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# A Fully Discrete Finite Element Scheme for the Kelvin-Voigt Model

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**Abstract.** In this paper, we study convergence of a fully discrete scheme for the two-dimensional nonstationary Kelvin-Voigt model. This scheme is based on a finite element approximation for space discretization and the Crank-Nicolson-type scheme for time discretization, which is a two step method. Moreover, we obtain error estimates of velocity and pressure. At last, the applicability and effectiveness of the present algorithm are illustrated by numerical experiments.

## 1. Introduction

In this article, we discuss convergence of a fully discrete scheme for the following system of motion arising in the Kelvin-Voigt fluids:

$$\begin{aligned}
&\mathbf{u}_t - \kappa\Delta\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x},t), \quad \mathbf{x}\in\Omega,\ t\in[0,T],\\
&\nabla\cdot\mathbf{u} = 0, \quad \mathbf{x}\in\Omega,\ t\in[0,T],\\
&\mathbf{u} = 0, \quad \text{on } \partial\Omega,\ t\in(0,T],\\
&\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \text{in } \mathbf{x}\in\Omega,
\end{aligned} \tag{1}$$

where $\Omega$ is a convex bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ represents the velocity vector, $p = p(\mathbf{x},t)$ is the pressure, $\nu$ is the kinematic coefficient of viscosity and $\kappa$ is the retardation time. The Kelvin-Voigt model was first introduced and studied by Oskolkov [21] as a model for certain viscoelastic fluids known as Kelvin-Voigt fluids. This model is applied not only in the study of organic polymers and food industry, but also in the mechanism of diffuse axonal injury that is unexplained by traumatic brain injury models [7].

Firstly, we present a quick review of some related literature on the Kelvin-Voigt model. Based on the analysis of Ladyzenskaya [18] for the solvability of the Navier-Stokes equations, Oskolkov [20, 21] has proved the global existence of the Navier-Stokes classical solution in finite time interval for the initial and boundary values problem (1). The investigations on solvability are further continued by Oskolkov and Shadiev [22, 24], who have discussed the existence and uniqueness results on the entire semiaxis in time.

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Recently, Bajpai et al. [1] have found that the additional term $\kappa\Delta\mathbf{u}_t$ which is the only difference with Navier-Stokes system takes care of the relaxation property of the fluid and they also have obtained an exponential decay property in time.

Secondly, for numerical methods, the finite element analysis of (1) has been studied in many articles. In [16], semidiscrete approximation for (1) is discussed, keeping the time variable continuous. Moreover, Pany et al. [25] have derived a priori optimal error estimates for the velocity in $L^\infty(L^2)$ norm as well as velocity in $L^\infty(H^1)$ norm and for the pressure term in $L^\infty(L^2)$ norm. Under the assumption that the solution is asymptotically stable as $t \to \infty$, Oskolkov [23] has proved that the Galerkin approximation to the problem (1) is convergent in semitime axis $t \geq 0$. Kuberry et al. [15] have found that the Voigt regularization provides accurate reduced order models for Navier-Stokes equations for fluid flow. In [3, 4], two fully discrete two-grid schemes have been applied to problem (1), where the second order accurate backward difference scheme and Crank-Nicolson scheme have been employed for time discretization. In [2], Bajpai et al. have analyzed both backward Euler scheme and backward difference scheme for full discretization of the problem (1) with the forcing function $\mathbf{f} = 0$. In continuation to the investigation, Pany et al. [27] have employed a backward Euler method along with its linearized version for the time discretization of the problem (1), which is first-order in time scheme. In [26], Pany has used a second-order backward difference method for time discretization and discussed the order of convergence with different $\kappa$.

In this article, we will study a fully discrete Crank-Nicolson-type scheme based on finite element approximation for numerically solving the Kelvin-Voigt equations. The rest of this paper is organized as follows. In the next section, some preliminaries used in the subsequent sections are presented. In Section 3, we show a finite element approximation in spatial direction and the Crank-Nicolson-type scheme in time direction for the considered problem. Moreover, convergence of the fully discrete scheme is given. Numerical experiments are shown to verify the theoretical results completely in Section 4.

## 2. Preliminaries

In this paper, we denote the usual $L^2(\Omega)$ norm and its inner product by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. The $L^p(\Omega)$ norm and $W^{m,p}(\Omega)$ norm are denoted by $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{m,p}(\Omega)}$ respectively for $m \in \mathbb{N}^+$, $1 \leq p \leq \infty$. In particular, $H^m(\Omega)$ is used to represent the space $W^{m,2}(\Omega)$ and $\|\cdot\|_{H^m}$ denotes the norm in $H^m(\Omega)$.

For $X$ being a normed function space in $\Omega$, $L^p(0,t;X)$ is the space of all functions defined on $(0,t)\times\Omega$ for which the norm

$$\|\cdot\|_{L^p(0,t;X)} = \left(\int_0^t \|\cdot\|_X^p dt\right)^{\frac{1}{p}},\ p \in [1,\infty)$$

is finite. For $p = \infty$, the usual modification is used in the definition of this space.

The natural function spaces for the problem are defined by

$$\mathbf{X} := H_0^1(\Omega)^2 = \{\mathbf{v} \in L^2(\Omega)^2 : \nabla\mathbf{v} \in (L^2(\Omega))^{2\times 2},\ \mathbf{v} = 0\ on\ \partial\Omega\},$$

$$Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}.$$

For $\mathbf{f}$ an element in the dual space of $\mathbf{X}$, its norm is defined by

$$\|\mathbf{f}\|_{-1} = \sup_{\mathbf{v}\in\mathbf{X}} \frac{|(\mathbf{f},\mathbf{v})|}{\|\nabla\mathbf{v}\|}.$$

The space of divergence free functions is given by

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{X} : (\nabla\cdot\mathbf{v}, q) = 0,\ \forall q \in Q\}.$$

Furthermore, in the rest of the paper, we adopt a bilinear form:

$a(\mathbf{u},\mathbf{v}) := (\nabla\mathbf{u},\nabla\mathbf{v}),$

and a skew-symmetrized trilinear form [11, 12]:

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v}) = (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) + \frac{1}{2}((\nabla\cdot\mathbf{u})\mathbf{v},\mathbf{w}). \tag{2}$$

Later analysis will require upper bounds on the nonlinear term, given in the following lemma.

**Lemma 2.1.** *For $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}$, we have*

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \le C\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|, \quad |b(\mathbf{u},\mathbf{v},\mathbf{w})| \le C\|\mathbf{u}\|^{\frac{1}{2}}\|\nabla\mathbf{u}\|^{\frac{1}{2}}\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|.$$

*In addition, if $\mathbf{v}\in L^\infty(\Omega)^2$ and $\nabla\mathbf{v}\in(L^\infty(\Omega))^{2\times2}$, then*

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \le C(\|\mathbf{v}\|_{L^\infty(\Omega)} + \|\nabla\mathbf{v}\|_{L^\infty(\Omega)})\|\mathbf{u}\|\|\nabla\mathbf{w}\|,$$
$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \le C(\|\mathbf{u}\|\|\nabla\mathbf{v}\|_{L^\infty(\Omega)} + \|\nabla\mathbf{u}\|\|\mathbf{v}\|_{L^\infty(\Omega)})\|\mathbf{w}\|,$$

*where $C>0$ is a constant depending on $\Omega$.*

The proof of Lemma 2.1 can be found in [17]. Subsequently, $C$ will denote a positive constant depending at most on the data $\kappa$, $\Omega$, $\nu$, $T$, $\mathbf{u}$ and $p$, and may stand for different values at its different occurrences. It is well known that the Gronwall's inequality plays an important role in the study of differential systems of various kind. The following variation on the discrete Gronwall Lemma is given in [10].

