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Complement of the Generalized Total Graph of $\mathbb{Z}_n$

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Abstract. Let R be a commutative ring with identity and H be a nonempty proper multiplicative prime subset of R. *The generalized total graph* of R is the (undirected) simple graph $GT_H(R)$ with all elements of R as the vertex set and two distinct vertices x and y are adjacent if and only if $x + y \in H$. *The complement of the generalized total graph* $\overline{GT_H(R)}$ of R is the (undirected) simple graph with vertex set R and two distinct vertices x and y are adjacent if and only if $x + y \notin H$. In this paper, we investigate certain domination properties of $\overline{GT_H(R)}$. In particular, we obtain the domination number, independence number and a characterization for γ-sets in $\overline{GT_P(\mathbb{Z}_n)}$ where P is a prime ideal of $\mathbb{Z}_n$. Further, we discuss properties like Eulerian, Hamiltonian, planarity, and toroidality of $\overline{GT_P(\mathbb{Z}_n)}$.

1. Introduction

Through out this paper R denotes a commutative ring with nonzero identity, $Z(R)$ its set of all zero-divisors, $Z^*(R) = Z(R) \setminus \{0\}$ and $U(R)$ its set of all units. Anderson and Livingston [3] introduced the *zero-divisor graph* of R, denoted by $\Gamma(R)$, as the (undirected) simple graph with vertex set $Z^*(R)$ and two distinct vertices $x, y \in Z^*(R)$ are adjacent if and only if $xy = 0$. Subsequently, Anderson and Badawi [5] introduced the concept of the *total graph* of a commutative ring. The *total graph* $T_\Gamma(R)$ of R is the undirected graph with vertex set R and for distinct $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$. Akbari et al. [1], Anderson and Badawi [4], Petrović et al. [11] and Tamizh Chelvam and Asir [6, 13, 14] have extensively studied about various graph theoretical aspects of the total graph of commutative rings.

Recently, Anderson and Badawi [2] introduced the concept of the generalized total graph of a commutative ring. A nonempty proper subset H of a commutative ring R is said to be a multiplicative prime subset of R if the following two conditions hold: (i) $ab \in H$ for every $a \in H$ and $b \in R$; (ii) if $ab \in H$ for $a, b \in R$, then either $a \in H$ or $b \in H$. For example, every prime ideal, union of prime ideals and $H = R \setminus U(R)$ are some of the multiplicative prime subsets of R. For a multiplicative prime subset H of R, the *generalized total graph* $GT_H(R)$ of R is the (undirected) simple graph with vertex set R and two distinct vertices x and y are adjacent if and only if $x + y \in H$, As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the ring of integers and ring of integers modulo n.

Let $G = (V, E)$ be a graph. We say that G is connected if there is a path between any two distinct vertices of G. For a graph $G = (V, E)$ and a subset $S \subseteq V$, the neighbor set of S in G to be the set of all vertices adjacent

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to vertices in S; and $deg(v)$ is the degree of a vertex v. $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of vertices in G respectively. K_n denotes the complete graph on n vertices and $K_{m,n}$ denotes the complete bipartite graph. A nonempty subset S of V is called a *dominating set* if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. A graph G is called *excellent* if, for every vertex $v \in V(G)$, there is a γ-set S containing v. For the terms in graph theory which are not explicitly mentioned here, one can refer [8] and for the terms regarding algebra one can refer [7].

In this paper, we study about the complement of a class of generalized total graphs on $\mathbb{Z}_n$. In particular, we investigate the structure of $\overline{GT_H(\mathbb{Z}_n)}$, where H is a prime ideal $P =< p >$ for a prime element $p \in \mathbb{Z}_n$. More specifically, we determine the domination number of $\overline{GT_P(\mathbb{Z}_n)}$. Having determined the domination number γ of $\overline{GT_P(\mathbb{Z}_n)}$, we characterize all γ-sets in $\overline{GT_P(\mathbb{Z}_n)}$. In Section 2, we study some properties namely degree of the vertices, Eulerian and Hamiltonian of $\overline{GT_P(\mathbb{Z}_n)}$. Further we obtain the independence and covering numbers of $\overline{GT_P(\mathbb{Z}_n)}$. In section 3, we characterize all integers n for which $\overline{GT_P(\mathbb{Z}_n)}$ is either planar or toroidal. In Section 4, we study some standard domination parameters of $\overline{GT_P(\mathbb{Z}_n)}$.

2. Basic Properties of $\overline{GT_P(\mathbb{Z}_n)}$

In this section, we first obtain some results on the degree of the vertices in the complement of the generalized total graph of $\mathbb{Z}_n$. Later, we discuss about some graph theoretical properties of $\overline{GT_P(\mathbb{Z}_n)}$. More specifically, we discuss about Eulerian and Hamiltonian characterizations of $\overline{GT_P(\mathbb{Z}_n)}$. Let p be a prime number in $\mathbb{Z}$ which divides n. Then $x \in < p > \subseteq \mathbb{Z}_n$ if and only if $(x, p) \neq 1$for $x \in \mathbb{Z}_n$, where (x, p) is the gcd of x and p. We recall the following structure theorem for generalized total graphs of commutative rings. Hereafter we take $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ for distinct primes $p_j, 1 \leq j \leq k$ with $p_1 < p_2 < \ldots < p_k$ and the prime ideal $P =< p_j >$ for some j.

Theorem 2.1. [5, Theorem 2.2] *Let H be a prime ideal of a commutative ring R, and let $|H| = \lambda$ and $|\frac{R}{H}| = \mu$.*

(i) *If $2 \in H$, then $GT_H(R \setminus H)$ is the union of $\mu - 1$ disjoint K_λ's;*

(ii) *If $2 \notin H$, then $GT_H(R \setminus H)$ is the union of $\frac{\mu-1}{2}$ disjoint $K_{\lambda,\lambda}$'s.*

Using Theorem 2.1, one can write $GT_H(R)$ is the union of μ disjoint K_λ's if $2 \in H$; and $GT_H(R)$ is the union of $\frac{\mu-1}{2}$ disjoint $K_{\lambda,\lambda}$'s and a K_λ if $2 \notin H$. Now, we obtain degrees of the vertices in the complement of the generalized total graph of $\mathbb{Z}_n$ with respect to a prime ideal $P =< p >$ of $\mathbb{Z}_n$.

Lemma 2.2. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ where $p_1 < p_2 < \ldots < p_k$, p_j's are prime, α_j's are positive integers and $P =< p_j >$ for some j. Then the following are true in $\overline{GT_P(\mathbb{Z}_n)}$:*

(i) *If $n = 2$, then $deg(v) = 1$, for every $v \in \mathbb{Z}_n$.*

(ii) *If n is an odd prime p, then* $deg(v) = \begin{cases} n-1 & \text{if } v = 0; \\ n-2 & \text{if } v \neq 0. \end{cases}$

(iii) *If n is composite and $2 \in P$, then $deg(v) = \frac{n}{2}$ for every $v \in \mathbb{Z}_n$.*

(iv) *If n is composite and $2 \notin P$, then*
$$deg(v) = \begin{cases} n - \frac{n}{p_j} & \text{for } v \in P; \\ n - \frac{n}{p_j} - 1 & \text{for } v \in \mathbb{Z}_n \setminus P. \end{cases}$$

The following is an immediate consequence of Lemma 2.2.

