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Strong Convergent Iterative Techniques for 2-generalized Hybrid Mappings and Split Equilibrium Problems

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Abstract. In this paper, we suggest a new iterative scheme for finding a common element of the set of solutions of a split equilibrium problem and the set of fixed points of 2-generalized hybrid mappings in Hilbert spaces. We show that the iteration converges strongly to a common solution of the considered problems. A numerical example is illustrated to verify the validity of the proposed algorithm. The results obtained in this paper extend and improve some known results in the literature.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. F_1: $C_1 \times C_1 \to \mathbb{R}$ and F_2: $C_2 \times C_2 \to \mathbb{R}$ are two equilibrium functions, where C_1 and C_2 are nonempty closed convex subsets of H_1 and H_2, respectively. If $A : H_1 \to H_2$ is a bounded linear operator, then split equilibrium problem (SEP) is defined as follows:

Find $x^* \in C_1$ such that

$$F_1(x^*, x) \geq 0 \ \ \forall x \in C_1, \tag{1}$$

and $y^* = Ax^* \in C_2$ such that

$$F_2(y^*, y) \geq 0 \quad \forall\, y \in C_2. \tag{2}$$

The set of all solutions of this split equilibrium problem is denoted by Ω, i.e,

$$\Omega = \{z \in C : z \in EP(F_1) \quad \text{such that } Az \in EP(F_2)\},$$

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where $EP(F_1)$ and $EP(F_2)$ denote the sets of all solutions of the equilibrium problems (1) and (2), respectively.

Equilibrium problem has received much attention due to its applications in a large variety of problems arising in physics, optimizations, economics and some others. The split equilibrium problem (1)-(2) constitute a pair of equilibrium problems where is the generalization of split feasibility problems. Some iterative methods have been rapidly established for solving these problems (see [1–10]).

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be:

(1) *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|, \forall x, y \in C$;

(2) *quasi-nonexpansive* if $\|T(x) - p\| \leq \|x - p\|$ for all $x \in C$ and $p \in F(T)$, where $F(T)$ denotes the set of fixed points of T;

(3) *nonspreading* if

$$2\|T(x) - T(y)\|^2 \leq \|T(x) - y\|^2 + \|T(y) - x\|^2, \forall x, y \in C;$$

(4) *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \forall x, y \in C;$$

It is obvious that the above inequality is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \forall x, y \in C; \tag{3}$$

(5) *α-inverse strongly monotone* if there exists $\alpha > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \alpha\|Tx - Ty\|^2, \quad \forall x, y \in C;$$

(6) *hybrid* if

$$3\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \forall x, y \in C;$$

(7) (α, β)-*generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|T(x) - T(y)\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2, \forall x, y \in C;$$

(8) 2-*generalized hybrid mapping* if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ such that for all $x, y \in C$

$$\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2,$$

such a mapping is called a $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping.

It is also easy to see that

a $(1, 0)$-generalized hybrid mapping is nonexpansive.

a $(2, 1)$-generalized hybrid mapping is nonspreading.

a $(3/2, 1/2)$-generalized hybrid mapping is hybrid.

a $(0, \alpha_2, 0, \beta_2)$-generalized hybrid mapping is (α_2, β_2)-generalized hybrid.

a 2-generalized hybrid mapping is quasi-nonexpansive.

In [11], Hojo et al. give two examples of 2-generalized hybrid mappings which are not generalized hybrid mapping.

Recently, the existence of fixed points and the convergence theorems of hybrid mappings have been studied by many authors (see [12–20]).

Very recently, Alizadeh and Moradlou [21–23] have obtained some weak convergence theorems for 2-generalized hybrid mapping and equilibrium problems.

Motivated by the above works, in this paper we introduce and consider a new iterative algorithm for a common element of the sets of solutions of the split equilibrium problems and common fixed points of 2-generalized hybrid mapping in Hilbert spaces. Under suitable conditions, some strong convergence for the sequences generated by the algorithm to a common solution of the problems is proved. The results presented in the paper extend and improve the corresponding results announced by Alizadeh and Moradlou [21], and some others.

2. Preliminaries and lemmas

In this section, we give some definitions and preliminaries which will be used in the sequel.

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The operator P_C denotes the Metric projection from H onto C. It is a fact that P_C is a firmly nonexpansive mapping from H onto C. Further, for any $x \in H$, z=$P_C x$ if and only if $\langle x - z, z - y\rangle \geq 0, \ \forall y \in C$.