**Lemma 2.2.** *(Discrete Gronwall lemma). For the integer $m\ge1$, let $\tau, C, a_m, b_m, c_m, r_m$, be nonnegative numbers such that*

$$a_n + \tau\sum_{m=1}^{n} b_m \le \tau\sum_{m=1}^{n} r_m a_m + \tau\sum_{m=1}^{n} c_m + C.$$

*Suppose that $\tau r_m < 1$ for all $m$ and set $\sigma_m = (1-\tau r_m)^{-1}$, then*

$$a_n + \tau\sum_{m=1}^{n} b_m \le \exp\left(\tau\sum_{m=1}^{n}\sigma_m r_m\right)\left(\sum_{m=1}^{n} c_m + C\right).$$

Next, let $\mathbf{X}^h\subset\mathbf{X}, Q^h\subset Q$ denote conforming velocity and pressure finite element spaces based on an edge to edge triangulations of $\Omega$ with maximum triangle diameter $h$. The velocity-pressure finite element spaces $(\mathbf{X}^h, Q^h)$ are assumed to satisfy the usual discrete inf-sup condition or $LBB^h$ condition for stability of the discrete pressure:

$$\inf_{q_h\in Q^h}\sup_{\mathbf{v}_h\in\mathbf{X}^h}\frac{(q_h,\nabla\cdot\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|\|q_h\|} = \beta^h > 0,$$

where $\beta^h$ is a constant that is independent of $h$. The discrete divergence free subspace of $\mathbf{X}^h$ is

$$\mathbf{V}^h := \{\mathbf{v}_h\in\mathbf{X}^h : (\nabla\cdot\mathbf{v}_h, q_h) = 0,\ \forall q_h\in Q^h\}.$$

To help simplify a very technical analysis, we choose $(\mathbf{X}^h, Q^h) = (P_m, P_{m-1})$ finite element pairs to approximate velocity-pressure spaces. Besides, one has the following approximation properties typical of piecewise polynomials of degree $(m, m-1)$ as follows [5]:

$$\begin{cases} \inf\limits_{\mathbf{v}_h\in\mathbf{X}^h}\|\mathbf{u}-\mathbf{v}_h\| \le Ch^{m+1}\|\mathbf{u}\|_{m+1}, & \mathbf{u}\in H^{m+1}(\Omega)^2,\\ \inf\limits_{\mathbf{v}_h\in\mathbf{X}^h}\|\nabla(\mathbf{u}-\mathbf{v}_h)\| \le Ch^{m}\|\mathbf{u}\|_{m+1}, & \mathbf{u}\in H^{m+1}(\Omega)^2,\\ \inf\limits_{q_h\in Q^h}\|p-q_h\| \le Ch^{m}\|p\|_{m}, & p\in H^{m}(\Omega). \end{cases} \tag{3}$$

We will also use the following inequality [8], which holds under $LBB^h$ condition and for all $\mathbf{u}\in\mathbf{V}$:

$$\inf_{\mathbf{v}_h\in\mathbf{V}^h}\|\nabla(\mathbf{u}-\mathbf{v}_h)\| \le C\inf_{\mathbf{v}_h\in\mathbf{X}^h}\|\nabla(\mathbf{u}-\mathbf{v}_h)\|.$$

Throughout the paper, we use the following Stokes projection [17].

**Definition 2.3.** *(Stokes projection). The Stokes projection operator* $P_s : (\mathbf{X}, Q) \to (\mathbf{X}^h, Q^h)$, $P_s(\mathbf{u}, p) = (\tilde{\mathbf{u}}, \tilde{p})$, *satisfies*

$$\begin{aligned} \nu a(\mathbf{u} - \tilde{\mathbf{u}}, \mathbf{v}_h) - (p - \tilde{p}, \nabla \cdot \mathbf{v}_h) = 0, \\ (\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), q_h) = 0, \end{aligned}$$

*for any* $\mathbf{v}_h \in \mathbf{X}^h, q_h \in Q^h$.

In $(\mathbf{V}^h, Q^h)$ this formulation reads: given $(\mathbf{u}, p) \in (\mathbf{X}, Q)$, find $\tilde{\mathbf{u}} \in \mathbf{V}^h$ satisfying

$$\nu a(\mathbf{u} - \tilde{\mathbf{u}}, \mathbf{v}_h) - (p - q_h, \nabla \cdot \mathbf{v}_h) = 0, \tag{4}$$

for any $\mathbf{v}_h \in \mathbf{V}^h, q_h \in Q^h$.

The following results can easily obtained by Taylor series expansion.

**Lemma 2.4.** *Let* $\tau = t_{n+1} - t_n$ *and* $\xi(\cdot, t)$ *be a function such that* $\xi_{tt} \in L^2(0, T; L^2(\Omega))$. *Then there exists* $\theta \in (0, 1)$ *such that*

$$\left\| \frac{\xi(\cdot, t_{n+1}) + \xi(\cdot, t_{n-1})}{2} - \xi(\cdot, t_n) \right\| \le C\tau^2 \|\xi_{tt}(\cdot, t_{n+\theta})\|.$$

*Besides, if* $\xi_{ttt} \in L^2(0, T; L^2(\Omega))$, *then there exists* $\theta \in (0, 1)$ *such that*

$$\left\| \frac{\xi(\cdot, t_{n+1}) - \xi(\cdot, t_{n-1})}{2\tau} - \xi_t(\cdot, t_n) \right\| \le C\tau^2 \|\xi_{ttt}(\cdot, t_{n+\theta})\|.$$

## 3. A fully discrete finite element scheme for Kelvin-Voigt equations

In this section, we will construct a Crank-Nicolson-type scheme of the problem (1) base on finite element discretization and show the error estimates of the fully discrete scheme. Let $\{t_n\}_{n=0}^N$ be a uniform partition of $[0, T]$ and $t_n = n\tau$, where $\tau > 0$ is time step.

**Algorithm 3.1.** Step I: Find $(\mathbf{u}_h^1, p_h^1) \in (\mathbf{X}^h, Q^h)$ such that for all $(\mathbf{v}_h, q_h) \in (\mathbf{X}^h, Q^h)$:

$$\begin{aligned} &\left(\frac{\mathbf{u}_h^1 - \mathbf{u}_h^0}{\tau}, \mathbf{v}_h\right) + \kappa a\left(\frac{\mathbf{u}_h^1 - \mathbf{u}_h^0}{\tau}, \mathbf{v}_h\right) + \nu a\left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, \mathbf{v}_h\right) + b\left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, \frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, \mathbf{v}_h\right) \\ &\quad - \left(\frac{p_h^1 + p_h^0}{2}, \nabla \cdot \mathbf{v}_h\right) = (\mathbf{f}(t_{\frac{1}{2}}), \mathbf{v}_h), \\ &(\nabla \cdot \mathbf{u}_h^1, q_h) = 0, \end{aligned} \tag{5}$$

where $t_{\frac{1}{2}} = \frac{1}{2}(t_1 + t_0)$. Let $\mathbf{u}_h^0$ be the Stokes Projection of $\mathbf{u}_0(\mathbf{x})$ into $\mathbf{V}^h$.

