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Wave Front Sets with Respect to Banach Spaces of Ultradistributions. Characterisation via the Short-Time Fourier Transform

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Abstract. We define ultradistributional wave front sets with respect to translation-modulation invariant Banach spaces of ultradistributions having solid Fourier image. The main result is their characterisation by the short-time Fourier transform.

1. Introduction

Hörmander [15] introduced the Sobolev wave front set of a distribution f as the set of points x and directions ξ at which f does not behave as an element of the Sobolev space H^s; i.e. it is not H^s micro-regular at (x, ξ). It is one of the most powerfull tools in studying the regularity of solutions of PDEs with wide range of applications in mathematical physics. Many authors considered various generalisations and characterisations of the Sobolev wave front set and other similar variants; see [18, 22–24, 26, 27] and the references there in. This concept was further generalised recently in [5, 6] where the wave front set is defined with respect to a general Banach spaces of distributions satisfying appropriate assumptions. In the setting of ultradistributions, the wave front set with respect to Fourier-Lebesgue spaces having sub-exponential weights was considered in [7, 16] where the authors also gave a discrete characterisation of it.

The goal of this article is to define the wave front set in the setting of non quasi-analytic ultradistributions with respect to a Banach space of ultradistributions satisfying appropriate assumptions; this generalisation is in the spirit of [5], where the distributional case was considered. The main result of the article (Theorem 3.10) is its characterisation by the short-time Fourier transform (cf. [21] for a similar characterisation of the Sobolev wave front set in the distributional setting).

2. Preliminaries

Let M_p, $p \in \mathbb{N}$, be a sequence of positive numbers satisfying $M_0 = M_1 = 1$ for which the following conditions hold true:

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$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$, $p \in \mathbb{Z}_+$;
$(M.2)$ there exist $c_0, H \geq 1$ such that $M_p \leq c_0 H^p \min_{0\leq q\leq p} M_q M_{p-q}$, $p \in \mathbb{N}$;
$(M.3)$ there exists $c_0 \geq 1$ such that $\sum_{q=p+1}^{\infty} M_{q-1}/M_q \leq c_0 p M_p/M_{p+1}$, $p \in \mathbb{Z}_+$;
$(M.4)$ $M_p^2/p!^2 \leq (M_{p-1}/(p-1)!) \cdot (M_{p+1}/(p+1)!)$.

The sequence $M_p = p!^s$, $s > 1$, satisfies all of the above conditions. When $\alpha \in \mathbb{N}^d$, we set $M_\alpha = M_{|\alpha|}$. The associated function to the sequence M_p is defined by $M(\lambda) = \sup_{p\in\mathbb{N}} \ln_+(\lambda^p/M_p)$, $\lambda > 0$ (see [17]). It is continuous, non-negative, monotonically increasing function, it vanishes for sufficiently small $\lambda > 0$ and increases more rapidly than $\ln \lambda^p$ as λ tends to infinity for any $p \in \mathbb{N}$.

Given an open set $U \subseteq \mathbb{R}^d$, we refer to Komatsu [17] for the definition and the basic properties of the locally convex spaces (from now on abbreviated as l.c.s.) $\mathcal{E}^{(M_p)}(U)$ and $\mathcal{E}^{\{M_p\}}(U)$ of ultradifferentiable functions of Beurling and Roumieu type respectively, as well as the corresponding spaces $\mathcal{D}^{(M_p)}(U)$ and $\mathcal{D}^{\{M_p\}}(U)$ of ultradifferentiable functions having compact support in U. Their strong duals are the corresponding spaces of ultradistributions of Beurling and Roumieu type. We also denote by $\mathcal{D}_K^{(M_p)}$ and $\mathcal{D}_K^{\{M_p\}}$ the spaces consisting of all elements of $\mathcal{E}^{(M_p)}(U)$ and $\mathcal{E}^{\{M_p\}}(U)$ respectively, supported by the compact set $K \subset U$ (cf. [17]). The common notation for the symbols (M_p) and $\{M_p\}$ will be $*$. If $\varphi \in \mathcal{E}^*(U)$ never vanishes than $1/\varphi$ also belongs to $\mathcal{E}^*(U)$. More precisely, we have the following result.

Lemma 2.1. *Let $\varphi \in \mathcal{E}^*(U)$ and let $V \subseteq U$ be the open set where $\varphi \neq 0$. The function $x \mapsto 1/\varphi(x)$ belongs to $\mathcal{E}^*(V)$.*

Proof. The proof relies on the multidimensional Faá di Bruno formula [3, Corollary 2.10] applied to the composition of the functions $\lambda \mapsto 1/\lambda$ and φ and the condition $(M.4)$ on M_p; it is similar to the proof of [20, Lemma 7.5] and we omit it (see [1, Theorem 4.1] for the one dimensional Beurling case and [28, Theorem 3] for the one dimensional Roumieu case). □

The entire function $P(z) = \sum_{\alpha\in\mathbb{N}^d} c_\alpha z^\alpha$, $z \in \mathbb{C}^d$, is an ultrapolynomial of class (M_p) (resp. of class $\{M_p\}$), whenever the coefficients c_α satisfy the estimate $|c_\alpha| \leq CL^{|\alpha|}/M_\alpha$, $\alpha \in \mathbb{N}^d$, for some $C, L > 0$ (resp. for every $L > 0$ and some $C = C(L) > 0$). The corresponding operator $P(D) = \sum_\alpha c_\alpha D^\alpha$ is called an ultradifferential operator of class (M_p) (resp. of class $\{M_p\}$) and it acts continuously on $\mathcal{E}^{(M_p)}(U)$ and $\mathcal{D}^{(M_p)}(U)$ (resp. on $\mathcal{E}^{\{M_p\}}(U)$ and $\mathcal{D}^{\{M_p\}}(U)$) and the corresponding spaces of ultradistributions.