Lemma 2.3. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $p_1 < p_2 < \dots < p_k$, p_j's are prime, α_j's are positive integers and $P =< p_j >$ for some j. Then*

(i) $\overline{GT_P(\mathbb{Z}_n)}$ *contains no isolated vertex;*

(ii) $\overline{GT_P(\mathbb{Z}_n)}$ *contains a vertex of degree $n - 1$ if and only if n is a prime integer;*

(iii) $\overline{GT_P(\mathbb{Z}_n)}$ *is regular if and only if $n = 2$ or $2 \in P$;*

(iv) $\overline{GT_P(\mathbb{Z}_n)}$ *is biregular if and only if n is odd. Moreover in this case, $\Delta(\overline{GT_P(\mathbb{Z}_n)}) = \delta(\overline{GT_P(\mathbb{Z}_n)}) + 1$;*

(v) $\overline{GT_P(\mathbb{Z}_n)}$ *is a nontrivial connected graph.*

The following observation follows from Theorem 2.1 and is useful throughout this paper.

Lemma 2.4. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $p_1 < p_2 < \dots < p_k$, p_j's are prime and α_j's are positive integers. If $P =< p_1 >$ and $p_1 = 2$, then $\overline{GT_P(\mathbb{Z}_n)} = K_{\frac{n}{2},\frac{n}{2}}$.*

Remark 2.5. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $p_1 < p_2 < \dots < p_k$, p_j's are prime and α_j's are positive integers. If p_j is an odd prime and $P =< p_j >$, then two distinct vertices x and y are adjacent in $\overline{GT_P(\mathbb{Z}_n)}$ if and only if $x \in i + P$ and $y \in \mathbb{Z}_n \setminus (p_j - i + P)$ for some i and $1 \le i < p_j$.

A *circuit* in a graph G is a closed trail of length 3 or more. A circuit C is called an *Eulerian circuit* if C contains every edge of G. A connected graph G is said to be *Eulerian* if it contains an Eulerian circuit. The following characterization for Eulerian graphs is used for characterization of $\overline{GT_P(\mathbb{Z}_n)}$ to be Eulerian.

Corollary 2.6. [8, Theorem 6.1] *A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.*

Using Corollary 2.6, in the following lemma, we obtain a characterization for $\overline{GT_P(\mathbb{Z}_n)}$ to be Eulerian.

Lemma 2.7. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $p_1 < p_2 < \dots < p_k$, p_j's are prime, α_j's are positive integers and $P =< p_j >$ for some j. Then the following are true:*

(i) *Let n be composite, $P =< p_1 >$ and $p_1 = 2$. Then $\overline{GT_P(\mathbb{Z}_n)}$ is Eulerian if and only if $n = 4k$ for some positive integer k.*

(ii) *If n is prime or $P =< p_j >$ for $p_j \neq 2$, then $\overline{GT_P(\mathbb{Z}_n)}$ is not Eulerian.*

Proof. Proof of (i) follows from Lemma 2.4, where as proof of (ii) follows from Lemma 2.3(v) and Lemma 2.2(ii) and (iv). □

A graph G is said to be *Hamiltonian* if it has a circuit which contains all the vertices of G. The following corollary is useful in proving $\overline{GT_P(\mathbb{Z}_n)}$ is always Hamiltonian.

Corollary 2.8. [8, Corollary 6.7] *Let G be a graph of order $n \ge 3$. If $deg(v) \ge \frac{n}{2}$ for each vertex v of G, then G is Hamiltonian.*

Lemma 2.9. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} > 3$ where $p_1 < p_2 < \dots < p_k$, p_j's are prime, α_j's are positive integers and $P =< p_j >$ for some j. Then $\overline{GT_P(\mathbb{Z}_n)}$ is Hamiltonian.*

Proof. Let $G = \overline{GT_P(\mathbb{Z}_n)}$. Since $n > 3$, we have $n - 2 \ge \frac{n}{2}$.

If n is prime, by Lemma 2.2(ii), $\delta(G) = n - 2 \ge \frac{n}{2}$. By Corollary 2.8, $\overline{GT_P(\mathbb{Z}_n)}$ is Hamiltonian.

Suppose n is a composite integer, $P =< p_j >$ and $p_j \neq 2$. Since $\frac{n}{2} \ge \frac{n}{p_j} + 1$, by Lemma 2.2(iv), $\delta(G) = n - \frac{n}{p_j} - 1 \ge \frac{n}{2}$. By Corollary 2.8, $\overline{GT_P(\mathbb{Z}_n)}$ is Hamiltonian.

Suppose n is a composite integer and $P =< 2 >$. By Lemma 2.4, $\overline{GT_P(\mathbb{Z}_n)} = K_{\frac{n}{2},\frac{n}{2}}$ and hence the proof follows from Corollary 2.8. □

A set S of vertices in a graph G is said to be *independent* if no two vertices in S are adjacent. The *vertex independence number* (or the *independence number*) $\beta(G)$ of G is the maximum cardinality of an independent set of G. A *vertex cover* in G is a set of vertices which covers all edges of G. The minimum number of vertices in a vertex cover of G is called the *vertex covering number* $\alpha(G)$ of G. Now, we obtain the independence domination number of $\overline{GT_P(\mathbb{Z}_n)}$.

Lemma 2.10. *Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$ where $p_1 < p_2 < \ldots < p_k$, $p_j's$ are prime, $\alpha_j's$ are positive integers and $P =< p_j >$ for some j. Then*

$$\beta(\overline{GT_P(\mathbb{Z}_n)}) = \begin{cases} 1 & \text{if } n = 2; \\ 2 & \text{if } n \text{ is an odd prime;} \\ \frac{n}{2} & \text{if } n \text{ is a composite integer and } p_j = 2; \\ \frac{n}{p_j} & \text{if } n \text{ is a composite integer and } p_j \neq 2. \end{cases}$$

Proof. Suppose $n = 2$. Then $\overline{GT_P(\mathbb{Z}_n)} = K_2$ and so $\beta(\overline{GT_P(\mathbb{Z}_n)}) = 1$.

Let n be an odd prime. Suppose $\beta(\overline{GT_P(\mathbb{Z}_n)}) \geq 3$. This gives that there exists a complete subgraph of order ≥ 3 in $GT_P(\mathbb{Z}_n) = K_1 \bigcup\limits_{\frac{n-1}{2}} K_2$, which is a contradiction. Hence $\beta(\overline{GT_P(\mathbb{Z}_n)}) \leq 2$. Note that, for $1 \leq i \leq p-1$, i is adjacent with $p - i$ only in $GT_P(\mathbb{Z}_n)$ and hence $\beta(\overline{GT_P(\mathbb{Z}_n)}) = 2$.