Lemma 2.1 ([24]). *Let H be a real Hilbert space and T: $H \to H$ be a nonexpansive mapping. Then for all $(x, y) \in H \times F(T)$, we have*

$$\langle x - T(x), y - T(x)\rangle \leq \frac{1}{2}\|x - T(x)\|^2.$$

Lemma 2.2 (Demiclosedness principle). *Let T be a nonexpansive mapping on a closed convex subset C of a real Hilbert space H. Then $I - T$ is demiclosed at any piont $y \in H$, that is, if $x_n \rightharpoonup x$ and $x_n - Tx_n \to y \in H$, then $x - Tx = y$.*

To obtain our main results, we need the following assumptions.

Assumption 2.3 ([25, 26]). *Let F: $C \times C \to \mathbb{R}$ be an equilibrium function satisfying the following assumptions:*

(1) $F(x, x)$=0, $\forall x \in C$;

(2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$, $\forall x, y \in C$;

(3) F is hemicontinuous with respect to the first variable, i.e., for each $x, y, z \in C$, $\limsup_{t\to 0^+} F(tz+(1-t)x, y) \leq F(x, y)$;

(4) for each $x \in C$, the function $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Lemma 2.4 ([27]). *Let C be a nonempty closed convex subset of a real Hilbert space H and F: $C \times C \to \mathbb{R}$ be an equilibrium function which satisfies the Assumption 2.3. Then for all $r > 0$, the resolvent of the equilibrium function $T_r^F : H \to C$ defined by*

$$T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \ \forall y \in C\}, \forall x \in H,$$

is well defined and satisfies the following conditions:

(1) $T_r^F(x)$ is nonempty and single-valued for each $x \in H$;

(2) T_r^F is firmly nonexpansive, i.e. for any x,y $\in H$,

$$\|T_r^F(x) - T_r^F(y)\|^2 \leq \langle T_r^F(x) - T_r^F(y), x - y\rangle;$$

(3) $F(T_r^F) = EP(F)$;

(4) the set $EP(F)$ is closed and convex;

(5) for $r, s > 0$ and for all $x, y \in H$, one has

$$\|T_r^F(x) - T_s^F(y)\|^2 \leq \|x - y\| + |1 - \frac{s}{r}|\|T_r^F(x) - x\|.$$

Lemma 2.5 ([28]). *Let H be a a real Hilbert space. For all $x, y \in H$,*

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2, \forall \alpha \in \mathbb{R}.$$

Now, we give a new iterative scheme as follows:

Let C_1 be a nonempty closed convex subset of a real Hilbert space H_1 and $S : C_1 \to C_1$ is a 2-generalized hybrid mapping.

For an initial point $x_0 \in C_1$, let $x_1 = P_{C_1}x_0$ and $D_1 = C_1$. Then

$$\begin{cases} u_n = T_{r_n}^{F_1}[I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n, \\ v_n = (1 - \beta_n)u_n + \frac{\beta_n}{n}\sum_{k=0}^{n-1} S^k u_n \\ y_n = (1 - \alpha_n)u_n + \frac{\alpha_n}{n}\sum_{k=0}^{n-1} S^k v_n \\ D_{n+1} = \{x \in D_n : \|y_n - x\| \leq \|x_n - x\|\}, \\ x_{n+1} = P_{D_{n+1}}x_0, \ \forall n \geq 1 \ . \end{cases} \tag{4}$$

For this iterative scheme, we will discuss its strong convergence and also prove that its limit point belongs to $F(S) \cap \Omega$, where $F(S)$ is a set of fixed points of S.

3. The main results

In this section, we show some strong convergence theorems for finding a common element of the solution set of split equilibrium problems and the set of fixed points of 2-generalized hybrid mapping in a Hilbert space.

Throughout this section we need the following assumptions:

(A1) $C_1 \subset H_1$ and $C_2 \subset H_2$ are nonempty closed convex subsets of the real Hilbert spaces H_1 and H_2, respectively.

(A2) $A : H_1 \to H_2$ is a bounded linear mapping.

(A3) $S : C_1 \to C_1$ is a 2-generalized hybrid mapping.

(A4) $F_1 : C_1 \times C_1 \to R$, $F_2 : C_2 \times C_2 \to R$ are two equilibrium functions such that Assumption 2.3 holds.

(A5) $T_{r_n}^{F_1} : H_1 \to C_1$, $T_{r_n}^{F_2} : H_2 \to C_2$ are the resolvent of the equilibrium functions F_1 and F_2, respectively.

We also need the following lemma.