Step II: For $n \ge 1$, given $(\mathbf{u}_h^{n-1}, p_h^{n-1}), (\mathbf{u}_h^n, p_h^n) \in (\mathbf{X}^h, Q^h)$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in (\mathbf{X}^h, Q^h)$ such that for all $(\mathbf{v}_h, q_h) \in (\mathbf{X}^h, Q^h)$:

$$\begin{aligned} &\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n-1}}{2\tau}, \mathbf{v}_h\right) + \kappa a\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n-1}}{2\tau}, \mathbf{v}_h\right) + \nu a\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}, \mathbf{v}_h\right) \\ &\quad + b\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}, \frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}, \mathbf{v}_h\right) - \left(\frac{p_h^{n+1} + p_h^{n-1}}{2}, \nabla \cdot \mathbf{v}_h\right) = (\mathbf{f}(t_n), \mathbf{v}_h), \\ &(\nabla \cdot \mathbf{u}_h^{n+1}, q_h) = 0. \end{aligned} \tag{6}$$

Furthermore, we will establish the bounds for the finite element discretization of the presented algorithm. Firstly, we consider the error estimate for velocity.

**Theorem 3.1.** *Let the finite element spaces $(X^h, Q^h)$ include continuous piecewise polynomials of degree m and $m-1$, respectively ($m \geq 2$), and satisfy the discrete inf-sup condition and approximation property* (3). *Assume that $\tau(h^{m-\frac{3}{2}} + \nu^{-1}) \leq 1$, $\mathbf{u} \in H^3(0,T;H^1(\Omega)^2) \cap L^2(0,T;H^{m+1}(\Omega)^2)$ and $p \in H^2(0,T;L^2(\Omega)) \cap L^2(0,T;H^m(\Omega))$. Then there is a C such that $\forall\, n \in \{1, ..., N-1\}$ the errors of velocity of Algorithm 3.1 satisfy*

$$\tau \sum_{i=1}^{n} \nu \left\| \nabla \left( \frac{(\mathbf{u}(t_{i+1}) - \mathbf{u}_h^{i+1}) + (\mathbf{u}(t_{i-1}) - \mathbf{u}_h^{i-1})}{2} \right) \right\|^2 \leq C(h^{2m} + \tau^4). \tag{7}$$

*Proof.* Consider the variational formulation of (1), for any time $t$,

$$(\mathbf{u}_t, \mathbf{v}_h) + \kappa a(\mathbf{u}_t, \mathbf{v}_h) + \nu a(\mathbf{u}, \mathbf{v}_h) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) - (p, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathbf{X}^h. \tag{8}$$

Subtract (6) from (8) at $t = t_n$ and thereby write

$$\begin{aligned} &\left(\mathbf{u}_t(t_n) - \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n-1}}{2\tau}, \mathbf{v}_h\right) + \kappa a\left(\mathbf{u}_t(t_n) - \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n-1}}{2\tau}, \mathbf{v}_h\right) + \nu a\left(\mathbf{u}(t_n) - \frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}, \mathbf{v}_h\right) \\ &\quad + b\left(\mathbf{u}(t_n), \mathbf{u}(t_n), \mathbf{v}_h\right) - b\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}, \frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}, \mathbf{v}_h\right) - \left(p(t_n) - \frac{p_h^{n+1} + p_h^{n-1}}{2}, \nabla \cdot \mathbf{v}_h\right) = 0. \end{aligned} \tag{9}$$

Then, we denote the corresponding error by

$$\mathbf{e}^n := \mathbf{u}(t_n) - \mathbf{u}_h^n = (\mathbf{u}(t_n) - \mathbf{U}^n) - (\mathbf{u}_h^n - \mathbf{U}^n) =: \boldsymbol{\eta}^n - \boldsymbol{\phi}_h^n, \tag{10}$$

where $\mathbf{U}^n$ is the Stokes Projection of $\mathbf{u}(t_n)$ into $\mathbf{V}_h$. Add and subtract

$$\begin{aligned} &\left(\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_{n-1})}{2\tau}, \mathbf{v}_h\right) + \kappa a\left(\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_{n-1})}{2\tau}, \mathbf{v}_h\right) \\ &\quad + b\left(\mathbf{u}(t_n) + \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_{n-1})}{2} + \frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}, \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_{n-1})}{2}, \mathbf{v}_h\right) \\ &\quad + \nu a\left(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_{n-1})}{2}, \mathbf{v}_h\right) - \left(\frac{p(t_{n+1}) + p(t_{n-1})}{2}, \nabla \cdot \mathbf{v}_h\right), \end{aligned}$$

to (9) to get the error equation

$$\begin{aligned} &\left(\frac{\mathbf{e}^{n+1} - \mathbf{e}^{n-1}}{2\tau}, \mathbf{v}_h\right) + \kappa a\left(\frac{\mathbf{e}^{n+1} - \mathbf{e}^{n-1}}{2\tau}, \mathbf{v}_h\right) + \nu a\left(\frac{\mathbf{e}^{n+1} + \mathbf{e}^{n-1}}{2}, \mathbf{v}_h\right) \\ &= \left(\frac{p(t_{n+1}) + p(t_{n-1})}{2} - q_h, \nabla \cdot \mathbf{v}_h\right) - b\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}, \frac{\mathbf{e}^{n+1} + \mathbf{e}^{n-1}}{2}, \mathbf{v}_h\right) \\ &\quad - b\left(\frac{\mathbf{e}^{n+1} + \mathbf{e}^{n-1}}{2}, \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_{n-1})}{2}, \mathbf{v}_h\right) + R(\mathbf{u}, p; \mathbf{v}_h), \end{aligned} \tag{11}$$

where note that $(q_h, \nabla \cdot \mathbf{v}_h) = 0$ and

$$\begin{aligned} R(\mathbf{u}, p; \mathbf{v}_h) &= \left(\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_{n-1})}{2\tau} - \mathbf{u}_t(t_n), \mathbf{v}_h\right) + \kappa a\left(\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_{n-1})}{2\tau} - \mathbf{u}_t(t_n), \mathbf{v}_h\right) \\ &\quad + \nu a\left(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_{n-1})}{2} - \mathbf{u}(t_n), \mathbf{v}_h\right) + b\left(\mathbf{u}(t_n), \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_{n-1})}{2} - \mathbf{u}(t_n), \mathbf{v}_h\right) \\ &\quad + b\left(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_{n-1})}{2} - \mathbf{u}(t_n), \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_{n-1})}{2}, \mathbf{v}_h\right) - \left(\frac{p(t_{n+1}) + p(t_{n-1})}{2} - p(t_n), \nabla \cdot \mathbf{v}_h\right). \end{aligned} \tag{12}$$

Moreover, by choosing of the projection $\mathbf{U}^{n+1}$ and $\mathbf{U}^{n-1}$, it follows from (4) that

$$\nu a\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2},\mathbf{v}_h\right)-\left(\frac{p(t_{n+1})+p(t_{n-1})}{2}-q_h,\nabla\cdot\mathbf{v}_h\right)=0. \tag{13}$$

