \forall v \in Y, \exists \lambda > 0 \ s.t. \ y + [0, \lambda]v \subset A\}.$$

We denote this by $GHe(f, S, K, \epsilon)$.

Let $Y = f(S)$, $C = \epsilon + \mathbb{R}^p_+, \epsilon \in K = \mathbb{R}^p_+, \varepsilon \geqslant 0$ and $f(S) + C(\varepsilon)$ is a convex set. From Theorem 4.1 in [20], we know that $Ge(f, S, \mathbb{R}^p_+, \epsilon) = Be(f, S, \mathbb{R}^p_+, \epsilon)$, combine with the definition of $Ben(f, S, C, \varepsilon)$, it is obvious that $Be(f, S, \mathbb{R}^p_+, \epsilon\varepsilon) \subset Ben(f, S, \epsilon + \mathbb{R}^p_+, \varepsilon)$. According to the Remark 3.2 and the implication relation among approximate proper efficiency in [20], we can get the relationship between $\varepsilon-$Proximal efficiency with more approximate proper efficiency in this special case.

$$Se(f, S, \mathbb{R}^p_+, \frac{\varepsilon}{\lambda}\epsilon) \subset GHe(f, S, \mathbb{R}^p_+, \epsilon\varepsilon) \subset Ge(f, S, \mathbb{R}^p_+, \epsilon\varepsilon)$$

$$= Be(f, S, \mathbb{R}^p_+, \epsilon\varepsilon) \subset Ben(f, S, \epsilon + \mathbb{R}^p_+, \varepsilon) = Pr(f, S, \epsilon + \mathbb{R}^p_+, \varepsilon).$$

Example 3.4 also show that $\varepsilon-$Borwein efficiency may not be $\varepsilon-$Proximal efficiency without the convexity assumption of $Y + C(\varepsilon)$. But we can establish the following cone characterization for $\varepsilon-$Borwein efficiency under the convexity assumption of C.

Theorem 3.4. *Let C be a convex set in $\mathbb{R}^p$. If $\bar{y} \in Bor[Y, C(\varepsilon)]$, then $N_c(Y + C(\varepsilon), \bar{y}) \cap -C(0)^{s+} \neq \emptyset$.*

Proof. If $\bar{y} \in Bor[Y, C(\varepsilon)]$, then $T(Y+C(\varepsilon), \bar{y}) \cap -C(0) \subseteq \{0\}$. Using the relation of (2.1), we have $T_c(Y+C(\varepsilon), \bar{y}) \cap -C(0) \subseteq \{0\}$. Lemma 2.2 (ii) implies $T_c(Y + C(\varepsilon), \bar{y}) \cap -C(0)^+ \supseteq (\{0\})^+$. And from the definition of Clarke normal cone, we have $-N_c(Y + C(\varepsilon), \bar{y}) + (-C(0))^+ \supseteq \mathbb{R}^p$, that is

$$-N_c(Y + C(\varepsilon), \bar{y}) - C(0)^+ = \mathbb{R}^p. \tag{3.3}$$

Let $x \in C(0)^{s+} \subseteq \mathbb{R}^p$, then (3.3) implies that there exist $h \in N_c(Y + C(\varepsilon), \bar{y})$ and $c \in C(0)^+$ such that $-h - c = x$. From Lemma 2.2, we have that $-h = x + c \in C(0)^{s+} + (C(0))^+ \subseteq C(0)^{s+}$. Hence, $h \in N_c(Y + C(\varepsilon), \bar{y}) \cap -C(0)^{s+}$. Which implies $N_c(Y + C(\varepsilon), \bar{y}) \cap -C(0)^{s+} \neq \emptyset$. □

The following example show that the reverse of Theorem 3.4 may not be hold.

Example 3.5. Consider the set C and Y in Example 3.4. Taking $\bar{y} = (0, 1)$, $\varepsilon = \frac{1}{2}$, then $T_c(Y + C(\varepsilon), \bar{y}) = \mathbb{R}^2_+$. And we have that $N_c(Y + C(\varepsilon), \bar{y}) \cap -C(0)^{s+} \neq \emptyset$. Since $T(Y + C(\varepsilon), \bar{y}) \cap -C(0) \nsubseteq \{0\}$, $\bar{y} \notin Bor[Y, C(\varepsilon)]$.

If we assume $Y + C(\varepsilon)$ is tangentially regular, then the reverse of Theorem 3.4 can be established.