The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is given by $\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} f(x)dx$, $\xi \in \mathbb{R}^d$.

For $m > 0$, we denote by $\mathcal{S}^{M_p,m}(\mathbb{R}^d)$ the (B)-space of all $\varphi \in C^\infty(\mathbb{R}^d)$ for which the norm $\|\varphi\|_m = \sup_{\alpha\in\mathbb{N}^d} m^{|\alpha|} \|e^{M(m|\cdot|)} D^\alpha \varphi\|_{L^\infty(\mathbb{R}^d)}/M_\alpha$ is finite. The spaces of sub-exponentially decreasing ultradifferentiable functions of Beurling and Roumieu type are defined by

$$\mathcal{S}^{(M_p)}(\mathbb{R}^d) = \lim_{\substack{\longleftarrow\\ m\to\infty}} \mathcal{S}^{M_p,m}(\mathbb{R}^d) \text{ and } \mathcal{S}^{\{M_p\}}(\mathbb{R}^d) = \lim_{\substack{\longrightarrow\\ m\to 0}} \mathcal{S}^{M_p,m}(\mathbb{R}^d),$$

respectively and their strong duals $\mathcal{S}'^{(M_p)}(\mathbb{R}^d)$ and $\mathcal{S}'^{\{M_p\}}(\mathbb{R}^d)$ are the spaces of tempered ultradistributions of Beurling and Roumieu type, respectively. When $M_p = p!^s$, $s > 1$, $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ is just the Gelfand-Shilov space $\mathcal{S}_s^s(\mathbb{R}^d)$. The ultradifferential operators of class $*$ act continuously on $\mathcal{S}^*(\mathbb{R}^d)$ and $\mathcal{S}'^*(\mathbb{R}^d)$ and the Fourier transform is a topological isomorphism on them. We refer to [2] for the topological properties of $\mathcal{S}^*(\mathbb{R}^d)$ and $\mathcal{S}'^*(\mathbb{R}^d)$.

For $f \in \mathcal{S}'^*(\mathbb{R}^d)$ and $0 \neq \chi \in \mathcal{S}^*(\mathbb{R}^d)$, the short-time Fourier transform of f with window χ (from now on abbreviated as STFT [12]; it is also known as the wave-packet transform first introduced by Córdoba and Fefferman [4]) is defined by $V_\chi f(x,\xi) = \mathcal{F}_{t\to\xi}(f(t)\overline{\chi(t-x)})$. For fixed window χ, $f \mapsto V_\chi f$ is a continuous operator from $\mathcal{S}'^*(\mathbb{R}^d)$ into $\mathcal{S}'^*(\mathbb{R}^{2d})$ and it restricts to a continuous operator from $\mathcal{S}^*(\mathbb{R}^d)$ into $\mathcal{S}^*(\mathbb{R}^{2d})$. Furthermore, when $f \in \mathcal{S}'^*(\mathbb{R}^d)$, $V_\chi f$ is smooth and, in fact, it is an element of $\mathcal{E}^*(\mathbb{R}^{2d})$. If the window χ is in $\mathcal{D}^*(\mathbb{R}^d)$, we can extend the definition of $V_\chi f$ even when $f \in \mathcal{D}'^*(\mathbb{R}^d)$ by $V_\chi f(x,\xi) = \langle e^{-i\xi\cdot} f, \overline{\chi(\cdot - x)}\rangle$ and one can easily verify that $V_\chi f \in \mathcal{E}^*(\mathbb{R}^{2d})$ in this case as well (see Remark 3.1 below).

We denote by T_x and M_ξ the translation and modulation operators: $T_x f = f(\cdot - x)$, $M_\xi f = e^{i\xi\cdot} f(\cdot)$. They act continuously on $\mathcal{S}^*(\mathbb{R}^d)$ and, by duality, on $\mathcal{S}'^*(\mathbb{R}^d)$ as well.

We end the section by recalling the definition and some of the important properties of translation-modulation invariant (B)-spaces of ultradistributions [9].

Definition 2.2. *([9, Definition 3.1]) A (B)-space E is said to be a translation-modulation invariant (B)-space of ultradistributions (in short: TMIB)) of class $*$ if it satisfies the following three conditions:*

(*a*) *The continuous and dense inclusions $\mathcal{S}^*(\mathbb{R}^d) \hookrightarrow E \hookrightarrow \mathcal{S}'^*(\mathbb{R}^d)$ hold.*

(*b*) *$T_x(E) \subseteq E$ and $M_\xi(E) \subseteq E$ for all $x, \xi \in \mathbb{R}^d$.*

(*c*) *There exist $\tau, C > 0$ (for every $\tau > 0$ there exists $C_\tau > 0$), such that*[1)]

$$\omega_E(x) := \|T_x\|_{\mathcal{L}_b(E)} \leq Ce^{M(\tau|x|)} \quad \text{and} \quad \nu_E(\xi) := \|M_{-\xi}\|_{\mathcal{L}_b(E)} \leq Ce^{M(\tau|\xi|)}, \tag{1}$$

where $\|\cdot\|_{\mathcal{L}_b(E)}$ stands for the norm on $\mathcal{L}(E) = \mathcal{L}(E, E)$ induced by $\|\cdot\|_E$ (the norm on E).

The functions $\omega_E : \mathbb{R}^d \to (0, \infty)$ and $\nu_E : \mathbb{R}^d \to (0, \infty)$ defined in (1) are called the weight functions of the translation and modulation groups of E, respectively (in short its weight functions).