Employing the error decomposition (10) and setting $\mathbf{v}_h=\frac{1}{2}(\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1})$ in (11) gives

$$\begin{aligned}&\frac{1}{4\tau}\left(\|\boldsymbol{\phi}_h^{n+1}\|^2-\|\boldsymbol{\phi}_h^{n-1}\|^2\right)+\frac{\kappa}{4\tau}\left(\|\nabla\boldsymbol{\phi}_h^{n+1}\|^2-\|\nabla\boldsymbol{\phi}_h^{n-1}\|^2\right)+\nu\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\|^2\\&=\left(\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^{n-1}}{2\tau},\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)+\kappa a\left(\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^{n-1}}{2\tau},\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\\&\quad-b\left(\frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1}}{2},\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2},\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\\&\quad-b\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2},\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2},\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\\&\quad+b\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2},\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2},\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)+R\left(\mathbf{u},p;\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right),\end{aligned} \tag{14}$$

since equation (13) and $b(\frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1}}{2},\frac{1}{2}(\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}),\frac{1}{2}(\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}))=0$ because of (2). Then applying the Cauchy-Schwarz and Young's inequalities to two linear terms on the right-hand side (RHS) of (14) yields

$$\begin{aligned}&\left|\left(\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^{n-1}}{2\tau},\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)+\kappa a\left(\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^{n-1}}{2\tau},\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right|\\&\le C\nu^{-1}\left\|\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^{n-1}}{2\tau}\right\|^2+C\nu^{-1}\kappa^2\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^{n-1}}{2\tau}\right)\right\|^2+\frac{\nu}{4}\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\|^2\end{aligned} \tag{15}$$

For clarity, we now analyze each of the remaining nonlinear terms on the RHS of (14) separately. We start with the first nonlinear term in (14). Applying Lemma 2.1, the regularity assumptions on $\mathbf{u}$ and Young's inequality, we obtain

$$\begin{aligned}&\left|b\left(\frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1}}{2},\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2},\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right|\\&\le\left|b\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2},\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2},\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right|+\left|b\left(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2},\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2},\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right|\\&\le\frac{\nu}{16}\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\|^2+C\nu^{-1}\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|^4+C\nu^{-1}\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|^2\\&\quad+C\left\|\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right\|^{\frac{1}{2}}\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\|^{\frac{1}{2}}\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\|.\end{aligned} \tag{16}$$

In light of inverse inequality, one can get

$$\left\|\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right\|\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\|\le C(\|\boldsymbol{\phi}_h^{n+1}\|+\|\boldsymbol{\phi}_h^{n-1}\|)(\|\nabla\boldsymbol{\phi}_h^{n+1}\|+\|\nabla\boldsymbol{\phi}_h^{n-1}\|)\le Ch^{-1}(\|\boldsymbol{\phi}_h^{n+1}\|+\|\boldsymbol{\phi}_h^{n-1}\|)^2.$$

Hence, the fourth term of (16) becomes

$$\begin{aligned}
&\left\|\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right\|^{\frac{1}{2}}\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\|^{\frac{1}{2}}\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\| \\
&\leq Ch^{-\frac{3}{2}}\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|(\|\boldsymbol{\phi}_h^{n+1}\|+\|\boldsymbol{\phi}_h^{n-1}\|)^2.
\end{aligned} \tag{17}$$

Further, putting (17) back into (16), one has

$$\begin{aligned}
&\left|b\left(\frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1}}{2},\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2},\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right| \leq \frac{\nu}{16}\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\|^2 + C\nu^{-1}\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|^4 \\
&\quad + C\nu^{-1}\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|^2 + Ch^{-\frac{3}{2}}\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|(\|\boldsymbol{\phi}_h^{n-1}\|^2+\|\boldsymbol{\phi}_h^{n+1}\|^2).
\end{aligned} \tag{18}$$

For the second trilinear term, using Lemma 2.1 and the regularity assumptions on $\mathbf{u}$, we invoke Young's inequality, resulting in

$$\begin{aligned}
&\left|b\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2},\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2},\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right| \\
&\leq C\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\| \leq \frac{\nu}{16}\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\|^2 + C\nu^{-1}\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|^2.
\end{aligned} \tag{19}$$

Similarly, the third trilinear term on the RHS of (14) is bounded, i.e.,

$$\begin{aligned}
&\left|b\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2},\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2},\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right| \\
&\leq C\left\|\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right\|\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\|\left(\left\|\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2}\right\|_{L^\infty(\Omega)}+\left\|\nabla\left(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2}\right)\right\|_{L^\infty(\Omega)}\right) \\
&\leq C\left\|\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right\|\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\| \leq \frac{\nu}{16}\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\|^2 + C\nu^{-1}(\|\boldsymbol{\phi}_h^{n+1}\|^2+\|\boldsymbol{\phi}_h^{n-1}\|^2).
\end{aligned} \tag{20}$$

Now, in light of (15) and (18)-(20), the error equation (14) can be rewritten as

$$\begin{aligned}
&\frac{1}{4\tau}\left(\|\boldsymbol{\phi}_h^{n+1}\|^2-\|\boldsymbol{\phi}_h^{n-1}\|^2\right)+\frac{\kappa}{4\tau}\left(\|\nabla\boldsymbol{\phi}_h^{n+1}\|^2-\|\nabla\boldsymbol{\phi}_h^{n-1}\|^2\right)+\frac{9\nu}{16}\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\|^2 \\
&\leq C\nu^{-1}\left\|\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^{n-1}}{2\tau}\right\|^2 + C\nu^{-1}\kappa^2\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^{n-1}}{2\tau}\right)\right\|^2 + C\nu^{-1}\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|^2 \\
&\quad + C\nu^{-1}\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|^4 + C\nu^{-1}(\|\boldsymbol{\phi}_h^{n}\|^2+\|\boldsymbol{\phi}_h^{n-1}\|^2) + \left|R\left(\mathbf{u},p;\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right| \\
&\quad + Ch^{-\frac{3}{2}}\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|(\|\boldsymbol{\phi}_h^{n+1}\|^2+\|\boldsymbol{\phi}_h^{n-1}\|^2).
\end{aligned} \tag{21}$$

Thus, what is left is to bound $R(\mathbf{u},p;\frac{1}{2}(\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}))$. Each of its four linear terms can be bounded by the Cauchy-Schwarz, Young's and Poincaré inequalities, together with the estimates in Lemma 2.4, i.e.,

$$\left|\left(\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_{n-1})}{2\tau}-\mathbf{u}_t(t_n),\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right| \leq \frac{\nu}{16}\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{n+1}+\boldsymbol{\phi}_h^{n-1}}{2}\right)\right\|^2 + C\nu^{-1}\tau^4\|\mathbf{u}_{ttt}(t_{n+\theta})\|^2, \tag{22}$$

$$\left|\kappa a\left(\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_{n-1})}{2\tau}-\mathbf{u}_t(t_n),\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right|\le\frac{\nu}{16}\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|^2+C\nu^{-1}\kappa^2\tau^4\|\nabla\mathbf{u}_{ttt}(t_{n+\theta})\|^2, \quad (23)$$

$$\left|\nu a\left(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2}-\mathbf{u}(t_n),\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right|\le\frac{\nu}{16}\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|^2+C\nu\tau^4\|\nabla\mathbf{u}_{tt}(t_{n+\theta})\|^2, \quad (24)$$

and

$$\left|\left(\frac{p(t_{n+1})+p(t_{n-1})}{2}-p(t_n),\nabla\cdot\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right)\right|\le\frac{\nu}{16}\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|^2+C\nu^{-1}\tau^4\|p_{tt}(t_{n+\theta})\|^2. \quad (25)$$