Theorem 3.5. *Let C be a convex set in $\mathbb{R}^p$, if $Y+C(\varepsilon)$ is tangentially regular at $\bar{y}$ and $N_c(Y+C(\varepsilon), \bar{y}) \cap -C(0)^{s+} \neq \emptyset$, then $\bar{y} \in Bor[Y, C(\varepsilon)]$.*

Proof. If $Y + C(\varepsilon)$ is tangentially regular at $\bar{y}$, then $(T(Y + C(\varepsilon), \bar{y}))^+ = (T_c(Y + C(\varepsilon), \bar{y}))^+$, that is

$$N(Y + C(\varepsilon), \bar{y}) = N_c(Y + C(\varepsilon), \bar{y}).$$

By the given assumption, we have $N(Y+C(\varepsilon), \bar{y}) \cap -C(0)^{s+} \neq \emptyset$. Let $h \in N(Y+C(\varepsilon), \bar{y}) \cap -C(0)^{s+}$, and suppose that $\bar{y} \notin Bor[Y, C(\varepsilon)]$, then there exists $d \neq 0$, such that $d \in T(Y + C(\varepsilon), \bar{y}) \cap -C(0)$. Since $h \in N(Y + C(\varepsilon), \bar{y})$ and $d \in T(Y + C(\varepsilon), \bar{y})$, we have that $\langle h, d \rangle \leqslant 0$.

On the other hand, since $d \in -C(0) \backslash \{0\}$ and $h \in -C(0)^{s+}$, we have that $\langle h, d \rangle > 0$. Which leads to a contradiction. Hence, $\bar{y} \in Bor[Y, C(\varepsilon)]$. □

It is worthy to emphasize that the above theorem may not necessarily hold for $\varepsilon-$Benson properly efficient point, even if $Y + C(\varepsilon)$ is tangentially regular. See the following example.

Example 3.6. Consider the set C and Y in Example 3.4. It was observed that $\bar{y} = (0, \frac{1}{2})^{\mathrm{T}} \notin Ben[Y, C(\varepsilon)]$. Also Y is tangentially regular at $\bar{y}$ as $T(Y + C(\varepsilon), \bar{y}) = T_c(Y + C(\varepsilon), \bar{y}) = \{(y_1, y_2)^{\mathrm{T}} | y_2 \geqslant -y_1, y_1 \geqslant 0\}$. Since $N(Y + C(\varepsilon), \bar{y}) = \{(y_1, y_2)^{\mathrm{T}} | y_1 \leqslant y_2 \leqslant 0, y_1 \leqslant 0\}$, we have that $N_c(Y + C(\varepsilon), \bar{y}) \cap -C(0)^{s+} \neq \emptyset$.

4. Linear Scalarization of Approximate Proper Efficiency

In this section, we study the linear scalar characterization for $\varepsilon-$Proximal properly efficient points. Let C be a closed pointed co-radiant set in $\mathbb{R}^p$, Y be a closed set in $\mathbb{R}^p$ and

$$S_\mu[Y, \varepsilon] = \{\bar{y} \in Y | \langle \mu, y \rangle + \varepsilon \geqslant \langle \mu, \bar{y} \rangle, \forall y \in Y\}.$$

Definition 4.1[21] *A set C in $\mathbb{R}^p$ is said to be locally starshaped at $\bar{c} \in C$ if for all $c \in C$, there exists $\alpha(c, \bar{c})$ where $0 < \alpha(c, \bar{c}) \leqslant 1$ such that*

$$\lambda \bar{c} + (1 - \lambda)c \in C, \ \forall\ 0 < \lambda < \alpha(c, \bar{c}).$$

Theorem 4.1. *Let $\varepsilon \geq 0$, $\bar{y} \in Y \cap Y + C(\varepsilon)$. For any $\mu \in C(0)^{s+} \cap \{\mu \in \mathbb{R}^p | \langle \mu, d \rangle > 1, \forall d \in C\}$, if $\bar{y} \in S_\mu[Y + C(\varepsilon), \varepsilon]$, then $\bar{y} \in P_r[Y, C(\varepsilon)]$.*

Proof. If $\bar{y} \in S_\mu[Y + C(\varepsilon), \varepsilon]$, it follows from the definition, we have that

$$\langle \mu, y \rangle + \varepsilon \geqslant \langle \mu, \bar{y} \rangle, \forall y \in Y \cap Y + C(\varepsilon). \tag{4.1}$$

And for all $q \in C(\varepsilon)$, we have that $q = \varepsilon d, d \in C$. Therefore,

$$\langle -\mu, q \rangle = \langle -\mu, \varepsilon d \rangle = \varepsilon \langle -\mu, d \rangle < -\varepsilon. \tag{4.2}$$