These spaces enjoy a number of important properties; we recall only the necessary ones here and refer to [9] for the complete account (see also [8, 10]). We start by pointing out that E is separable and the weight functions ω_E and ν_E are measurable. Moreover, the translation and modulation operators on E form both C_0-groups, i.e. $x \mapsto T_x f$ and $x \mapsto M_x f$, $\mathbb{R}^d \mapsto E$, are continuous for each $f \in E$. Also E is a Banach convolution module over the Beurling (convolution) algebra $L^1_{\omega_E}(\mathbb{R}^d)$ (the weighted L^1 space of measurable functions g such that $\|g\|_{L^1_{\omega_E}} := \|g\omega_E\|_{L^1} < \infty$) and a Banach multiplication module over the Wiener-Beurling (multiplication) algebra $\mathcal{F}L^1_{\nu_E}$ (see [9, Proposition 3.2]). In particular, multiplication by elements of $\mathcal{S}^*(\mathbb{R}^d)$ is a well defined and continuous operation on E. Furthermore, the Fourier image of E, which we denote by $\mathcal{F}E$, is again a TMIB space of class $*$ with norm $\|\mathcal{F}f\|_{\mathcal{F}E} = \|f\|_E$ and, consequently, it enjoys all of the properties we mentioned above; in particular, its weight functions $\omega_{\mathcal{F}E}$ and $\nu_{\mathcal{F}E}$ are measurable and satisfy the estimate (1), with E replaced by $\mathcal{F}E$.

3. The wave front set with respect to a TMIB space of class $*$. Characterisations via the STFT

Let E be TMIB space of class $*$ over $\mathbb{R}^d$. Besides the properties (*a*), (*b*) and (*c*) of Definition 2.2 we additionally assume that it satisfies the following:

(*d*) $\mathcal{F}E$ is a solid space (cf. [11]), i.e. $\mathcal{F}E \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^d)$ [2)] and there exists $C_0 > 0$ such that if $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $f \in \mathcal{F}E$ and $|g(x)| \leq |f(x)|$ a.e, then $g \in \mathcal{F}E$ and $\|g\|_{\mathcal{F}E} \leq C_0\|f\|_{\mathcal{F}E}$.

Notice that the solidity implies that if $f \in \mathcal{F}E$ then $|f| \in \mathcal{F}E$ and $\||f|\|_{\mathcal{F}E} \leq C_0\|f\|_{\mathcal{F}E}$. Consequently, if $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $f_1, \ldots, f_k \in \mathcal{F}E$ are such that $|g(x)| \leq \sum_{j=1}^k |f_j(x)|$ a.e, then $g \in \mathcal{F}E$ and $\|g\|_{\mathcal{F}E} \leq C_0^2 \sum_{j=1}^k \|f_j\|_{\mathcal{F}E}$.

Following Hörmander [13] (cf. [14, Section 8.1, p. 253]), for $f \in \mathcal{E}'^*(\mathbb{R}^d)$ we define the set $\Sigma_E(f) \subseteq \mathbb{R}^d\backslash\{0\}$ as follows: $\xi \in \mathbb{R}^d\backslash\{0\}$ does not belong to $\Sigma_E(f)$ if and only if there exists a cone neighbourhood Γ of ξ such that

$$\theta_\Gamma \mathcal{F}f \in \mathcal{F}E, \tag{2}$$

where θ_Γ denotes the characteristic function of Γ. Clearly $\Sigma_E(f)$ is a closed cone in $\mathbb{R}^d\backslash\{0\}$. From now on, for a measurable subset $G \subseteq \mathbb{R}^d$, θ_G will always stand for the characteristic function of G.

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1)The closed graph theorem together with the conditions (a) and (b) yield that $T_x, M_\xi \in \mathcal{L}(E)$, for all $x, \xi \in \mathbb{R}^d$ (see the proof of [10, Lemma 3.1]); hence, we can take their operator norms in (1).

2)Since $\mathcal{F}E$ is continuously included into $\mathcal{D}'^*(\mathbb{R}^d)$, the closed graph theorem for Fréchet spaces immediately implies that the inclusion $\mathcal{F}E \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is continuous.

Remark 3.1. *Before we state the next result we make the following general observation. For every* $f \in \mathcal{E}'^*(\mathbb{R}^d)$, $\mathcal{F}f \in \mathcal{E}^*(\mathbb{R}^d)$. *Furthermore, if B is a bounded subset of* $\mathcal{E}'^*(\mathbb{R}^d)$ *then there exist* $C_1, h_1 > 0$ *(resp. for every* $h_1 > 0$ *there exists* $C_1 > 0$*) such that* $|\mathcal{F}f(x)| \leq C_1 e^{M(h_1|x|)}$, $\forall x \in \mathbb{R}^d$, $\forall f \in B$. *This easily follows from [17, Proposition 5.11] and [17, Theorem 8.1 and Theorem 8.7].*

On the other hand, if $f \in \mathcal{E}'^*(\mathbb{R}^d)$ *satisfies the following estimate: for every* $h_1 > 0$ *there exists* $C_1 > 0$ *(resp. there exist* $h_1, C_1 > 0$*) such that* $|\mathcal{F}f(x)| \leq C_1 e^{-M(h_1|x|)}$, $\forall x \in \mathbb{R}^d$, *then a straightforward computation gives* $f \in \mathcal{D}^*(\mathbb{R}^d)$.

We recall the following lemmas from [19] which will be used in the proof of Proposition 3.4; we state only the Beurling case of these results since this is the only part we need for the proof of Proposition 3.4.