Then, to estimate two nonlinear terms in (12), we use Lemma 2.1, 2.4 and Young's inequality, and obtain

$$\begin{aligned}
&\left|b\left(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2}-\mathbf{u}(t_n),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2},\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right|\\
&\quad+\left|b\left(\mathbf{u}(t_n),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2}-\mathbf{u}(t_n),\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right|\\
&\le C\left\|\nabla\left(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2}\right)\right\|\left\|\nabla\left(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2}-\mathbf{u}(t_n)\right)\right\|\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\| \quad (26)\\
&\quad+C\|\nabla\mathbf{u}(t_n)\|\left\|\nabla\left(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2}-\mathbf{u}(t_n)\right)\right\|\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|\\
&\le C\tau^2\|\nabla\mathbf{u}_{tt}(t_{n+\theta})\|\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|\le\frac{\nu}{16}\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|^2+C\nu^{-1}\tau^4\|\nabla\mathbf{u}_{tt}(t_{n+\theta})\|^2.
\end{aligned}$$

Consequently, making use of (22)-(26), we arrive at

$$\begin{aligned}
\left|R\left(\mathbf{u},p;\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right|\le\frac{5\nu}{16}\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|^2&+C\nu^{-1}\tau^4\|\mathbf{u}_{ttt}(t_{n+\theta})\|^2\\
&+C\nu^{-1}\kappa^2\tau^4\|\nabla\mathbf{u}_{ttt}(t_{n+\theta})\|^2+C\nu\tau^4\|\nabla\mathbf{u}_{tt}(t_{n+\theta})\|^2\\
&+C\nu^{-1}\tau^4\|\nabla\mathbf{u}_{tt}(t_{n+\theta})\|^2+C\nu^{-1}\tau^4\|p_{tt}(t_{n+\theta})\|^2.
\end{aligned}$$

So that error equation (21) becomes

$$\begin{aligned}
&\frac{1}{4\tau}\left(\|\phi_h^{n+1}\|^2-\|\phi_h^{n-1}\|^2\right)+\frac{\kappa}{4\tau}\left(\|\nabla\phi_h^{n+1}\|^2-\|\nabla\phi_h^{n-1}\|^2\right)+\frac{\nu}{4}\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|^2\\
&\le C\nu^{-1}\left\|\frac{\eta^{n+1}-\eta^{n-1}}{2\tau}\right\|^2+C\nu^{-1}\kappa^2\left\|\nabla\left(\frac{\eta^{n+1}-\eta^{n-1}}{2\tau}\right)\right\|^2+C\nu^{-1}\left\|\nabla\left(\frac{\eta^{n+1}+\eta^{n-1}}{2}\right)\right\|^2\\
&\quad+C\nu^{-1}\left\|\nabla\left(\frac{\eta^{n+1}+\eta^{n-1}}{2}\right)\right\|^4+C\nu^{-1}(\|\phi_h^{n+1}\|^2+\|\phi_h^{n-1}\|^2)+Ch^{-\frac{3}{2}}(\|\phi_h^{n+1}\|^2+\|\phi_h^{n-1}\|^2)\left\|\nabla\left(\frac{\eta^{n+1}+\eta^{n-1}}{2}\right)\right\| \quad (27)\\
&\quad+C\nu^{-1}\tau^4\|\mathbf{u}_{ttt}(t_{n+\theta})\|^2+C\nu^{-1}\kappa^2\tau^4\|\nabla\mathbf{u}_{ttt}(t_{n+\theta})\|^2+C\nu\tau^4\|\nabla\mathbf{u}_{tt}(t_{n+\theta})\|^2\\
&\quad+C\nu^{-1}\tau^4\|\nabla\mathbf{u}_{tt}(t_{n+\theta})\|^2+C\nu^{-1}\tau^4\|p_{tt}(t_{n+\theta})\|^2.
\end{aligned}$$

By help of (3), one gets

$$C\nu^{-1}\left\|\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^{n-1}}{2\tau}\right\|^2 = C\nu^{-1}\left\|\frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}}\frac{\partial\boldsymbol{\eta}}{\partial t}(s)ds\right\|^2 \leq C\nu^{-1}\left(\frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}}1\cdot\|\frac{\partial\boldsymbol{\eta}}{\partial t}(s)\|ds\right)^2$$
$$\leq C\nu^{-1}\left(\frac{\int_{t_{n-1}}^{t_{n+1}}1^2ds}{2\tau}\right)\left(\frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}}\|\frac{\partial\boldsymbol{\eta}}{\partial t}(s)\|^2ds\right) \tag{28}$$
$$\leq C\nu^{-1}h^{2m+2}\frac{1}{2\tau}\|\mathbf{u}_t\|^2_{L^2(t_{n-1},t_{n+1};H^{m+1}(\Omega)^2)} \leq C\nu^{-1}h^{2m+2}\frac{1}{\tau}\|\mathbf{u}_t\|^2_{L^2(0,T;H^{m+1}(\Omega)^2)}.$$

Multiplying both sides of (27) by $4\tau$, using the bounds (28) and summing over the time levels from 1 to $n$, and choosing $\mathbf{U}^0 = \mathbf{u}_h^0$, one arrives at

$$\|\boldsymbol{\phi}_h^{n+1}\|^2 + \kappa\|\nabla\boldsymbol{\phi}_h^{n+1}\|^2 + \tau\sum_{i=1}^{n}\nu\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{i+1}+\boldsymbol{\phi}_h^{i-1}}{2}\right)\right\|^2$$
$$\leq \|\boldsymbol{\phi}_h^1\|^2 + \kappa\|\nabla\boldsymbol{\phi}_h^1\|^2 + C\nu^{-1}h^{2m+2}\|\mathbf{u}_t\|^2_{L^2(0,T;H^{m+1}(\Omega)^2)} + C\nu^{-1}\kappa^2h^{2m}\|\mathbf{u}_t\|^2_{L^2(0,T;H^{m+1}(\Omega)^2)}$$
$$+ C\nu^{-1}h^{2m}\|\mathbf{u}\|^2_{L^2(0,T;H^{m+1}(\Omega)^2)} + C\nu^{-1}h^{4m}\|\mathbf{u}\|^4_{L^2(0,T;H^{m+1}(\Omega)^2)} + C\nu^{-1}\tau^4\|\mathbf{u}_{ttt}\|^2_{L^2(0,T;L^2(\Omega)^2)}$$
$$+ C\nu^{-1}\kappa^2\tau^4\|\nabla\mathbf{u}_{ttt}\|^2_{L^2(0,T;L^2(\Omega)^2)} + C\nu\tau^4\|\nabla\mathbf{u}_{tt}\|^2_{L^2(0,T;L^2(\Omega)^2)} + C\nu^{-1}\tau^4\|\nabla\mathbf{u}_{tt}\|^2_{L^2(0,T;L^2(\Omega)^2)}$$
$$+ C\nu^{-1}\tau^4\|p_{tt}\|^2_{L^2(0,T;L^2(\Omega))} + C\nu^{-1}\tau\sum_{i=1}^{n}(\|\boldsymbol{\phi}_h^{i+1}\|^2 + \|\boldsymbol{\phi}_h^{i-1}\|^2) + Ch^{m-\frac{3}{2}}\tau\sum_{i=1}^{n}\|\mathbf{u}\|_{m+1}(\|\boldsymbol{\phi}_h^{i+1}\|^2 + \|\boldsymbol{\phi}_h^{i-1}\|^2).$$