Suppose n is composite and $P =< 2 >$. By Lemma 2.4, $\beta(\overline{GT_P(\mathbb{Z}_n)}) = \frac{n}{2}$.

Suppose n is composite and $p_j \neq 2$. Then P is an independent set in $\overline{GT_P(\mathbb{Z}_n)}$ and $|P| = \frac{n}{p_j} \geq 2$. Let $S \subseteq \mathbb{Z}_n \setminus P$ be an independent subset of $\overline{GT_P(\mathbb{Z}_n)}$.

Claim : $|S| \leq 2$.

Let x_1, x_2 and x_3 be three distinct elements in S such that $x_1 \in i_1 + P$, $x_2 \in i_2 + P$ and $x_3 \in i_3 + P$ for $1 \leq i_1, i_2, i_3 \leq p_j - 1$.

Assume that at least two of i_1, i_2 and i_3 are equal. Without loss of generality, let us take $i_1 = i_2$. Then x_1, x_2 are adjacent in $\overline{GT_P(\mathbb{Z}_n)}$ and so K_2 is a subgraph of $\overline{GT_P(\mathbb{Z}_n)}$. Hence S is not an independent in $\overline{GT_P(\mathbb{Z}_n)}$.

Assume that i_1, i_2 and i_3 are all distinct. Suppose the sum of at least any two of i_1, i_2 and i_3 is p_j. Without loss of generality, let $i_1 + i_2 = p_j$. Then x_3 is adjacent with both x_1 and x_2. Hence S is not an independent set in $\overline{GT_P(\mathbb{Z}_n)}$. Suppose the sum of any two of i_1, i_2 and i_3 is not equal to p_j. Then the subgraph induced by $\{x_1, x_2, x_3\}$ is K_3, which is a contradiction to S is an independent set in $\overline{GT_P(\mathbb{Z}_n)}$. Hence $|S| \leq 2$. Therefore P is a maximal independent set in $\overline{GT_P(\mathbb{Z}_n)}$ having order $\frac{n}{p_j} \geq 2$. ☐

Corollary 2.11. [8, Corollary 8.8] *For every graph G of order n containing no isolated vertices, $\alpha(G) + \beta(G) = n$.*

Using Corollary 2.11, we obtain the following on the vertex covering number.

Corollary 2.12. *Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$ where $p_1 < p_2 < \ldots < p_k$, $p_j's$ are prime, $\alpha_j's$ are positive integers and $P =< p_j >$ for some j. Then the vertex covering number*

$$\alpha(\overline{GT_P(\mathbb{Z}_n)}) = \begin{cases} 1 & \text{if } n = 2; \\ n-2 & \text{if } n \text{ is an odd prime;} \\ \frac{n}{2} & \text{if } n \text{ is a composite integer and } p_j = 2; \\ n - \frac{n}{p_j} & \text{if } n \text{ is a composite integer and } p_j \neq 2. \end{cases}$$

3. Characterization of genus for $\overline{GT_P(\mathbb{Z}_n)}$

In section, we study about the genus of $\overline{GT_P(\mathbb{Z}_n)}$. More specifically we characterize all integers n for which $\overline{GT_P(\mathbb{Z}_n)}$ is either planar or toroidal. Let k be a non-negative integer and S_k an orientable surface of genus n. The genus of the graph G, denoted by $g(G)$, is the smallest k such that G embeds into S_k. If H is a subgraph of G, then $g(H) \leq g(G)$. Graphs with genus 0 are planar and graphs of genus 1 are toroidal. Maimani et al. [10], Pucanović [12] and Tamizh Chelvam et al. [15] have studied about the genus of total graphs and other graphs associated with commutative rings. Let us first recall some known results connecting genus of graphs.

Theorem 3.1. [16, Euler formula] *If G is a finite connected graph with n vertices, e edges, and of genus g, then* $n - e + f = 2 - 2g$, *where f is a number of faces obtained when G is embedded in* S_n.

Theorem 3.2. [16, Theorems 6.37 & 6.38] *The following statements hold:*

(i) *For* $n \geq 3$, $g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$.

(ii) *For* $m, n \geq 2$, $g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$.

Note that a graph G is planar if and only if G does not contain either K_5 or $K_{3,3}$[8, Theorem 9.7]. According to Theorem 3.2, if $n = 5, 6, 7$ then $g(K_n) = 1$. Further, $g(K_{4,4}) = g(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$ and $g(K_{5,4}) = g(K_{6,4}) = g(K_{m,4}) = 2$ if $m = 7, 8, 9, 10$. Now, we obtain in the following theorem, a characterization for $\overline{GT_P(\mathbb{Z}_n)}$ to be planar.

Theorem 3.3. *Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ *where* $p_1 < p_2 < \ldots < p_k$, p_j's *are prime and* α_j's *are positive integers. Then the following are true:*

(i) *Let n be composite,* $p_1 = 2$ *and* $P =< p_1 >$. *Then* $\overline{GT_P(\mathbb{Z}_n)}$ *is planar if and only if* $n = 4$;

(ii) *Let n be composite,* $p_j \neq 2$ *and* $P =< p_j >$. *Then* $\overline{GT_P(\mathbb{Z}_n)}$ *is planar if and only if* $n = 6$;

(iii) *Let n be prime,* $n = p$ *and* $P =< p >$. *Then* $\overline{GT_P(\mathbb{Z}_p)}$ *is planar if and only if* $p \in \{2, 3, 5\}$.

Proof. (i) Assume that n is composite, $p_1 = 2$ and $P =< p_1 >$. Suppose $n = 4$. One can see that $\overline{GT_P(\mathbb{Z}_4)} = K_{2,2}$ and hence planar. Conversely assume that $\overline{GT_P(\mathbb{Z}_n)}$ is planar. Suppose $n > 4$. Since $p_1 = 2$ and p_1 divides n, we have $n \geq 6$. By Lemma 2.4, we have $\overline{GT_P(\mathbb{Z}_n)} = K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2} \geq 3$ which implies that $\overline{GT_P(\mathbb{Z}_n)}$ contains $K_{3,3}$, which is a contradiction.

(ii) Assume that n is composite, $p_j \neq 2$ and $P =< p_j >$. If $n = 6$, then $p_j = 3$. A planar embedding of $\overline{GT_P(\mathbb{Z}_6)}$ is given in Figure 1 and so it is planar.