Lemma 3.2. *[19, Lemma 2.1] Let* $r' \geq 1$ *and* $k > 0$. *There exists an ultrapolynomial* $P(z)$ *of class* (M_p) *such that* P *does not vanish on* $\mathbb{R}^d$ *and satisfies the following estimate: there exists* $C > 0$ *such that*

$$|D^\alpha(1/P(x))| \leq C\alpha! r'^{-|\alpha|} e^{-M(k|x|)}, \quad \forall x \in \mathbb{R}^d,\ \forall \alpha \in \mathbb{N}^d.$$

Lemma 3.3. *[19, Lemma 2.4] Let* $r > 0$.

(*i*) *For each* $\chi, \varphi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d)$ *and* $\psi \in \mathcal{S}^{M_p,r}(\mathbb{R}^d)$ *it holds that* $\chi * (\varphi\psi) \in \mathcal{S}^{(M_p)}(\mathbb{R}^d)$.

(*ii*) *Let* $\varphi, \chi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d)$ *with* $\varphi(0) = 1$ *and* $\int_{\mathbb{R}^d} \chi(x)dx = 1$. *For each* $n \in \mathbb{Z}_+$ *define* $\chi_n(x) = n^d\chi(nx)$ *and* $\varphi_n(x) = \varphi(x/n)$. *Then there exists* $k \geq 2r$ *such that the operators* $\tilde{Q}_n : \psi \mapsto \chi_n * (\varphi_n\psi)$ *are continuous as mappings from* $\mathcal{S}^{M_p,k}(\mathbb{R}^d)$ *to* $\mathcal{S}^{M_p,r}(\mathbb{R}^d)$, *for all* $n \in \mathbb{Z}_+$. *Moreover* $\tilde{Q}_n \to \mathrm{Id}$, *as* $n \to \infty$, *in* $\mathcal{L}_b(\mathcal{S}^{M_p,k}(\mathbb{R}^d), \mathcal{S}^{M_p,r}(\mathbb{R}^d))$.

Proposition 3.4. *Let* $\psi \in \mathcal{D}^*(\mathbb{R}^d)$ *and* $f \in \mathcal{E}'^*(\mathbb{R}^d)$. *Then* $\Sigma_E(\psi f) \subseteq \Sigma_E(f)$.

Proof. Let $0 \neq \xi_0 \notin \Sigma_E(f)$. There exists a cone neighbourhood Γ_1 of ξ_0 such that (2) holds. Pick a cone neighbourhood Γ of ξ_0 such that $\overline{\Gamma} \subseteq \Gamma_1 \cup \{0\}$. We have

$$\theta_\Gamma(\xi)\mathcal{F}(\psi f)(\xi) = (2\pi)^{-d}\theta_\Gamma(\xi)\mathcal{F}\psi * \mathcal{F}f(\xi) = I_1(\xi)/(2\pi)^d + I_2(\xi)/(2\pi)^d,$$

where

$$\begin{aligned} I_1(\xi) &= \theta_\Gamma(\xi)\int_{\mathbb{R}^d} \mathcal{F}\psi(\eta)(1-\theta_{\Gamma_1}(\xi-\eta))\mathcal{F}f(\xi-\eta)d\eta,\\ I_2(\xi) &= \theta_\Gamma(\xi)\int_{\mathbb{R}^d} \mathcal{F}\psi(\eta)\theta_{\Gamma_1}(\xi-\eta)\mathcal{F}f(\xi-\eta)d\eta. \end{aligned}$$

Clearly $I_1, I_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. We prove that both I_1 and I_2 belong to $\mathcal{F}E$ which, in turn, will yield the claim in the proposition. For this purpose we make the following observations: there exist $0 < c < 1$ such that

$$\{\eta \in \mathbb{R}^d \,|\, \exists \xi \in \Gamma,\ |\eta - \xi| \leq c|\xi|\} \subseteq \Gamma_1. \tag{3}$$

For I_1, we avail ourselves of (3) by noticing that if $\xi \in \Gamma$ and $\xi - \eta \notin \Gamma_1$, then $|\eta| > c|\xi|$. Thus, applying Remark 3.1 together with [17, Proposition 3.6] we infer

$$\begin{aligned} e^{M(h|\xi|)}|I_1(\xi)| &\leq C_1\int_{\mathbb{R}^d} |\mathcal{F}\psi(\eta)|e^{M(h|\eta|/c)}e^{M(h_1(1+c^{-1})|\eta|)}d\eta\\ &\leq c_0C_1\int_{\mathbb{R}^d} |\mathcal{F}\psi(\eta)|e^{M((hc^{-1}+h_1c^{-1}+h_1)H|\eta|)}d\eta. \end{aligned}$$

Hence $e^{M(h|\cdot|)}I_1 \in L^\infty(\mathbb{R}^d)$ for every $h > 0$ in the (M_p) case and for some $0 < h \leq 1$ in the $\{M_p\}$ case (in the $\{M_p\}$ case we can take h_1 arbitrarily small in the above estimates). Because of Lemma 3.2, we can find an ultrapolynomial $P(z)$ of class (M_p) which does not vanish on the real axis and satisfies the following estimate: there exists $C' > 0$ such that $|D^\alpha(1/P(x))| \leq C'\alpha! e^{-M(|x|)}$, $\forall x \in \mathbb{R}^d$. As P is of class (M_p), there exist $\tilde{C}, s \geq 1$ such that $|P(x)| \leq \tilde{C}e^{M(s|x|)}$, $\forall x \in \mathbb{R}^d$ (see [17, Proposition 4.5]). Thus, in the $\{M_p\}$ case, $x \mapsto 1/P(hx/s)$ belongs to $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and $|I_1(\xi)| \leq C''/|P(h\xi/s)|$, $\forall \xi \in \mathbb{R}^d$, for some $C'' > 0$. The solidity of $\mathcal{F}E$ gives $I_1 \in \mathcal{F}E$