Combining with (4.1) and (4.2), we have

$$\langle -\mu, y + q - \bar{y} \rangle \leqslant 0.$$

Hence, there exists $\delta > 0$, such that

$$\langle -\mu, y + q - \bar{y} \rangle \leqslant 0 \leqslant \delta \parallel y + q - \bar{y} \parallel^2 .$$

This implies $-\mu \in N_p(Y + C(\varepsilon), \bar{y})$. Since $\mu \in C(0)^{s+}$, $\bar{y} \in P_r[Y, C(\varepsilon)]$. □

Remark 4.1. If $\varepsilon = 0$, Theorem 4.1 cannot be reduced to Theorem 4.1 in [11], since the condition is different. And the following example which extracted in [14] can illustrate the result of Theorem 4.1.

Example 4.1. Let $C = \{(y_1, y_2)^{\mathrm{T}} | y_1 + y_2 \geqslant 1, y_1 \geqslant 0, y_2 \geqslant 0\}$ and $Y = \{(y_1, y_2)^{\mathrm{T}} | y_2 \geqslant -y_1,\ 0 \leqslant y_1 \leqslant 2\} \cup \{(y_1, y_2)^{\mathrm{T}} | y_2 \geqslant 2, -3 \leqslant y_1 \leqslant 0\}$. Taking $\varepsilon = \frac{1}{2}, \mu = (2, 1)^{\mathrm{T}}$, then for any $d = (d_1, d_2)^{\mathrm{T}} \in C$, we have that $\langle \mu, d \rangle = 2d_1 + d_2 > 1$. Which satisfies the condition in Theorem 4.1. According to the definition of $S_\mu[Y + C(\varepsilon), \varepsilon]$, we know that $S_\mu[Y + C(\varepsilon), \varepsilon] = \{(\bar{y_1}, \bar{y_2})^{\mathrm{T}} | \bar{y_1} \in [-3, 2.5], \bar{y_2} = -\bar{y_1} - 0.5\}$. It is clear that $S_\mu[Y + C(\varepsilon), \varepsilon] \subseteq P_r[Y, C(\varepsilon)]$. (see Figure 4).

In Remark 4.1, we sate that the condition of Theorem 4.1 is different from the exact case. We use Example 4.1 to state the necessity of the assumption in Theorem 4.1. In fact, if we take $\varepsilon = \frac{1}{2}, \mu = (\frac{1}{2}, \frac{1}{2})^{\mathrm{T}}$, then there exists $d = (\frac{1}{2}, 1)^{\mathrm{T}} \in C$ such that $\langle \mu, d \rangle \leq 1$. Which implies it does not satisfy the condition in Theorem 4.1. Furthermore, we have that $S_\mu[Y + C(\varepsilon), \varepsilon] = \{(\bar{y_1}, \bar{y_2})^{\mathrm{T}} | \bar{y_1} \in [-3, 2.5], \bar{y_2} = -\bar{y_1} - 0.5\} \cup \{(\bar{y_1}, \bar{y_2})^{\mathrm{T}} | -2.5 \leq \bar{y_1} \leq -\frac{7}{4}, \bar{y_2} = 2\}$. It is obvious that $\bar{y} = (-2, 2)^{\mathrm{T}} \notin P_r[Y, C(\varepsilon)]$.

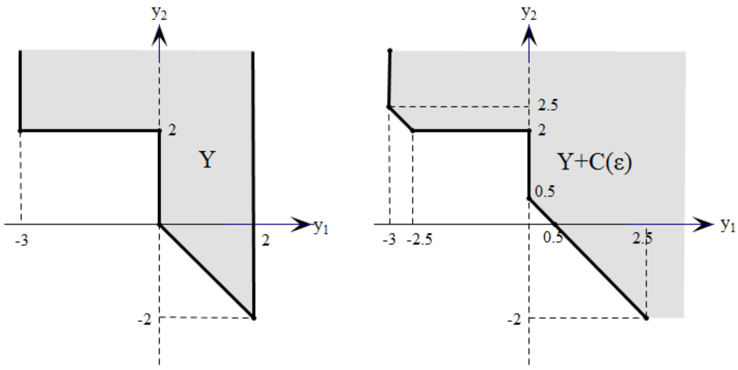

Figure 4: The image spaces of Example 4.1

Theorem 4.2. *Let $d(0, C) \leqslant \delta$. If $\bar{y} \in P_r[Y, C(\varepsilon)]$, and $Y + C(\varepsilon)$ is locally starshaped at $\bar{y}$, then $\bar{y} \in S_\mu[Y + C(\varepsilon), \varepsilon\delta]$.*