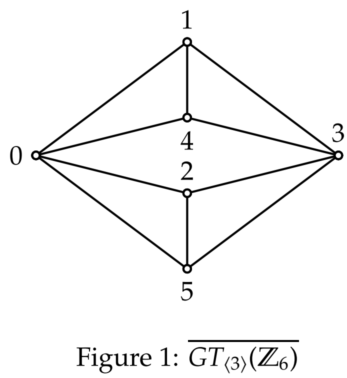

Figure 1: $\overline{GT_{\langle 3 \rangle}(\mathbb{Z}_6)}$

Conversely assume that $\overline{GT_P(\mathbb{Z}_n)}$ is planar, $P =< p_j >$ and $p_j \neq 2$. Suppose $n > 6$.

If $p_j = 3$, then $|P| = |1 + P| = |2 + P| = \frac{n}{3} \geq 3$. Clearly the induced subgraph $< P \cup (1 + P) >$ contains $K_{3,3}$ as a subgraph and so $\overline{GT_P(\mathbb{Z}_n)}$ is non-planar, a contradiction.

If $p_j \geq 5$, then $n \geq 10$ and $|P| = |i + P| \geq 2$ for $1 \leq i \leq p_j - 1$. Consider $S = \{x\} \cup (1 + P) \cup (2 + P)$ where $x \in P$. Then $|S| \geq 5$ and $< S >$ contains K_5 as a subgraph and so K_5 is a subgraph of $\overline{GT_P(\mathbb{Z}_n)}$, a contradiction to $\overline{GT_P(\mathbb{Z}_n)}$ is planar. Hence $n = 6$.

(iii) It is easy to check that $\overline{GT_P(\mathbb{Z}_p)}$ is planar for $P =< p >$ and $p \in \{2, 3, 5\}$. Conversely assume that $\overline{GT_P(\mathbb{Z}_p)}$ is planar where $P =< p >$ and p is a prime number. Suppose $p = 7$. By Lemma 2.2 (iv), $deg(0) = 6$ and $deg(v) = 5$ for $v \neq 0$. This implies that $\overline{GT_P(\mathbb{Z}_7)}$ contains $m = 18$ edges and $m = 18 > 15 = 3p - 6$. By [8, Theorem 9.2], $\overline{GT_P(\mathbb{Z}_7)}$ is not planar. Suppose $p \geq 11$ and p is an odd prime integer. Note that the induced subgraph induced by $\{0, 1, 2, 3, 4\}$ of $\overline{GT_P(\mathbb{Z}_p)}$ is K_5 a contradiction. Hence $p \in \{2, 3, 5\}$. □

Now, we obtain in the following theorem, a characterization for $\overline{GT_P(\mathbb{Z}_n)}$ to be toroidal.

Theorem 3.4. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ where $p_1 < p_2 < \ldots < p_k$, p_j's are prime and α_j's are positive integers. Then the following are true:*

(i) *Let n be composite, $p_1 = 2$ and $P =< p_1 >$. Then $\overline{GT_P(\mathbb{Z}_n)}$ is toroidal if and only if $n \in \{6, 8\}$;*

(ii) *Let $n \geq 9$ be composite, $p_j \neq 2$ and $P =< p_j >$. Then $\overline{GT_P(\mathbb{Z}_n)}$ is not toroidal.*

(iii) *Let n be prime and $P =< n >$. Then $\overline{GT_P(\mathbb{Z}_p)}$ is toroidal if and only if $n = 7$.*

Proof. (i) Assume that n is composite, $p_1 = 2$ and $P =< p_1 >$. One can see that $\overline{GT_P(\mathbb{Z}_6)} = K_{3,3}$ and $\overline{GT_P(\mathbb{Z}_8)} = K_{4,4}$. By Theorem 3.2, $g(\overline{GT_P(\mathbb{Z}_6)}) = 1$ and $g(\overline{GT_P(\mathbb{Z}_8)}) = 1$. Hence $\overline{GT_P(\mathbb{Z}_p)}$ is toroidal if $p = 6$ or $p = 8$.

Conversely assume that n is composite, $p_1 = 2$ and $P =< p_1 >$. Suppose $\overline{GT_P(\mathbb{Z}_n)}$ is toroidal. By Theorem 3.3, $n \geq 6$. Suppose $n \geq 9$. Since $p_1 = 2$ and p_1 divides n, we have $n \geq 10$. By Lemma 2.4, we have $\overline{GT_P(\mathbb{Z}_n)} = K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2} \geq 5$ which implies that $\overline{GT_P(\mathbb{Z}_n)}$ contains $K_{5,5}$ as a subgraph, which is a contradiction. Hence $n \in \{6, 8\}$.

(ii) Let n be composite, $p_j \neq 2$ and $P =< p_j >$.

Case 1. Consider the case that $n = 9$. Then $p_j = 3$ and so $|P| = |1 + P| = |2 + P| = \frac{n}{3} = 3$. The graph $\overline{GT_P(\mathbb{Z}_9)}$ is given in Figure 2 and one can see that $K_{3,6}$ is a subgraph of $\overline{GT_P(\mathbb{Z}_9)}$ with vertex partition $V_1 = \{0, 3, 6\}$ and $V_2 = \{1, 2, 4, 5, 7, 8\}$. By Theorem 3.2, $g(K_{3,6}) = 1$ and hence one can fix an embedding of $K_{3,6}$ on the surface of torus. Note that, there are 9 faces in the embedding of $K_{3,6}$, say $\{f_1, \ldots, f_9\}$. Let n_i be the length of the face f_i. Then $n_i \geq 4$ for every i and $\sum_{i=1}^{9} n_i = 36$. Thus implies that $n_i = 4$ for every i. Now, the induced subgraph $< S >=< \{1, 4, 7\} > \subseteq V(\overline{GT_{<3>}(\mathbb{Z}_9)})$ is K_3. Also, edges of the induced subgraph $< S >$ are disjoint from edges of $K_{3,6}$. Since K_3 cannot be embedded in the torus along with an embedding with only rectangles as faces, one cannot have an embedding of K_3 and $K_{3,6}$ together in the torus. This implies that $g(\overline{GT_3(\mathbb{Z}_9)}) \geq 2$. Hence $\overline{GT_P(\mathbb{Z}_n)}$ is not toroidal

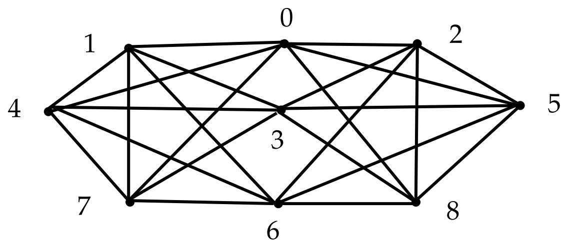

Figure 2 : $\overline{GT_3(\mathbb{Z}_9)}$

Case 2. Consider the case that $n > 9$.