in the $\{M_p\}$ case. For the (M_p) case, since $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ is continuously included into $\mathcal{F}E$, there exist $C', h' \geq 1$ such that $\|\varphi\|_{\mathcal{F}E} \leq C'\|\varphi\|_{h'}$, $\forall \varphi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d)$, which in turn yields that the closure of $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ in $\mathcal{S}^{M_p,h'}(\mathbb{R}^d)$, which we denote by $X_{h'}$ for short, is continuously included into $\mathcal{F}E$. Now Lemma 3.3 gives the existence of $h'' > h'$ such that (in the notations of Lemma 3.3) $\tilde{Q}_n\phi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d)$, $\forall \phi \in \mathcal{S}^{M_p,h''}(\mathbb{R}^d)$, and $\tilde{Q}_n\phi \to \phi$, as $n \to \infty$, in the topology of $\mathcal{S}^{M_p,h'}(\mathbb{R}^d)$. We conclude that $\mathcal{S}^{M_p,h''}(\mathbb{R}^d)$ is continuously included into $X_{h'}$. If $P(z)$ is the same ultrapolynomial as before, than $x \mapsto 1/P(h''x)$ belongs to $\mathcal{S}^{M_p,h''}(\mathbb{R}^d)$ (notice that $(M.3)$ gives $h''^{|\alpha|}\alpha! \leq C_2 h''^{-|\alpha|}M_\alpha$, $\forall \alpha \in \mathbb{N}^d$, for some $C_2 > 0$) and consequently in $\mathcal{F}E$ as well. Since $|I_1(\xi)| \leq C''/|P(h''\xi)|$, $\forall \xi \in \mathbb{R}^d$, the solidity of $\mathcal{F}E$ proves that $I_1 \in \mathcal{F}E$ in the (M_p) case as well.

We turn our attention to I_2 next. Let $\eta \in \mathbb{R}^d$ be fixed. Then

$$\theta_\Gamma(\xi)\theta_{\Gamma_1}(\xi-\eta)|\mathcal{F}f(\xi-\eta)| \leq |T_\eta(\theta_{\Gamma_1}\mathcal{F}f)(\xi)|, \ \forall \xi \in \mathbb{R}^d.$$

Since $\theta_{\Gamma_1}\mathcal{F}f \in \mathcal{F}E$, the solidity of $\mathcal{F}E$ implies that $\eta \mapsto \mathbf{F}(\eta) = \theta_\Gamma T_\eta(\theta_{\Gamma_1}\mathcal{F}f)$, $\mathbb{R}^d \to \mathcal{F}E$, is well defined $\mathcal{F}E$-valued mapping and

$$\|\mathbf{F}(\eta)\|_{\mathcal{F}E} \leq C_0\omega_{\mathcal{F}E}(\eta)\|\theta_{\Gamma_1}\mathcal{F}f\|_{\mathcal{F}E}. \tag{4}$$

For $\eta, \eta_0 \in \mathbb{R}^d$, we have

$$|\theta_\Gamma(\xi)T_\eta(\theta_{\Gamma_1}\mathcal{F}f)(\xi) - \theta_\Gamma(\xi)T_{\eta_0}(\theta_{\Gamma_1}\mathcal{F}f)(\xi)| \leq |T_\eta(\theta_{\Gamma_1}\mathcal{F}f)(\xi) - T_{\eta_0}(\theta_{\Gamma_1}\mathcal{F}f)(\xi)|, \ \forall \xi \in \mathbb{R}^d.$$

Again, the solidity of $\mathcal{F}E$ implies

$$\|\mathbf{F}(\eta) - \mathbf{F}(\eta_0)\|_{\mathcal{F}E} \leq C_0\|T_\eta(\theta_{\Gamma_1}\mathcal{F}f) - T_{\eta_0}(\theta_{\Gamma_1}\mathcal{F}f)\|_{\mathcal{F}E} \to 0, \text{ as } \eta \to \eta_0.$$

Consequently, $\mathbf{F}$ is continuous and hence strongly measurable. Now, (4) proves that $\eta \mapsto \mathcal{F}\psi(\eta)\mathbf{F}(\eta)$, $\mathbb{R}^d \to \mathcal{F}E$, is Bochner integrable. We claim

$$I_2 = \int_{\mathbb{R}^d} \mathcal{F}\psi(\eta)\mathbf{F}(\eta)d\eta \in \mathcal{F}E. \tag{5}$$

To verify this, fix $\varphi \in \mathcal{D}^*(\mathbb{R}^d)$. Then

$$\begin{aligned}\left\langle \int_{\mathbb{R}^d} \mathcal{F}\psi(\eta)\mathbf{F}(\eta)d\eta, \varphi \right\rangle &= \int_{\mathbb{R}^d} \mathcal{F}\psi(\eta)\langle \mathbf{F}(\eta), \varphi\rangle d\eta \\ &= \int_{\mathbb{R}^{2d}} \mathcal{F}\psi(\eta)\theta_\Gamma(\xi)\theta_{\Gamma_1}(\xi-\eta)\mathcal{F}f(\xi-\eta)\varphi(\xi)d\xi d\eta,\end{aligned}$$

where, the very last integral is absolutely convergent. We conclude

$$\left\langle \int_{\mathbb{R}^d} \mathcal{F}\psi(\eta)\mathbf{F}(\eta)d\eta, \varphi \right\rangle = \langle I_2, \varphi\rangle.$$

As $\varphi \in \mathcal{D}^*(\mathbb{R}^d)$ is arbitrary, we deduce (5), which completes the proof of the proposition. □

Following Hörmander [13] (cf. [14, Section 8.1, p. 253]), for $f \in \mathcal{D}'^*(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we define

$$\Sigma_{x,E}(f) = \bigcap_{\chi \in \mathcal{D}^*(\mathbb{R}^d),\, \chi(x) \neq 0} \Sigma_E(\chi f).$$

Clearly $\Sigma_{x,E}(f)$ is a closed cone subset of $\mathbb{R}^d \backslash \{0\}$.