Since $\mathbf{u} \in L^\infty(0,T;H^{m+1}(\Omega)^2)$, we can combine the last two sums as $C\left(h^{m-\frac{3}{2}}\tau + \nu^{-1}\tau\right)\sum_{i=1}^{n}\|\boldsymbol{\phi}_h^{i+1}\|^2$. The error equation finally takes the form

$$\|\boldsymbol{\phi}_h^{n+1}\|^2 + \kappa\|\nabla\boldsymbol{\phi}_h^{n+1}\|^2 + \tau\sum_{i=1}^{n}\nu\left\|\nabla\left(\frac{\boldsymbol{\phi}_h^{i+1}+\boldsymbol{\phi}_h^{i-1}}{2}\right)\right\|^2 \tag{29}$$
$$\leq \|\boldsymbol{\phi}_h^1\|^2 + \kappa\|\nabla\boldsymbol{\phi}_h^1\|^2 + C\nu^{-1}(1+\kappa^2+h^2+h^{2m})h^{2m} + C(\nu^{-1}+\nu^{-1}\kappa^2+\nu)\tau^4 + C(h^{m-\frac{3}{2}}+\nu^{-1})\tau\sum_{i=1}^{n}\|\boldsymbol{\phi}_h^{i+1}\|^2.$$

Subtract (5) from (8) at $t = t_{\frac{1}{2}}$, we have

$$b\left(\mathbf{u}(t_{\frac{1}{2}}),\mathbf{u}(t_{\frac{1}{2}}),\mathbf{v}_h\right) - b\left(\frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2},\frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2},\mathbf{v}_h\right)$$
$$= b\left(\mathbf{u}(t_{\frac{1}{2}}),\mathbf{u}(t_{\frac{1}{2}}) - \frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\mathbf{v}_h\right) + b\left(\frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2},\frac{\mathbf{e}^1+\mathbf{e}^0}{2},\mathbf{v}_h\right) \tag{30}$$
$$+ b\left(\frac{\mathbf{e}^1+\mathbf{e}^0}{2},\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\mathbf{v}_h\right) + b\left(\mathbf{u}(t_{\frac{1}{2}}) - \frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\mathbf{v}_h\right).$$

Taking $\mathbf{v}_h = \frac{1}{2}(\boldsymbol{\phi}_h^1 + \boldsymbol{\phi}_h^0)$, the second and third terms in (30) can be treated exactly as in (18), (19) and (20). The first and last are similar, since, after application of Lemma 2.1 and regularity assumptions on $\mathbf{u}$, both can be bounded as

$$C\left\|\nabla\left(\mathbf{u}(t_{\frac{1}{2}}) - \frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2}\right)\right\|\|\nabla\boldsymbol{\phi}_h^1\| \leq \frac{\nu}{2}\|\nabla\boldsymbol{\phi}_h^1\|^2 + C\nu^{-1}\tau^4,$$

with the help of Lemma 2.4 and Young's inequality. This leads to the upper bound:

$$\|\boldsymbol{\phi}_h^1\|^2 + \kappa\|\nabla\boldsymbol{\phi}_h^1\|^2 + \nu\tau\|\nabla\boldsymbol{\phi}_h^1\|^2 \tag{31}$$
$$\leq C(\nu^{-1}\tau h^{2m} + \nu^{-1}\tau\kappa^2h^{2m} + \nu\tau h^{2m} + \nu^{-1}\tau h^{2m+2} + \nu^{-1}\tau h^{4m} + \nu^{-1}\tau^5 + \nu^{-1}\kappa^2\tau^5 + \nu\tau^5).$$

Inserting above estimate into (29) yields

$$\begin{aligned}&\|\phi_h^{n+1}\|^2+\kappa\|\nabla\phi_h^{n+1}\|^2+\tau\sum_{i=1}^{n}\nu\left\|\nabla\left(\frac{\phi_h^{i+1}+\phi_h^{i-1}}{2}\right)\right\|^2\\ &\le C(\nu^{-1}+\nu+\nu^{-1}\kappa^2)(h^{2m}+\tau^4)+C(h^{m-\frac{3}{2}}+\nu^{-1})\tau\sum_{i=1}^{n}\|\phi_h^{i+1}\|^2.\end{aligned}$$

Hence, arguing as in discrete Gronwall Lemma, there exists $C$ such that for any $n$

$$\|\phi_h^{n+1}\|^2+\kappa\|\nabla\phi_h^{n+1}\|^2+\tau\sum_{i=1}^{n}\nu\left\|\nabla\left(\frac{\phi_h^{i+1}+\phi_h^{i-1}}{2}\right)\right\|^2\le C(h^{2m}+\tau^4). \tag{32}$$

If we apply triangle inequality, then we arrive at (7) and complete the proof. □

We now estimate the error of pressure in $L^2(0,T;L^2(\Omega))$. This hinges on the error estimate for the time difference of the velocity error.

**Lemma 3.2.** *Under assumptions of Theorem 3.1, there exists a constant C such that*

$$\begin{aligned}&\nu\tau^2\left\|\nabla\left(\frac{\mathbf{e}^{n+1}-\mathbf{e}^{n-1}}{2\tau}\right)\right\|^2+\nu\left\|\nabla\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}\right)\right\|^2+\tau\sum_{i=1}^{n}\left\|\frac{\mathbf{e}^{i+1}-\mathbf{e}^{i-1}}{2\tau}\right\|^2\\ &+\kappa\tau\sum_{i=1}^{n}\left\|\nabla\left(\frac{\mathbf{e}^{i+1}-\mathbf{e}^{i-1}}{2\tau}\right)\right\|^2\le C(h^{2m}+\tau^4).\end{aligned}$$

*Proof.* Invoke the error decomposition (10). We begin this proof by setting $\mathbf{v}_h=\frac{1}{2\tau}(\phi_h^{n+1}-\phi_h^{n-1})\in\mathbf{V}^h$ in (11) and using (13) to obtain

$$\begin{aligned}&\left\|\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right\|^2+\kappa\left\|\nabla\left(\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right\|^2+\nu\frac{\|\nabla\phi_h^{n+1}\|^2-\|\nabla\phi_h^{n-1}\|^2}{4\tau}\\ &=\left(\frac{\eta^{n+1}-\eta^{n-1}}{2\tau},\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)+\kappa a\left(\frac{\eta^{n+1}-\eta^{n-1}}{2\tau},\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\\ &\quad+b\left(\frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1}}{2},\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2},\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)-R\left(\mathbf{u},p;\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\\ &\quad+b\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2},\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2},\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right).\end{aligned} \tag{33}$$

By virtue of the Lemma 2.4, we get

$$\begin{aligned}R\left(\mathbf{u},p;\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)&\le C\tau^2\left(\mathbf{u}_{ttt}(t_{n+\theta}),\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)+C\kappa\tau^2 a\left(\mathbf{u}_{ttt}(t_{n+\theta}),\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\\ &\quad+C\nu\tau^2 a\left(\mathbf{u}_{tt}(t_{n+\theta}),\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)+C\tau^2 b\left(\mathbf{u}(t_n),\mathbf{u}_{tt}(t_{n+\theta}),\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\\ &\quad+C\tau^2 b\left(\mathbf{u}_{tt}(t_{n+\theta}),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2},\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)+C\tau^2\left(p_{tt}(t_{n+\theta}),\nabla\cdot\left(\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right).\end{aligned}$$