Subcase 2.1. If $p_j = 3$, then $n \geq 12$ and so $|P| = |1 + P| = |2 + P| = \frac{n}{3} \geq 4$. Clearly subgraphs induced by cosets $1 + P$ and $2 + P$ are two disjoint $K_{\frac{n}{3}}$'s. Let $S = (1 + P) \cup \{y\}$ where $y \in 2 + P$. Now, the induced subgraph $< S >$ contains $K_4 \cup K_1$ and so the the subgraph induced by $P \cup S$ contains $K_{4,5}$ as a subgraph. Hence $g(\overline{GT_P(\mathbb{Z}_n)}) \geq 2$. Hence $\overline{GT_P(\mathbb{Z}_n)}$ is not toroidal

Subcase 2.2. If $p_j \geq 5$, then $n \geq 10$ and $|P| = |i + P| \geq 2$ for $1 \leq i \leq p_j - 1$. Let $S = \{x\} \cup (1 + P) \cup (2 + P)$ and $T = \{y\} \cup (4 + P) \cup (5 + P)$ where $x, y \in P$. Then $|S| = |T| \geq 5$ and $< S \cup T >$ contains $K_5 \cup K_5$ as a subgraph. Therefore $\overline{GT_P(\mathbb{Z}_n)}$ contains $2K_5$ and so $g(\overline{GT_P(\mathbb{Z}_n)}) \geq 2$ which implies that $\overline{GT_P(\mathbb{Z}_n)}$ is not toroidal

(iii) Assume that n is prime and $P =< n >$. An embedding of $\overline{GT_P(\mathbb{Z}_7)}$ where $P =< 7 >$ in S_1 is given in Figure 3 and so it is toroidal.

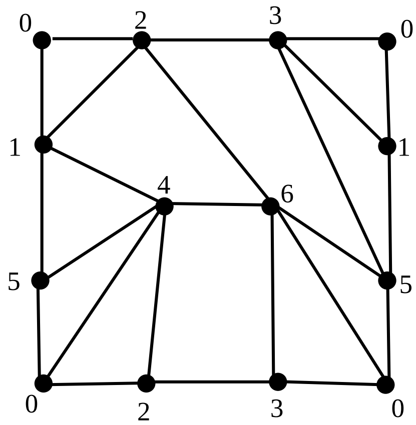

Figure 3 :$\overline{GT_P(\mathbb{Z}_7)}$

Conversely assume that $\overline{GT_P(\mathbb{Z}_p)}$ is toroidal for a prime number n and $P =< n >$. By Theorem 3.3, $p \geq 7$. Suppose $p \geq 11$. Consider the partition of $\mathbb{Z}_p = \{0\} \bigcup_{i=1}^{\frac{p-1}{2}} \{x_i\} \bigcup_{i=1}^{\frac{p-1}{2}} \{y_i\}$, where each x_i is the additive inverse of y_i. Note that $< \bigcup_{i=1}^{\frac{p-1}{2}} \{x_i\} >=< \bigcup_{i=1}^{\frac{p-1}{2}} \{y_i\} >= K_{\frac{p-1}{2}}$ and x_i, y_i are not adjacent in $\overline{GT((\mathbb{Z}_p)}$. Since $\frac{p-1}{2} \geq 5$, $\overline{GT_P(\mathbb{Z}_p)}$ contains $2K_5$ as a subgraph, a contradiction to $\overline{GT_P(\mathbb{Z}_p)}$ is toroidal. Hence $p = 7$. □

4. Domination Parameters of $\overline{GT_P(\mathbb{Z}_n)}$

In this section, we discuss about various domination parameters of $\overline{GT_P(\mathbb{Z}_n)}$. More specifically, we discuss about $\gamma_t, \gamma_c, \gamma_{cl}, \gamma_p, \gamma_{eff}, \gamma_s, \gamma_w$ and γ_i of $\overline{GT_P(\mathbb{Z}_n)}$.

For a graph $G = (V, E)$, a subset $S \subseteq V$ is called a *dominating set* if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. A subset $S \subseteq V$ is called a *total dominating set* if every vertex in $v \in V$ is adjacent to some vertex $u \in S$ and $v \neq u$. A dominating set S is called a *connected (or clique) dominating* set if the subgraph induced by S is connected (or complete). A dominating set S is called an *independent dominating set* if no two vertices of S are adjacent. A dominating set S is called a *perfect dominating set* if every vertex in $V \setminus S$ is adjacent to exactly one vertex in S. A dominating set S is called an *efficient dominating set* if S is both an independent and a perfect dominating set of G. A dominating set S is called a *strong* (or *weak*) dominating set if for every vertex $u \in V \setminus S$, there is a vertex $v \in S$ with $\deg_G(v) \geq \deg_G(u)$ (or $\deg_G(v) \leq \deg_G(u)$) and u is adjacent to v. The domination number γ of G is defined to be minimum cardinality of a dominating set in G and such a domination set is called γ-set in G. One can refer Haynes et al., [9] for definitions of other domination parameters like *total dominating number* γ_t, *connected dominating number* γ_c, *clique dominating number* γ_{cl}, *independent dominating number* $i(G)$, *perfect dominating number* γ_p, *efficient dominating number* γ_{eff}, *strong dominating number* γ_s and *weak dominating number* γ_w. A graph G is called *excellent* if, for every vertex $v \in V$, there exists a γ-set S containing v. A *domatic partition* of G is a partition of V, into dominating sets in G. The maximum number of sets in a domatic partition is called a *domatic number* of G and is denoted by $d(G)$. The maximum number of sets in a domatic partition in which each partition is a total dominating set is called a *total domatic number* of G and is denoted by $d_t(G)$. A graph G is called *domatically full* if $d(G) = \delta(G) + 1$. The *bondage number* $b(G)$ is the minimum number of edges whose removal increases the domination number. The *independent number* $\beta_0(G)$ is the maximum cardinality of an independent set in G. A graph G is well-covered if $\beta_0(G) = i(G)$.

In the following lemma, we obtain the domination number of $\overline{GT_P(\mathbb{Z}_n)}$.

Lemma 4.1. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ where $p_1 < p_2 < \ldots < p_k$, p_j's are prime, α_j's are positive integers and $P =< p_j >$ for some j. Then*

$$\gamma(\overline{GT_P(\mathbb{Z}_n)}) = \begin{cases} 1 & \text{if } n \text{ is prime;} \\ 2 & \text{if } n \text{ is composite.} \end{cases}$$

Proof. Assume that n is prime. By Lemma 2.2(i) and (ii), $\overline{GT_P(\mathbb{Z}_n)}$ contains a vertex of degree $n-1$ and so $\gamma(\overline{GT_P(\mathbb{Z}_n)}) = 1$.

Suppose n is composite and $P =< 2 >$. Then $\overline{GT_P(\mathbb{Z}_n)} = K_{\frac{n}{2},\frac{n}{2}}$ and so $\gamma(\overline{GT_P(\mathbb{Z}_n)}) = 2$.