Proposition 3.5. *Let $f \in \mathcal{D}'^*(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and Γ be an open cone such that $\Sigma_{x,E}(f) \subseteq \Gamma$. There exists $\chi \in \mathcal{D}^*(\mathbb{R}^d)$ satisfying $\chi(x) \neq 0$ and having support arbitrarily close to x such that $\Sigma_E(\chi f) \subseteq \Gamma$. In particular, $\Sigma_{x,E}(f) = \emptyset$ if and only if there exists $\chi \in \mathcal{D}^*(\mathbb{R}^d)$, satisfying $\chi(x) \neq 0$, such that $\chi f \in E$.*

Proof. The proof is the same as in [14, Section 8.1, p. 253-254] but now applying Proposition 3.4 and Lemma 2.1. □

We can now define the wave front set of $f \in \mathcal{D}'^*(\mathbb{R}^d)$ with respect to E.

Definition 3.6. *For $f \in \mathcal{D}'^*(\mathbb{R}^d)$, we define the E-wave front set of f by*

$$WF_E(f) = \{(x,\xi) \in \mathbb{R}^d \times (\mathbb{R}^d \backslash \{0\}) | \, \xi \in \Sigma_{x,E}(f)\}.$$

Remark 3.7. *Clearly $WF_E(f)$ is a closed subset of $\mathbb{R}^d \times (\mathbb{R}^d \backslash \{0\})$ and it is conic in the second variable, i.e. if $(x,\xi) \in WF_E(f)$, then $(x, \lambda\xi) \in WF_E(f)$, $\forall \lambda > 0$. Hence, we can consider it as a closed subspace of $\mathbb{R}^d \times \mathbb{S}^{d-1}$.*

Remark 3.8. *When E is a Sobolev space $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, and $f \in \mathcal{D}'(\mathbb{R}^d)$, then the definition of $WF_E(f)$ coincides with the Sobolev wave front set of f as defined by Hörmander [15, Definition 8.2.5, p. 188; Proposition 8.2.6, p. 189].*

Remark 3.9. *For $f \in \mathcal{D}'^*(\mathbb{R}^d)$, we can define the set* $\operatorname{sing\,supp}_E f \subseteq \mathbb{R}^d$ *whose complement is given by the points at which f locally behaves as an element of E. More precisely, $x_0 \in \mathbb{R}^d$ does not belong to* $\operatorname{sing\,supp}_E f$ *if and only if there exists $\chi \in \mathcal{D}^*(\mathbb{R}^d)$ satisfying $\chi(x_0) \neq 0$ such that $\chi f \in E$ (because of Lemma 2.1, this is the same as if we furthermore require for χ to be identically equal to 1 on a neighbourhood of x_0). Clearly* $\operatorname{sing\,supp}_E f$ *is closed in $\mathbb{R}^d$ and Proposition 3.5 proves that the projection of $WF_E(f)$ on the first component is exactly* $\operatorname{sing\,supp}_E f$.

We can now formulate and prove the main result of the article.

Theorem 3.10. *Let $f \in \mathcal{D}'^*(\mathbb{R}^d)$ and $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \backslash \{0\})$. The following conditions are equivalent.*

(*i*) $(x_0, \xi_0) \notin WF_E(f)$.

(*ii*) *There exist a cone neighbourhood Γ of ξ_0 and a compact neighbourhood K of x_0 such that the mapping $\chi \mapsto \theta_\Gamma \mathcal{F}(\chi f)$, $\mathcal{D}^*_K \to \mathcal{F}E$, is well-defined and continuous.*

(*iii*) *There exist a cone neighbourhood Γ of ξ_0 and a compact neighbourhood K of x_0 such that*

$$\theta_\Gamma V_\chi f(x,\cdot) \in \mathcal{F}E, \ \ \forall \chi \in \mathcal{D}^*_{K-\{x_0\}}, \ \forall x \in K,$$

the mapping $x \mapsto \theta_\Gamma V_\chi f(x,\cdot)$, $K \to \mathcal{F}E$, is continuous and the mapping

$$\chi \mapsto \theta_\Gamma V_{\bar{\chi}} f, \ \ \mathcal{D}^*_{K-\{x_0\}} \to C(K;\mathcal{F}E), \tag{6}$$

is continuous.[3)]

(*iv*) *There exist a cone neighbourhood Γ of ξ_0, a compact neighbourhood K of x_0 and $\chi \in \mathcal{D}^*(\mathbb{R}^d)$, satisfying $\chi(0) \neq 0$ such that $\theta_\Gamma V_\chi f(x,\cdot) \in \mathcal{F}E$, $\forall x \in K$, and $\sup_{x\in K} \|\theta_\Gamma V_\chi f(x,\cdot)\|_{\mathcal{F}E} < \infty$.*