Adding and subtracting $b\left(\frac{1}{2}(\mathbf{e}^{n+1}+\mathbf{e}^{n-1}), \frac{1}{2}(\mathbf{e}^{n+1}+\mathbf{e}^{n-1}), \frac{1}{2\tau}(\phi_h^{n+1}-\phi_h^{n-1})\right)$ to the first nonlinear term in (33), integrating by parts and applying Hölder's inequality yields

$$\begin{aligned}\left|b\left(\frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1}}{2}, \frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}, \frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right| &\le \left|b\left(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2}, \frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}, \frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right| \\ &+ \left|b\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}, \frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}, \frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right| + \left|b\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}, \frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}, \frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right|.\end{aligned} \tag{34}$$

Now, we bound three nonlinear terms on the RHS of (34) separately. For the first term, by the help of Lemma 2.1, Cauchy-Schwarz and Young's inequalities yields

$$\begin{aligned}&\left|b\left(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2}, \frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}, \frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right| \\ &\le C\left\|\nabla\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}\right)\right\|\left\|\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right\| \le \frac{1}{4}\left\|\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right\|^2 + C\left\|\nabla\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}\right)\right\|^2.\end{aligned}$$

We next bound the second and third nonlinear terms using Lemma 2.1 and Young's inequality.

$$\begin{aligned}&\left|b\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}, \frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}, \frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right| \\ &\le C\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|\left\|\nabla\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}\right)\right\|\left\|\nabla\left(\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right\| \\ &\le \frac{3\kappa}{16}\left\|\nabla\left(\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right\|^2 + Ch^{2m}\left\|\nabla\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}\right)\right\|^2.\end{aligned}$$

Similarly, by using (32), we arrive at

$$\begin{aligned}&\left|b\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}, \frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}, \frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right| \\ &\le C\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|\left\|\nabla\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}\right)\right\|\left\|\nabla\left(\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right\| \\ &\le \frac{3\kappa}{16}\left\|\nabla\left(\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right\|^2 + C(h^{2m}+\tau^4)\left\|\nabla\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}\right)\right\|^2.\end{aligned}$$

Finally, we consider the second nonlinear term in (33) . Using the regularity assumption of $\mathbf{u}$, we get

$$\begin{aligned}&\left|b\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}, \frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2}, \frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right| \\ &\le \left|b\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}, \frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2}, \frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right| \\ &\quad + \left|b\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}, \frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_{n-1})}{2}, \frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right| \\ &\le C\left\|\nabla\left(\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n-1}}{2}\right)\right\|\left\|\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right\| + C\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|\left\|\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right\| \\ &\le \frac{1}{4}\left\|\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right\|^2 + C\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|^2 + Ch^{2m}.\end{aligned}$$

Insert above estimate into (33). Then, the bound on $R(\mathbf{u}, p; \frac{1}{2\tau}(\phi_h^{n+1} - \phi_h^{n-1})$ is obtained by the same technique in the proof of Theorem 3.1.

$$\begin{aligned} &\frac{1}{4}\left\|\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right\|^2 + \frac{\kappa}{4}\left\|\left(\nabla\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right\|^2 + \nu\frac{\|\nabla\phi_h^{n+1}\|^2 - \|\nabla\phi_h^{n-1}\|^2}{4\tau} \\ &\leq C\left\|\nabla\left(\frac{\mathbf{e}^n+\mathbf{e}^{n-1}}{2}\right)\right\|^2 + C\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|^2 + C(h^{2m}+\tau^4) + C(h^{2m}+\tau^4)\left\|\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}\right\|^2. \end{aligned} \tag{35}$$

At the $t = t_{\frac{1}{2}}$ time level, choosing $\mathbf{v}_h = \frac{1}{\tau}(\phi_h^1 - \phi_h^0)$ and $\mathbf{U}^0 = \mathbf{u}_h^0$ in the initial error decomposition gives $\phi_h^0$, we obtain

$$\begin{aligned} &\frac{1}{2}\left\|\frac{\phi_h^1-\phi_h^0}{\tau}\right\|^2 + \frac{\kappa}{2}\left\|\nabla\left(\frac{\phi_h^1-\phi_h^0}{\tau}\right)\right\|^2 + \nu\frac{\|\nabla\phi_h^1\|^2 - \|\nabla\phi_h^0\|^2}{2\tau} \\ &\leq C(h^{2m}+\tau^4) + \tau^2\left|b\left(\mathbf{u}_{tt}(t_\theta), \frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2}, \frac{\phi_h^1-\phi_h^0}{\tau}\right)\right|. \end{aligned} \tag{36}$$

Sum (35) over the time levels $n \geq 1$ and add to (36). Multiply by $4\tau$, we have

$$\begin{aligned} &\tau\sum_{i=1}^n\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{2\tau}\right\|^2 + \kappa\tau\sum_{i=1}^n\left\|\nabla\left(\frac{\phi_h^{i+1}-\phi_h^{i-1}}{2\tau}\right)\right\|^2 + \nu\|\nabla\phi_h^{n+1}\|^2 \\ &\leq C\tau\sum_{i=1}^n\left\|\nabla\left(\frac{\phi_h^{i+1}+\phi_h^{i-1}}{2}\right)\right\|^2 + C(h^{2m}+\tau^4) + C\tau^3\left|b\left(\mathbf{u}_{tt}(t_\theta), \frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2}, \frac{\phi_h^1-\phi_h^0}{\tau}\right)\right|. \end{aligned} \tag{37}$$

For any $n \geq 1$, add the inequality (37) to itself at the $n-1$ time level. Using the identity

$$\|\nabla\phi_h^{n+1}\|^2 + \|\nabla\phi_h^{n-1}\|^2 = \frac{1}{2}\tau^2\left\|\nabla\left(\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right\|^2 + 2\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|^2,$$

we obtain

$$\begin{aligned} &\frac{1}{2}\nu\tau^2\left\|\nabla\left(\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right\|^2 + 2\nu\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|^2 + \tau\sum_{i=1}^n\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{2\tau}\right\|^2 + \kappa\tau\sum_{i=1}^n\left\|\nabla\left(\frac{\phi_h^{i+1}-\phi_h^{i-1}}{2\tau}\right)\right\|^2 \\ &\leq C\tau\sum_{i=1}^n 2\nu\left\|\nabla\left(\frac{\phi_h^{i+1}+\phi_h^{i-1}}{2}\right)\right\|^2 + C\tau^3\left|b\left(\mathbf{u}_{tt}(t_\theta), \frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2}, \frac{\phi_h^1-\phi_h^0}{\tau}\right)\right| + C(h^{2m}+\tau^4). \end{aligned} \tag{38}$$

Using Lemma 2.1 and Young's inequality yields

$$2\tau^3\left|b\left(\mathbf{u}_{tt}(t_\theta), \frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2}, \frac{\phi_h^1-\phi_h^0}{\tau}\right)\right| \leq \frac{\tau}{2}\left\|\frac{\phi_h^1-\phi_h^0}{\tau}\right\|^2 + C\tau^5.$$

By the help of discrete Gronwall Lemma and (31), we arrive that

$$\nu\tau^2\left\|\nabla\left(\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\tau}\right)\right\|^2 + \nu\left\|\nabla\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}\right)\right\|^2 + \tau\sum_{i=1}^n\left\|\frac{\phi_h^{i+1}-\phi_h^{i-1}}{2\tau}\right\|^2 + \kappa\tau\sum_{i=1}^n\left\|\nabla\left(\frac{\phi_h^{i+1}-\phi_h^{i-1}}{2\tau}\right)\right\|^2 \leq C(h^{2m}+\tau^4).$$

The proof of the lemma is now concluded by the triangle inequality. □

We finish this section by deriving error estimate of the pressure.