Lemma 3.1 ([26]). *Assume that the assumptions $(A1-A5)$ are satisfied and $r_n \in (r, +\infty)$ with $r > 0, \gamma \in (0, \frac{1}{L})$, where L is the spectral radius of A^*A. Then $A^*(I - T_{r_n}^{F_2})A$ is a $\frac{1}{L}$-inverse strongly monotone mapping and $I - \gamma A^*(I - T_{r_n}^{F_2})A$ is a nonexpansive mapping.*

Theorem 3.2. *Assume that the assumptions $(A1 - A5)$ are satisfied and $0 < \alpha < \alpha_n, \beta_n < \beta < 1, r < r_n < \infty$, for $\alpha, \beta \in (0, 1)$, $r > 0$, $\gamma \in (0, \frac{1}{L})$, where L is the spectral radius of A^*A. In addition, if $\Theta = F(S) \cap \Omega \neq \emptyset$, then for any $x_0 \in C_1$, the sequence $\{x_n\}$ defined by (4) converges strongly to some point $p \in \Theta$.*

Proof. We shall divide the proof into five steps.

Step (I): $\Theta \subset D_n$, $\forall n \geq 1$.

Obviously, $\Theta \subset D_1 = C_1$. By induction, assume that $\Theta \subset D_n$ for some $n \geq 1$. We only need to show that $\Theta \subset D_{n+1}$. For any $p \in \Theta$, we have $p = T_{r_n}^{F_1} p$ and $(I - \gamma A^*(I - T_{r_n}^{F_2})A)p = p$ from Lemma 2.4. The Lemma 3.1 results in

$$\begin{aligned} \|u_n - p\| &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)p\| \\ &\leq \|(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - (I - \gamma A^*(I - T_{r_n}^{F_2})A)p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{5}$$

Since $p \in F(S)$ and S is quasi-nonexpansive, we get

$$\begin{aligned} \|Sv_n - p\| &\leq \|v_n - p\| \\ &= \|(1 - \beta_n)u_n + \frac{\beta_n}{n} \sum_{k=0}^{n-1} S^k u_n - p\| \\ &\leq (1 - \beta_n)\|u_n - p\| + \frac{\beta_n}{n} \| \sum_{k=0}^{n-1} (S^k u_n - p)\| \\ &\leq (1 - \beta_n)\|u_n - p\| + \frac{\beta_n}{n} \sum_{k=0}^{n-1} \|(S^k u_n - p)\| \\ &\leq (1 - \beta_n)\|u_n - p\| + \frac{\beta_n}{n} \sum_{k=0}^{n-1} \|(S u_n - p)\| \\ &\leq \|u_n - p\|. \end{aligned} \tag{6}$$

Combining (4), (5), (6) and Lemma 2.5, we obtain

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\alpha_n)u_n + \frac{\alpha_n}{n}\sum_{k=0}^{n-1} S^k v_n - p\|^2 \\
&= \|(1-\alpha_n)(u_n - p) + \alpha_n(\frac{1}{n}\sum_{k=0}^{n-1} S^k v_n - p)\|^2 \\
&= (1-\alpha_n)\|u_n - p\|^2 + \alpha_n\|\frac{1}{n}\sum_{k=0}^{n-1}(S^k v_n - p)\|^2 - \alpha_n(1-\alpha_n)\|u_n - \frac{1}{n}\sum_{k=0}^{n-1} S^k v_n\|^2 \\
&\le (1-\alpha_n)\|u_n - p\|^2 + \alpha_n\frac{1}{n}\sum_{k=0}^{n-1}\|S^k v_n - p\|^2 - \alpha_n(1-\alpha_n)\|u_n - \frac{1}{n}\sum_{k=0}^{n-1} S^k v_n\|^2 \\
&\le (1-\alpha_n)\|u_n - p\|^2 + \alpha_n\|v_n - p\|^2 - \alpha_n(1-\alpha_n)\|u_n - \frac{1}{n}\sum_{k=0}^{n-1} S^k v_n\|^2 \\
&\le \|u_n - p\|^2 - \alpha_n(1-\alpha_n)\|u_n - \frac{1}{n}\sum_{k=0}^{n-1} S^k v_n\|^2 \\
&\le \|u_n - p\|^2 \\
&\le \|x_n - p\|^2, \qquad (7)
\end{aligned}$$

which implies that $p \in D_{n+1}$. Therefore, $\Theta \subset D_{n+1}$.

Step (II): The sequence $\{x_n\}$ is a Cauchy sequence.