Suppose n is composite and $P =< p_j >$, $p_j \neq 2$. By Lemma 2.3(ii), $\gamma(\overline{GT_P(\mathbb{Z}_n)}) > 1$. Let $x \in P$ and $y \notin P$. Let $z \in \mathbb{Z}_n \setminus \{x, y\}$. Suppose $z \in P$. Then z, y are adjacent in $\overline{GT_P(\mathbb{Z}_n)}$ by Lemma 2.4. Suppose $z \notin P$. Then z, x are adjacent in $\overline{GT_P(\mathbb{Z}_n)}$ by Lemma 2.4. Therefore $\{x, y\}$ is a dominating set in $\overline{GT_P(\mathbb{Z}_n)}$. By Lemma 2.3, $\{x, y\}$ is a minimal dominating set in $\overline{GT_P(\mathbb{Z}_n)}$. □

In view of Lemma 2.4, we have the following characterization of γ-sets in $\overline{GT_P(\mathbb{Z}_n)}$.

Theorem 4.2. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ where $p_1 < p_2 < \ldots < p_k$, p_j's are prime, α_j's are positive integers and $P =< p_j >$ for some j.*

(i) *Let n be composite and $S = \{x, y\} \subseteq \mathbb{Z}_n$. Then S is a γ-set if and only if x, y are in two distinct cosets of P in $\mathbb{Z}_n$;*

(ii) *The set $S = \{0\}$ is a γ-set in $\overline{GT_P(\mathbb{Z}_n)}$ if and only if n is prime.*

Corollary 4.3. *Let n be a composite integer. Then $\gamma_t(\overline{GT_P(\mathbb{Z}_n)}) = \gamma_c(\overline{GT_P(\mathbb{Z}_n)}) = \gamma_{cl}(\overline{GT_P(\mathbb{Z}_n)}) = 2$.*

Proof. Assume that n is a composite integer. Let $x \in P$ and $y \in \mathbb{Z}_n \setminus P$. By Lemma 2.4, $S = \{x, y\}$ is a dominating set of $\overline{GT_P(\mathbb{Z}_n)}$. By Lemma 2.2, x, y are adjacent in $\overline{GT_P(\mathbb{Z}_n)}$. Therefore S is a total dominating set of $\overline{GT_P(\mathbb{Z}_n)}$. Therefore $\gamma_t(\overline{GT_P(\mathbb{Z}_n)}) = \gamma_c(\overline{GT_P(\mathbb{Z}_n)}) = \gamma_{cl}(\overline{GT_P(\mathbb{Z}_n)}) = 2$. □

Recall that when $p_1 = 2$ and $P =< p_1 >$, $\overline{GT_P(\mathbb{Z}_n)}$ is a complete bi-partite graph. Using this along with Theorem 4.2, we have the following result.

Lemma 4.4. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ where $p_1 < p_2 < \ldots < p_k$, p_j's are prime, α_j's are positive integers and $P =< p_j >$ for some j. Then the following are true:*

(i) *If n is composite, then $\overline{GT_P(\mathbb{Z}_n)}$ is excellent;*

(ii) *Let n be prime. Then $\overline{GT_P(\mathbb{Z}_n)}$ is excellent if and only if $n = 2$.*

Proof. (i) Assume that n is a composite integer. Let $x \in \mathbb{Z}_n$. Then either $x \in P$ or $x \in \mathbb{Z}_n \setminus P$. Without loss of generality $x \in P$. By Lemma 4.2(i), for any $y \in \mathbb{Z}_n \setminus P$, $\{x, y\}$ is a γ-set in $\overline{GT_P(\mathbb{Z}_n)}$.

(ii) When $n = 2$, $\overline{GT_P(\mathbb{Z}_n)} = K_2$ and hence it is excellent. Conversely suppose $\overline{GT_P(\mathbb{Z}_n)}$ is excellent for a prime number n. By Lemma 4.1, $\gamma(\overline{GT_P(\mathbb{Z}_n)} = 1$. By the assumption that $\overline{GT_P(\mathbb{Z}_n)}$ is excellent, degree of every vertex must be $n-1$ and hence $n-1 = 1$ which in turn implies that $n = 2$. □

Lemma 4.5. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $p_1 < p_2 < \dots < p_k$, p_j's are prime, α_j's are positive integers and $P =< p_1 >$. Then the following are true:*

(i) *If n is prime, then $\{0\}$ is a perfect dominating set in $\overline{GT_P(\mathbb{Z}_n)}$;*

(ii) *Let n be composite. Then a perfect dominating set exists in $\overline{GT_P(\mathbb{Z}_n)}$ if and only if $p_1 = 2$.*

Proof. (i) If n is prime, then by Theorem 4.2, $S = \{0\}$ is a perfect domination set in $\overline{GT_P(\mathbb{Z}_n)}$.

(ii) Let n be a composite integer. Assume that $p_1 \neq 2$. Suppose S is a perfect dominating set in $\overline{GT_P(\mathbb{Z}_n)}$. Since n is composite, $|P| = |i + P| \geq 2$, for $1 \leq i \leq p_j - 1$. Let x, y be two distinct elements in $\mathbb{Z}_n$. Suppose $x, y \in P$. Note that, every $z \in \mathbb{Z}_n \setminus P$ is adjacent to both x and y and hence both x and y cannot be in S.

Suppose $x \in P \cap S$ and $y \in P$ with $y \notin S$. Since P is an independent set, x, y are not adjacent in $\overline{GT_P(\mathbb{Z}_n)}$. Since S is dominating set, $\exists z_1 \in S$ such that z_1, y are adjacent in $\overline{GT_P(\mathbb{Z}_n)}$. By Lemma 2.4, $z_1 \in i + P$ for some i where $1 \leq i \leq p_j - 1$. Since $|i + P| \geq 2$, $\exists z_2 \in i + P$ with $z_2 \neq z_1$. If $z_2 \in S$, by Lemma 2.4 and $p_j \neq 2$, y is adjacent to z_1, z_2 which is a contradiction. Hence $z_2 \in V \setminus S$. By Lemma 2.4, x, z_2 are adjacent and z_1, z_2 are also adjacent in $\overline{GT_P(\mathbb{Z}_n)}$, which is a contradiction to S is a perfect dominating set.

Suppose $x, y \in S \setminus P$. Since every element in P is adjacent to both x, y again a contradiction. Hence $\overline{GT_P(\mathbb{Z}_n)}$ has no perfect dominating set.

Conversely, assume that $p_1 = 2$. Then $\overline{GT_P(\mathbb{Z}_n)} = K_{\frac{n}{2},\frac{n}{2}}$. Let $x \in P$ and $y \in 1 + P$. Then $S = \{x, y\}$ is a dominating set of $\overline{GT_P(\mathbb{Z}_n)}$. Clearly every odd number in $\mathbb{Z}_n$ is adjacent with only $x \in S$, and every even number is adjacent with only $y \in S$. Hence S is a γ_p-set of $\overline{GT_P(\mathbb{Z}_n)}$. □

In view of Lemma 4.5, we have the following corollary.