**Theorem 3.3.** *Under assumptions of Theorem 3.1, there exists a constant C such that*

$$\tau\sum_{i=1}^{n}\left\|\frac{p(t_{i+1})-p_h^{i+1}+(p(t_{i-1})-p_h^{i-1})}{2}\right\|\leq C(h^m+\tau^2).$$

*Proof.* Considering (11) and applying the discrete $LBB^h$ condition to obtain for any $n\geq 1$,

$$\begin{aligned}&\beta^h\left\|\frac{(p(t_{n+1})-p_h^{n+1})+(p(t_{n-1})-p_h^{n-1})}{2}\right\|\\&\leq C\left\|\frac{\mathbf{e}^{n+1}-\mathbf{e}^{n-1}}{2\tau}\right\|+\kappa\left\|\nabla\left(\frac{\mathbf{e}^{n+1}-\mathbf{e}^{n-1}}{2\tau}\right)\right\|+\nu\left\|\nabla\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}\right)\right\|\\&\quad+C\left\|\nabla\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}\right)\right\|+C\left\|\nabla\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}\right)\right\|^2+C\tau^2+C\nu\tau^2+C\kappa\tau^2\\&\leq\left\|\frac{\mathbf{e}^{n+1}-\mathbf{e}^{n-1}}{2\tau}\right\|+\kappa\left\|\nabla\left(\frac{\mathbf{e}^{n+1}-\mathbf{e}^{n-1}}{2\tau}\right)\right\|+C(h^m+\tau^2)\end{aligned}\tag{39}$$

Sum (39) over the time levels $n\geq 1$ and multiply by $\tau$ obtain

$$\begin{aligned}&\beta^h\tau\sum_{i=1}^{n}\left\|\frac{(p(t_{i+1})-p_h^{i+1})+(p(t_{i-1})-p_h^{i-1})}{2}\right\|\\&\leq C\tau\sum_{i=1}^{n}\left\|\frac{\mathbf{e}^{i+1}-\mathbf{e}^{i-1}}{2\tau}\right\|+\kappa\tau\sum_{i=1}^{n}\left\|\nabla\left(\frac{\mathbf{e}^{i+1}-\mathbf{e}^{i-1}}{2\tau}\right)\right\|+C(h^m+\tau^2).\end{aligned}\tag{40}$$

Using the discrete $LBB^h$ condition and (31) on the first time level, we derive the following bound:

$$\beta^h\tau\left\|\frac{(p(t_1)-p_h^1)+(p(t_0)-p_h^0)}{2}\right\|\leq C(h^m+\tau^2).\tag{41}$$

Add the inequality (41) to (40). The proof is concluded by applying the triangle inequality and the result of Lemma 3.2. □

## 4. Numerical experiments

In this section, we present numerical experiments to check the numerical theory developed in the previous sections and illustrate the efficiency of the presented algorithm. Firstly, using the Green-Taylor vortex problem, we confirm the predicted convergence rates. Further, test is then performed using the flow around a cylinder benchmark problem. In all experiments, the model is discretized in space using the finite element approximation with Taylor-Hood element ($P2-P1$) pair. Denote errors by

$$Err(u)=\left(\tau\sum_{k=1}^{n}\|\nabla(u(t_{k+1})-u_h^{k+1})\|\right)^{1/2},\quad Err(p)=\left(\tau\sum_{k=1}^{n}\|p(t_{k+1})-p_h^{k+1}\|\right)^{1/2}.$$

### 4.1. Green-Taylor vortex problem

Our first example is designed to test the predicted rates of convergence. The problem of simulating decay of the Green-Taylor vortex [9, 30], is an interesting test problem in which the true solution is known.

It has been used as a numerical test by Chorin [6], Tafti [29] and John and Layton [13]. The prescribed solution in $\Omega = [0,1] \times [0,1]$ has the form

$$u_1 = 10x^2(x-1)^2 y(y-1)(2y-1)\cos(t),$$
$$u_2 = -10x(x-1)(2x-1)y^2(y-1)^2\cos(t),$$
$$p = 10(2x-1)(2y-1)\cos(t).$$

Here, we choose $\nu = 1$, $\kappa = 0.01$ and $T = 1$. The results for the Kelvin-Voigt model are presented in Table 1. From the table, it can be easily to see that the presented method works well and keeps the convergence rates just like the theoretical analysis in the previous sections.

**Table 1:** Error and convergence rates for the considered scheme with $\tau = \mathbb{O}(h)$

| $h$ | $Err(u)$ | Rate | $Err(p)$ | Rate |
|---|---|---|---|---|
| 1/8 | 0.00430 | — | 0.01743 | — |
| 1/16 | 0.00107 | 2.007 | 0.00424 | 2.039 |
| 1/32 | 0.00026 | 2.029 | 0.00104 | 2.029 |

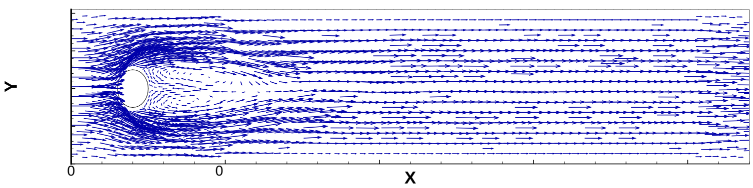
### 4.2. *Flow around a cylinder*

Our next numerical illustration is for the two-dimensional underresolved flow around a cylinder which is a well known benchmark problem [28]. A detailed numerical study of this problem for the Navier-Stokes equations is done by John [14]. This fluid flow problem is not turbulent but does have some interesting features. The flow patterns are driven by interaction of a fluid with a wall.

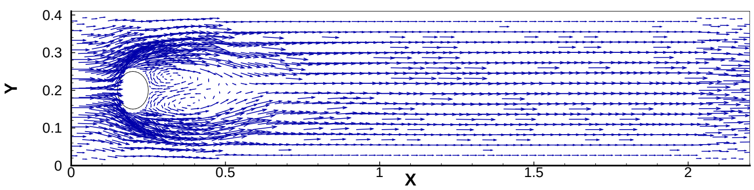
The time dependent inflow and outflow profile are

$$u_1(0,y,t) = u_1(2.2,y,t) = \frac{6}{0.41^2}\sin(\frac{\pi t}{8}y(0.41-y)), \quad u_2(0,y,t) = u_2(2.2,y,t) = 0.$$

No slip boundary conditions are prescribed along the top and bottom walls and the initial condition is $\mathbf{u}(x,y,0) = 0$. The viscosity is $\nu = 10^{-3}$ and the external force $\mathbf{f} = 0$.




(a)




(b)

Figure 1: (a) the results of the Navier-Stokes model ($\kappa = 0$) at $T = 4$ and (b) the results of the Kelvin-Voigt model ($\kappa = 0.01$) at $T = 5$.

The flows resulting from the Navier-Stokes model ($\kappa = 0$) at $T = 4$ and the Kelvin-Voigt model ($\kappa = 0.01$) at $T = 5$ are shown in Figure 1. From this figure, we can see that the Kelvin-Voigt model takes longer time than the Navier-Stokes model does to form two vortices behind the cylinder, which shows that the Kelvin-Voigt model approximations take longer to reach the steady state. Above numerical result is consistent with the phenomenon in [1, 19] very well. Hence, the Kelvin-Voigt model pays more attentions to the relaxation property of the fluid.

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