According to (4) and the Step (I), it is obvious that D_n is nonempty closed and convex subset of C_1. Since $\Theta \subset D_{n+1} \subset D_n$, for all $n \ge 1$, we obtain from $x_{n+1} = P_{D_{n+1}}x_0$ that

$$\|x_{n+1} - x_0\| = \|P_{D_{n+1}}x_0 - x_0\| \le \|p - x_0\|, \forall p \in \Theta,$$

which implies that $\{x_n\}$ is bounded. By $x_n = P_{D_n}x_0$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \ge 0.$$

Therefore,

$$0 \le \langle x_0 - x_n, x_n - x_{n+1}\rangle \quad \le \quad -\|x_n - x_0\|^2 + \|x_{n+1} - x_0\|\|x_0 - x_n\|.$$

Hence,

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\|, \qquad \forall n \ge 1,$$

which implies that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. For any $n > m \ge 1$, $x_m = P_{D_m}x_0$, we also have

$$\begin{aligned}
\|x_n - x_0\|^2 &= \|x_n - x_m + x_m - x_0\|^2 \\
&= \|x_n - x_m\|^2 + \|x_m - x_0\|^2 + 2\langle x_n - x_m, x_m - x_0\rangle \\
&\ge \|x_n - x_m\|^2 + \|x_m - x_0\|^2.
\end{aligned}$$

Therefore, we get

$$\|x_n - x_m\|^2 \le \|x_n - x_0\|^2 - \|x_m - x_0\|^2 \to 0 \qquad as\ n, m \to \infty. \qquad (8)$$

Hence $\{x_n\}$ is a Cauchy sequence. We may assume that

$$x_n \to x^*, \qquad as\ n \to \infty. \qquad (9)$$

Step (III): $\lim_{n\to\infty} \| u_n - \frac{1}{n}\sum_{k=0}^{n-1} S^k u_n \| = 0.$

Since $x_{n+1} \in D_{n+1} \subset D_n$, by the definition of D_{n+1}, we have

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|.$$

It follows from (8) that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{10}$$

Therefore, we obtain from (8) and (10)

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \quad as\ n \to \infty. \tag{11}$$

Further, from (7), we have

$$\|y_n - p\|^2 \leq \|u_n - p\|^2 - \alpha_n(1-\alpha_n)\|u_n - \frac{1}{n}\sum_{k=0}^{n-1} S^k v_n\|^2. \tag{12}$$

Therefore,

$$\begin{aligned} \alpha_n(1-\alpha_n)\|u_n - \frac{1}{n}\sum_{k=0}^{n-1} S^k v_n\|^2 &\leq \|u_n - p\|^2 - \|y_n - p\|^2, \\ &\leq \|x_n - p\|^2 - \|y_n - p\|^2, \\ &\leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|). \end{aligned}$$

By $0 < \alpha < \alpha_n < \beta < 1$ and (11), we have

$$\|u_n - \frac{1}{n}\sum_{k=0}^{n-1} S^k v_n\| \to 0, \ \ as\ n \to \infty. \tag{13}$$

Furthermore, $p \in \Theta$ ensures $p = T_{r_n}^{F_1} p$ and $p = (I - \gamma A^*(I - T_{r_n}^{F_2})A)p$. Therefore, we have from Lemma 3.1 and (3)

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}^{F_1}[I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n - T_{r_n}^{F_1}[I - \gamma A^*(I - T_{r_n}^{F_2})A]p\|^2 \\ &\leq \|[I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n - [I - \gamma A^*(I - T_{r_n}^{F_2})A]p\|^2 \\ &\quad -\|(I - T_{r_n}^{F_1})[I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n - (I - T_{r_n}^{F_1})[I - \gamma A^*(I - T_{r_n}^{F_2})A]p\|^2 \\ &= \|x_n - p\|^2 - 2\gamma\langle x_n - p, A^*(I - T_{r_n}^{F_2})Ax_n - A^*(I - T_{r_n}^{F_2})Ap\rangle \\ &\quad +\gamma^2\|A^*(I - T_{r_n}^{F_2})Ax_n - A^*(I - T_{r_n}^{F_2})Ap\|^2 \\ &\quad -\|(I - T_{r_n}^{F_1})[I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n - (I - T_{r_n}^{F_1})[I - \gamma A^*(I - T_{r_n}^{F_2})A]p\|^2 \\ &\leq \|x_n - p\|^2 + \gamma(\gamma - \frac{2}{L})\|A^*(I - T_{r_n}^{F_2})Ax_n\|^2 - \|(I - T_{r_n}^{F_1})[I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n\|^2. \end{aligned} \tag{14}$$

From (12) and (14), we get

$$\begin{aligned} \gamma(\frac{2}{L} - \gamma)\|A^*(I - T_{r_n}^{F_2})Ax_n\|^2 &+ \|(I - T_{r_n}^{F_1})[I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n\|^2 \\ &\leq \|x_n - p\|^2 - \|u_n - p\|^2 \\ &\leq \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|). \end{aligned} \tag{15}$$