Corollary 4.6. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ be an integer where $p_1 < p_2 < \dots < p_k$, p_j's are prime and α_j's are positive integers. Then the following are true:*

$$\gamma_p(\overline{GT_P(\mathbb{Z}_n)}) = \begin{cases} 1 & \text{if } n \text{ is a prime ;} \\ 2 & \text{if } n \text{ is a composite and } p_1 = 2; \\ 0 & \text{if } n \text{ is a composite and } p_1 \neq 2. \end{cases}$$

Lemma 4.7. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ be a composite integer where $p_1 < p_2 < \dots < p_k$, p_j's are prime and α_j's are positive integer. Then the following are true:*

(i) *If n is composite, $p_1 = 2$ and $P =< p_1 >$, then $\gamma_s(\overline{GT_P(\mathbb{Z}_n)}) = \gamma_w(\overline{GT_P(\mathbb{Z}_n)}) = 2$;*

(ii) *If n is composite, $p_j \neq 2$ and $P =< p_j >$ for some j, then $\gamma_s(\overline{GT_P(\mathbb{Z}_n)}) = \frac{n}{p_j}$ and $\gamma_w(\overline{GT_P(\mathbb{Z}_n)}) = 2$.*

Proof. (i) Suppose n is a composite integer and $p_1 = 2$. Then $\overline{GT_P(\mathbb{Z}_n)} = K_{\frac{n}{2},\frac{n}{2}}$. For $x \in P$ and $y \in 1+P$, $S = \{x, y\}$ is a dominating set of $\overline{GT_P(\mathbb{Z}_n)}$. Also $deg(v) = \frac{n}{2}$ for all $v \in \mathbb{Z}_n$. Therefore $\gamma_s(\overline{GT_P(\mathbb{Z}_n)}) = \gamma_w(\overline{GT_P(\mathbb{Z}_n)}) = 2$.

(ii) Suppose n is a composite integer and $p_j \neq 2$. By the definition of $\overline{GT_P(\mathbb{Z}_n)}$, each vertex $x \in P$ is adjacent with each vertex $y \in \mathbb{Z}_n \setminus P$. By Lemma 2.2(iv), we have $deg\,(x) = n - \frac{n}{p_j}$ and $deg\,(y) = n - \frac{n}{p_j} - 1$. Since P dominates $\overline{GT_P(\mathbb{Z}_n)}$ and $deg(x) > deg(y)$, P is a strong dominating set in $\overline{GT_P(\mathbb{Z}_n)}$. Then $\gamma_s(\overline{GT_P(\mathbb{Z}_n)}) = \frac{n}{p_j}$.

Let $x \in i + P$ and $y \in (p_j - i) + P$ where $1 \leq i \leq \frac{p_j - 1}{2}$. By Theorem 4.2, $S = \{x, y\}$ is a dominating set of $\overline{GT_P(\mathbb{Z}_n)}$ and by Lemma 2.2(iv), $deg(x) = deg(y) = n - \frac{n}{p_j} - 1 = \delta(\overline{GT_P(\mathbb{Z}_n)})$. Then $\{x, y\}$ is a weak dominating set in $\overline{GT_P(\mathbb{Z}_n)}$ and so $\gamma_w(\overline{GT_P(\mathbb{Z}_n)}) = 2$. □

Lemma 4.8. *Let $n > 1$ be a prime integer. Then the following are true:*

(i) *If $n = 2$, then $\gamma_s(\overline{GT_P(\mathbb{Z}_n)}) = \gamma_w(\overline{GT_P(\mathbb{Z}_n)}) = 1$.*

(ii) *If $n \neq 2$, then $\gamma_s(\overline{GT_P(\mathbb{Z}_n)}) = 1$ and $\gamma_t(\overline{GT_P(\mathbb{Z}_n)}) = \gamma_c(\overline{GT_P(\mathbb{Z}_n)}) = \gamma_{cl}(\overline{GT_P(\mathbb{Z}_n)}) = \gamma_w(\overline{GT_P(\mathbb{Z}_n)}) = 2$.*

Proof. (i) If $n = 2$, then $\overline{GT_P(\mathbb{Z}_n)} = K_2$ and so $\gamma_s(\overline{GT_P(\mathbb{Z}_n)}) = \gamma_w(\overline{GT_P(\mathbb{Z}_n)}) = 1$;

(ii) Let n be an odd prime. By Theorem 4.2, $S = \{0\}$ is a γ-set in $\overline{GT_P(\mathbb{Z}_n)}$. By Lemma 2.2(ii), $deg(0) > deg(v)$ for all $v \in \mathbb{Z}_n \setminus \{0\}$. This gives that $\gamma_s(\overline{GT_P(\mathbb{Z}_n)}) = 1$. If $a \neq 0$, then $S = \{0, a\}$ is a γ-set in $\overline{GT_P(\mathbb{Z}_n)}$. By Lemma 2.2(ii), $deg(a) \leq deg(v)$ for all $v \in \mathbb{Z}_n$. By Lemma 2.4, $0, a$ are adjacent in $\overline{GT_P(\mathbb{Z}_n)}$. Hence $\gamma_t(\overline{GT_P(\mathbb{Z}_n)}) = \gamma_c(\overline{GT_P(\mathbb{Z}_n)}) = \gamma_{cl}(\overline{GT_P(\mathbb{Z}_n)}) = \gamma_w(\overline{GT_P(\mathbb{Z}_n)}) = 2$. □

Now, we obtain the independent dominating number of $\overline{GT_P(\mathbb{Z}_n)}$.

Lemma 4.9. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ where $p_1 < p_2 < \ldots < p_k$, p_j's are prime, α_j's are positive integers and $P =< p_j >$ for some j. Then*

$$\gamma_i(\overline{GT_P(\mathbb{Z}_n)}) = \begin{cases} 1 & \text{if } n \text{ is prime;} \\ \frac{n}{2} & \text{if } n \text{ is composite and } p_1 = 2; \\ 2 & \text{if } n \text{ is composite and } p_j \neq 2. \end{cases}$$

Proof. Let n be a prime integer. By Theorem 4.2, $S = \{0\}$ is a γ-set in $\overline{GT_P(\mathbb{Z}_n)}$. Clearly S is an independent dominating set and so $\gamma_i(\overline{GT_P(\mathbb{Z}_n)}) = 1$.

Suppose n is a composite integer and $p_1 = 2$. By Lemma 2.4, $\overline{GT_P(\mathbb{Z}_n)} = K_{\frac{n}{2},\frac{n}{2}}$ and hence P is an independent dominating set. Therefore $\gamma_i(\overline{GT_P(\mathbb{Z}_n)}) = \frac{n}{2}$.