Proof. (*i*) ⇒ (*ii*). Pick a cone neighbourhood Γ_1 of ξ_0 and $\chi \in \mathcal{D}^*(\mathbb{R}^d)$ with $\chi(x_0) \neq 0$ such that $\theta_{\Gamma_1}\mathcal{F}(\chi f) \in \mathcal{F}E$. Let K_1 be a compact neighbourhood of x_0 such that χ never vanishes on K_1 and take a compact neighbourhood K of x_0 such that $K \subset \operatorname{int} K_1$. Fix an open cone $\Gamma \ni \xi_0$ satisfying $\overline{\Gamma} \subseteq \Gamma_1 \cup \{0\}$ and find $0 < c < 1$ such that (3) holds true. Repeating the proof of Proposition 3.4 verbatim with $\chi f \in \mathcal{E}'^*(\mathbb{R}^d)$ in place of f we conclude that $\theta_\Gamma \mathcal{F}(\psi\chi f) \in \mathcal{F}E$, for all $\psi \in \mathcal{D}^*_K$. Lemma 2.1 infers the function $x \mapsto 1/\chi(x)$, $\operatorname{int} K_1 \to \mathbb{C}$, belongs to $\mathcal{E}^*(\operatorname{int} K_1)$. Thus, for each $\psi \in \mathcal{D}^*_K$, we have $\psi f = (\psi/\chi)\chi f$, with $\psi/\chi \in \mathcal{D}^*_K$. We deduce that $\theta_\Gamma \mathcal{F}(\psi f) \in \mathcal{F}E$, for all $\psi \in \mathcal{D}^*_K$. Since $\psi \mapsto \theta_\Gamma \mathcal{F}(\psi f)$, $\mathcal{D}^*_K \to \mathcal{S}'^*(\mathbb{R}^d)$, is continuous we conclude that the mapping $\psi \mapsto \theta_\Gamma \mathcal{F}(\psi f)$, $\mathcal{D}^*_K \to \mathcal{F}E$, has closed graph ($\mathcal{F}E$ is continuously included into $\mathcal{S}'^*(\mathbb{R}^d)$). The Ptak closed graph theorem [25, Theorem 8.5, p. 166] implies that it is continuous ($\mathcal{D}^*_K$ is barrelled and $\mathcal{F}E$ is a (B)-space and consequently a Ptak space; see [25, Section 4.8, p. 162]).
(*ii*) ⇒ (*iii*). Let K_1 be a compact neighbourhood of x_0 and Γ a cone neighbourhood of ξ_0 such that $\chi \mapsto \theta_\Gamma \mathcal{F}(\chi f)$, $\mathcal{D}^*_{K_1} \to \mathcal{F}E$, is well-defined and continuous. Without losing of generality, we can assume

[3)] $C(K;\mathcal{F}E)$ stands for the (B)-space of all continuous functions $K \to \mathcal{F}E$.

that $K_1 = \overline{B(x_0, r)}$, for some $r > 0$. Let $K = \overline{B(x_0, r/4)}$. For every $x \in K$ and $\chi \in \mathcal{D}^*_{K-\{x_0\}}$, the function $t \mapsto \chi_x(t) = \overline{\chi(t-x)}$ belongs to $\mathcal{D}^*_{K_1}$ and thus $\theta_\Gamma V_\chi f(x,\cdot) = \theta_\Gamma \mathcal{F}(\chi_x f) \in \mathcal{F}E$. Fix $\chi \in \mathcal{D}^*_{K-\{x_0\}}$. Our immediate goal is to prove that the mapping $x \mapsto \theta_\Gamma V_\chi f(x,\cdot)$, $K \to \mathcal{F}E$, is continuous. Let $x' \in K$ be arbitrary but fixed. The Taylor formula yields

$$\overline{\chi(t-x)} - \overline{\chi(t-x')} = \sum_{|\beta|=1} (x'-x)^\beta \int_0^1 \overline{\partial^\beta \chi(t-x'+s(x'-x))} ds.$$

When $x \in K$, the function

$$t \mapsto \chi_{x,x',\beta}(t) = \int_0^1 \overline{\partial^\beta \chi(t-x'+s(x'-x))} ds$$

belongs to $\mathcal{D}^*_{K_1}$ and the set $\{\chi_{x,x',\beta} | x \in K, |\beta| = 1\}$ is bounded in $\mathcal{D}^*_{K_1}$. Thus, there exists $C' > 0$ such that

$$\|\theta_\Gamma \mathcal{F}(\chi_{x,x',\beta} f)\|_{\mathcal{F}E} \leq C', \ \forall x \in K, \forall |\beta| = 1.$$

Since

$$|\theta_\Gamma(\xi) V_\chi f(x,\xi) - \theta_\Gamma(\xi) V_\chi f(x',\xi)| = \left| \sum_{|\beta|=1} (x'-x)^\beta \theta_\Gamma(\xi) \mathcal{F}(\chi_{x,x',\beta} f)(\xi) \right|,$$

for all $\xi \in \mathbb{R}^d$, $x \in K$, the solidity of $\mathcal{F}E$ proves

$$\|\theta_\Gamma V_\chi f(x,\cdot) - \theta_\Gamma V_\chi f(x',\cdot)\|_{\mathcal{F}E} \leq C_0 C' d |x-x'| \to 0, \text{ as } x \to x',$$

which, in turn, verifies the continuity of $x \mapsto \theta_\Gamma V_\chi f(x,\cdot)$, $K \to \mathcal{F}E$. It remains to prove the continuity of the mapping (6). Let B be a bounded subset of $\mathcal{D}^*_{K-\{x_0\}}$. One easily verifies that $\{\overline{\chi_x} | x \in K, \chi \in B\}$ is a bounded subset of $\mathcal{D}^*_{K_1}$. As $\psi \mapsto \theta_\Gamma \mathcal{F}(\psi f)$, $\mathcal{D}^*_{K_1} \to \mathcal{F}E$, is continuous and $\theta_\Gamma V_{\bar{\chi}} f(x,\cdot) = \theta_\Gamma \mathcal{F}(\overline{\chi_x} f)$, $\forall x \in K$, $\forall \chi \in \mathcal{D}^*_{K-\{x_0\}}$, we infer that the set $\{\theta_\Gamma V_{\bar{\chi}} f(x,\cdot) | \chi \in B, x \in K\}$ is bounded in $\mathcal{F}E$ and consequently the image of B under the mapping (6) is bounded in $C(K; \mathcal{F}E)$. Since $\mathcal{D}^*_{K-\{x_0\}}$ is bornological, we conclude that (6) is continuous.