Proof. Let $\bar{y} \in P_r[Y, C(\varepsilon)]$, then there exists $h \in -C(0)^{s+}$ such that $h \in N_p(Y + C(\varepsilon), \bar{y})$. By Lemma 2.1, there exists $\delta_1 \geqslant 0$ such that

$$\langle h, \hat{y} - \bar{y}\rangle \leqslant \delta_1 \|\hat{y} - \bar{y}\|^2, \forall\ \hat{y} \in Y + C(\varepsilon). \tag{4.3}$$

Since $Y + C(\varepsilon)$ is locally starshaped at $\bar{y}$, for any $\hat{y} \in Y + C(\varepsilon)$ there exists $\alpha(\hat{y}, \bar{y}), 0 < \alpha \leqslant 1$ such that

$$\bar{y} + \lambda(\hat{y} - \bar{y}) \in Y + C(\varepsilon), \forall\ 0 < \lambda < \alpha(\hat{y}, \bar{y}).$$

Which combining with (4.3), we have

$$\langle h, \lambda(\hat{y} - \bar{y})\rangle \leqslant \delta_1 \lambda^2 \|\hat{y} - \bar{y}\|^2,$$

that is

$$\langle h, \hat{y} - \bar{y}\rangle \leqslant \delta_1 \lambda \|\hat{y} - \bar{y}\|^2.$$

Taking $\lambda \to 0$ in the above inequality, we have

$$\langle h, \hat{y} - \bar{y}\rangle \leqslant 0, \forall \hat{y} \in Y + C(\varepsilon).$$

Especially, taking $\mu = \frac{h}{\|h\|}$, for any given $\hat{y} = y + q, y \in Y, q \in C(\varepsilon)$,

$$\langle \mu, y + q - \bar{y}\rangle \leqslant 0.$$

Since $q \in C(\varepsilon)$, there exists $d \in C$ such that $q = \varepsilon d$. And the above inequality implies

$$\langle \mu, y - \bar{y}\rangle \leqslant \langle -\mu, q\rangle = \varepsilon\langle -\mu, d\rangle \leqslant \varepsilon\| - \mu\|\|d\|.$$

Since C is a closed set, there exists $d_1 \in C$ such that $\|d_1\| = d(0, C)$. If we take $q = \varepsilon d_1$, then from the assumption $d(0, C) \leqslant \delta$, the above inequality implies that

$$\langle -\mu, \bar{y}\rangle \leqslant \langle -\mu, y\rangle + \varepsilon\delta.$$

Since $-\mu \in C(0)^{s+}$, $\bar{y} \in S_\mu[Y + C(\varepsilon), \varepsilon\delta]$. □

The following example illustrate that the above theorem does not hold in the absence of local starshapedness.

Example 4.2. Considering the set C and Y in Example 4.1. It is easy to check that $d(0, C) \leqslant 1$ and $Y + C(\varepsilon)$ is not locally starshaped at $\bar{y} = (-3, 2.5)^{\mathrm{T}}$, since $(1 - \lambda)\bar{y} + \lambda y \notin Y + C(\varepsilon)$ for any $0 < \lambda < 1$ and $y = (0, \frac{1}{2})^{\mathrm{T}}$. $N_p(Y + C(\varepsilon), \bar{y}) \cap -C(0)^{s+} \neq \varnothing$ implies that $\bar{y} \in P_r[Y, C(\varepsilon)]$. But $\bar{y} \notin S_\mu[Y + C(\varepsilon), \varepsilon\delta]$. In fact, taking $\delta = 1, \varepsilon = \frac{1}{2}$, there exists $y = (-3, 2)^{\mathrm{T}}, \mu_2 = 2$, such that $-3\mu_1 + 2\mu_2 + \frac{1}{2} < -3\mu_1 + 2.5\mu_2$, which means that $\bar{y} \notin S_\mu[Y + C(\varepsilon), \varepsilon\delta]$.

5. Conclusion

The co-radiant set is a useful tool to characterize the approximate efficiency in multiobjective optimization problems. In this paper, we consider a new kind of approximate proper efficiency by using the proximal normal cone and co-radiant set. And the relation of the approximate efficiency and the approximate proper efficiency are discussed. At last, we also give the properties and linear scalarization results for this approximate proper efficiency. For the existing research in this paper, we consider that cone as a powerful tool for characterizing optimal conditions, we can further study the optimality conditions of $\varepsilon-$Proximal proper efficient solutions by using the proximal subdifferential and proximal normal cone.

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