Let n be composite and $p_j \neq 2$. For $x \in i + P$ and $y \in (p_j - i) + P$ where $1 \leq i \leq p_j$, by Lemma 4.2, $\{x, y\}$ is a dominating set in $\overline{GT_P(\mathbb{Z}_n)}$. By Lemma 2.4, x, y are not adjacent in $\overline{GT_P(\mathbb{Z}_n)}$. Hence $\gamma_i(\overline{GT_P(\mathbb{Z}_n)}) = 2$. □

Corollary 4.10. *Let $n > 1$ be a prime number. Then $\gamma_{eff}(\overline{GT_P(\mathbb{Z}_n)}) = 1$.*

Proof. Let $n > 1$ be a prime integer. By Theorem 4.2, $S = \{0\}$ is a γ-set in $\overline{GT_P(\mathbb{Z}_n)}$. Clearly S is both independent and perfect dominating set and so $\gamma_{eff}(\overline{GT_P(\mathbb{Z}_n)}) = 1$. □

A graph G is *well-covered* if $\beta(G) = \gamma_i(G)$.

Lemma 4.11. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ where $p_1 < p_2 < \ldots < p_k$, p_j's are prime and α_j's are positive integers. Then the following are true:*

(i) *If $n = 2$, $p_1 = 2$ and $P =< p_1 >$, then $\overline{GT_P(\mathbb{Z}_n)}$ is well-covered;*

(ii) *If n is composite, $p_1 = 2$ and $P =< p_1 >$, then $\overline{GT_P(\mathbb{Z}_n)}$ is well-covered;*

(iii) *Let n be composite, $p_j \neq 2$ and $P =< p_j >$. Then $\overline{GT_P(\mathbb{Z}_n)}$ is well-covered if and only if $n = 2p_j$.*

Proof. Proof of (i) and (ii) follows from Theorem 2.10 and Lemma 4.9.

(iii) Suppose $n = 2p_j$ is a composite integer and $p_j \neq 2$. Note that $\frac{n}{p_j} = 2$. By Theorem 2.10 and Lemma 4.9, $\gamma_i(\overline{GT_P(\mathbb{Z}_n)}) = \beta(\overline{GT_P(\mathbb{Z}_n)}) = 2$. Conversely assume that $n \neq 2p_j$. By Theorem 2.10 and Lemma 4.9, $\gamma_i(\overline{GT_P(\mathbb{Z}_n)}) \neq \beta(\overline{GT_P(\mathbb{Z}_n)})$(Since $\frac{n}{p_j} > 2$). Hence $\overline{GT_P(\mathbb{Z}_n)}$ is not well-covered. □

Lemma 4.12. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ where $p_1 < p_2 < \ldots < p_k$, p_j's are prime, α_j's are positive integers and $P =< p_j >$ for some j. Then*

$$d(\overline{GT_P(\mathbb{Z}_n)}) = \begin{cases} 2 & \text{if } n = 2,\ p_1 = 2 \text{ and } P =< p_1 >; \\ \frac{n+1}{2} & \text{if } n = p_j \text{ an odd prime and } P =< p_j >; \\ \frac{n}{2} & \text{if } n \text{ is composite, } p_1 = 2 \text{ and } P =< p_1 >; \\ \dfrac{n - \frac{n}{p_j}}{2} + 1 & \text{if } n \text{ is composite } p_j \neq 2 \text{ and } P =< p_j > . \end{cases}$$

Proof. For $n = 2$, $\overline{GT_P(\mathbb{Z}_2)} = K_2$ and hence $d(\overline{GT_P(\mathbb{Z}_n)}) = 2$.

Suppose n is an odd prime. Let a and b be such that $a \neq 0$ and $a + b = n$. Then $S = \{a, b\} \subseteq \mathbb{Z}_n$ is a dominating set and so $\mathbb{Z}_n = \{0\} \bigcup_{\frac{n-1}{2}} \{a, b\}$ is a maximal domatic partition of $\overline{GT_P(\mathbb{Z}_n)}$. This gives that $d(\overline{GT_P(\mathbb{Z}_n)}) = \frac{n+1}{2}$.

Suppose n is a composite integer and $p_1 = 2$. By Lemma 2.4, $\overline{GT_P(\mathbb{Z}_n)} = K_{\frac{n}{2},\frac{n}{2}}$. Select $a \in P$ and $b \in 1 + P$. Then $X = \bigcup_{\frac{n}{2}} \{a, b\}$ is a maximal domatic partition of $\overline{GT_P(\mathbb{Z}_n)}$ and hence $d(\overline{GT_P(\mathbb{Z}_n)}) = \frac{n}{2}$.

Suppose n is a composite integer and $p_j \neq 2$. Select i_1 and i_2 such that $1 \leq i_1 \leq p_j - 1$ and $i_1 + i_2 = p_j$. Let $a \in i_1 + P$ and $b \in i_2 + P$. Note that $\mathbb{Z}_n \setminus P = \bigcup_{\frac{n-\frac{n}{p_j}}{2}} \{a, b\}$ and hence $\mathbb{Z}_n = P \bigcup_{\frac{n-\frac{n}{p_j}}{2}} \{a, b\}$ is a maximal domatic partition of $\overline{GT_P(\mathbb{Z}_n)}$ and hence $d(\overline{GT_P(\mathbb{Z}_n)}) = \dfrac{n - \frac{n}{p_j}}{2} + 1$. □

Lemma 4.13. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ where $p_1 < p_2 < \ldots < p_k$, p_j's are prime, α_j's are positive integers and $P = < p_j >$ for some j. Then $\overline{GT_P(\mathbb{Z}_n)}$ is domatically full if and only if $n = 2, 3$.*

Proof. If part follows from Lemma 2.2(iv) and Lemma 4.12.

Conversely assume that $\overline{GT_P(\mathbb{Z}_n)}$ is domatically full. Suppose that n is prime and $n \neq 2$. Then n is an odd prime. By Lemma 2.2, $\delta(\overline{GT_P(\mathbb{Z}_n)}) = n - 2$ and by Lemma 4.12, $d(\overline{GT_P(\mathbb{Z}_n)}) = \frac{n+1}{2}$. By the assumption $n - 2 + 1 = \frac{n+1}{2}$, which in turn implies that $n = 3$.

Let n be a composite integer. If $p_j = 2$, by Lemma 2.2 and Lemma 4.12, $\delta(\overline{GT_P(\mathbb{Z}_n)}) = d(\overline{GT_P(\mathbb{Z}_n)})$. Therefore $\overline{GT_P(\mathbb{Z}_n)}$ is not domatically full. If $p_j \neq 2$, then by Lemma 2.2 and Lemma 4.12, $n - \frac{n}{p_j} - 1 + 1 = \delta(\overline{GT_P(\mathbb{Z}_n)}) + 1 = d(\overline{GT_P(\mathbb{Z}_n)}) = \dfrac{n - \frac{n}{p_j}}{2} + 1$, which is impossible. □

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