$(iii) \Rightarrow (iv)$ is trivial and $(iv) \Rightarrow (i)$ follows easily by specialising $x = x_0$. □

References

[1] J. Bruna, On inverse-closed algebras of infinitely differentiable functions, Studia Mathematica 69(1) (1981) 59–68.

[2] R. Carmichael, A. Kamiński and S. Pilipović, Boundary Values and Convolution in Ultradistribution Spaces, World Scientific Publishing Co. Pte. Ltd., 2007.

[3] G. M. Constantine, T. H. Savits, A multivariate Faà di Bruno formula with applications, Transactions of the American Mathematical Society 348(2) (1996) 503–520.

[4] A. Córdoba, C. Fefferman, Wave packets and Fourier integral operators, Communications in Partial Differential Equations 3 (1978) 979–1005

[5] S. Coriasco, K. Johansson, J. Toft, Local wave-front sets of Banach and Fréchet types, and pseudo-differential operators, Monatshefte für Mathematik 169(3-4) (2013) 1–32.

[6] S. Coriasco, K. Johansson, J. Toft, Global wave-front sets of Banach, Fréchet and modulation space types, and pseudo-differential operators, Journal of Differential Equations 254(8) (2013) 3228–3258.

[7] A. Debrouwere, J. Vindas, Discrete characterizations of wave front sets of FourierLebesgue and quasianalytic type, Journal of Mathematical Analysis and Applications 438(2) (2016) 889–908.

[8] P. Dimovski, S. Pilipović, B. Prangoski, J. Vindas, Convolution of ultradistributions and ultradistribution spaces associated to translation-invariant Banach spaces, Kyoto Journal of Mathematics 56 (2016) 401–440.

[9] P. Dimovski, S. Pilipović, B. Prangoski, J. Vindas, Translation-modulation invariant Banach spaces of ultradistributions, Journal of Fourier Analysis and Applications 25(3) (2019) 819–841.

[10] P. Dimovski, B. Prangoski, J. Vindas, On a class of translation-invariant spaces of quasianalytic ultradistributions, Novi Sad Journal of Mathematics 45 (2015) 143–175.

[11] H. G. Feichtinger, Compactness in translation invariant Banach spaces of distributions and compact multipliers, Journal of Mathematical Analysis and Applications 102 (1984) 289–327.

[12] K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser Boston, 2001.

[13] L. Hörmander, Fourier integral operators. I, Acta Mathematica 127 (1971) 79–183.

[14] L. Hörmander, The Analysis of Linear Partial Differential Operators I (second edition), Springer-Verlag, Berlin Heidelberg New York, 1990.
[15] L. Hörmander, Lectures on Nonlinear Hyperbolic Differential Equations, Mathématiques et Applications 26, Springer-Verlag, Berlin Heidelberg New York, 1997.
[16] K. Johansson, S. Pilipović, N. Teofanov, J. Toft, Micro-local analysis in some spaces of ultradistributions, Publications de l Institut Mathematique (Beograd) (N.S.) 92 (106) (2012) 1–24.
[17] H. Komatsu, Ultradistributions, I: Structure theorems and a characterization, Journal of the Faculty of Science, University of Tokyo. Section IA. Mathematics. 20(1) (1973) 25–105.
[18] S. Nakamura, Propagation of the homogeneous wave-front set for Schrödinger equations, Duke Mathematical Journal 126 (2005) 349–367.
[19] S. Pilipović, B. Prangoski, J. Vindas, On quasianalytic classes of Gelfand-Shilov type. Parametrix and convolution, Journal de Mathématiques Pures et Appliquées, 116 (2018) 174-210.
[20] S. Pilipović, B. Prangoski, Complex powers for a class of infinite order hypoelliptic operators, Dissertationes Mathematicae, 529 (2018) 1–58.
[21] S. Pilipović, B. Prangoski, On the characterisations of wave front sets via the short-time Fourier transform, Mathematical Notes 105 (2019), 153-157.
[22] S. Pilipovic, N. Teofanov, J. Toft, Micro-local analysis with Fourier Lebesgue spaces. Part I, Journal of Fourier Analysis and Applications 17 (2011) 374–407.
[23] L. Rodino, P. Wahlberg, The Gabor wave front set, Monatshefte für Mathematik 173 (2014) 625–655.
[24] M. Ruzhansky, V. Turunen, Quantization of pseudo-differential operators on the torus, Journal of Fourier Analysis and Applications 16 (2010) 943–982.
[25] H. H. Schaefer, Topological Vector Spaces, Springer-Verlag, New York-Berlin, 1971.
[26] R. Schulz, P. Wahlberg, The equality of the homogeneous and the Gabor wave front set, Communications in Partial Differential Equations 42(5) (2017) 703–730.
[27] R. Schulz, P. Wahlberg, Microlocal properties of Shubin pseudodifferential and localization operators, Journal of Pseudo-Differential Operators and Applications 7 (2016) 91–111.
[28] J. A. Siddiqi, Inverse-closed Carleman algebras of infinitely differentiable functions, Proceedings of the American Mathematical Society 109(2) (1990) 357–367.