Since $\gamma \in (0, \frac{1}{L})$, using (11), we have

$$\lim_{n\to\infty} \|A^*(I - T_{r_n}^{F_2})Ax_n\| = 0, \quad \lim_{n\to\infty} \|(I - T_{r_n}^{F_1})[I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n\| = 0. \tag{16}$$

Since $T_{r_n}^{F_1}$ is firmly nonexpansive, so we have

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}^{F_1}[I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n - T_{r_n}^{F_1}p\|^2 \\ &\le \|[I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n - p\|^2 \\ &= \|x_n - p\|^2 + \gamma^2\|A^*(I - T_{r_n}^{F_2})Ax_n\|^2 + 2\gamma\langle p - x_n, A^*(I - T_{r_n}^{F_2})Ax_n\rangle \\ &= \|x_n - p\|^2 + \gamma^2\|A^*(I - T_{r_n}^{F_2})Ax_n\|^2 + 2\gamma\langle Ap - Ax_n, (I - T_{r_n}^{F_2})Ax_n\rangle,\end{aligned}$$

and

$$\begin{aligned}\gamma^2\|A^*(I - T_{r_n}^{F_2})Ax_n\|^2 &= \gamma^2\langle (I - T_{r_n}^{F_2})Ax_n, AA^*(I - T_{r_n}^{F_2})Ax_n\rangle \\ &\le L\gamma^2\|(I - T_{r_n}^{F_2})Ax_n\|^2.\end{aligned}$$

We also have from Lemma 2.1

$$\begin{aligned}2\gamma\langle Ap - Ax_n, (I - T_{r_n}^{F_2})Ax_n\rangle &= 2\gamma\langle Ap - T_{r_n}^{F_2}Ax_n - (Ax_n - T_{r_n}^{F_2}Ax_n), (I - T_{r_n}^{F_2})Ax_n\rangle \\ &= 2\gamma\{\langle Ap - T_{r_n}^{F_2}Ax_n, Ax_n - T_{r_n}^{F_2}Ax_n\rangle - \|Ax_n - T_{r_n}^{F_2}Ax_n\|^2\} \\ &\le 2\gamma\{\frac{1}{2}\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 - \|Ax_n - T_{r_n}^{F_2}Ax_n\|^2\} \\ &= -\gamma\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|u_n - p\|^2 &\le \|x_n - p\|^2 + L\gamma^2\|(I - T_{r_n}^{F_2})Ax_n\|^2 - \gamma\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 \\ &= \|x_n - p\|^2 + \gamma(L\gamma - 1)\|(I - T_{r_n}^{F_2})Ax_n\|^2,\end{aligned}$$

which implies from (12) that

$$\begin{aligned}-\gamma(L\gamma - 1)\|(I - T_{r_n}^{F_2})Ax_n\|^2 &\le \|x_n - p\|^2 - \|u_n - p\|^2 \\ &\le \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|).\end{aligned}$$

Due to $\gamma \in (0, \frac{1}{L})$ and (11), we have

$$\lim_{n\to\infty} \|(I - T_{r_n}^{F_2})Ax_n\| = 0. \tag{17}$$

Hence, we obtain from (16)

$$\begin{aligned}\|u_n - x_n\| &= \|T_{r_n}^{F_1}[I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n - x_n\| \\ &\le \|T_{r_n}^{F_1}[I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n - [I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n\| + \|[I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n - x_n\| \\ &= \|(I - T_{r_n}^{F_1})[I - \gamma A^*(I - T_{r_n}^{F_2})A]x_n\| + \gamma\|A^*(I - T_{r_n}^{F_2})A]x_n\| \to 0, \quad \text{as } n \to \infty.\end{aligned} \tag{18}$$

Using (4) and Lemma 2.5, we have

$$\begin{aligned}\|v_n - p\|^2 &= \|(1 - \beta_n)u_n + \frac{\beta_n}{n}\sum_{k=0}^{n-1} S^k u_n - p\|^2 \\ &= \|(1 - \beta_n)(u_n - p) + \beta_n(\frac{1}{n}\sum_{k=0}^{n-1} S^k u_n - p)\|^2 \\ &= (1 - \beta_n)\|u_n - p\|^2 + \beta_n\|\frac{1}{n}\sum_{k=0}^{n-1}(S^k u_n - p)\|^2 - \beta_n(1 - \beta_n)\|u_n - \frac{1}{n}\sum_{k=0}^{n-1} S^k u_n\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\beta_n)\|u_n-p\|^2+\beta_n\frac{1}{n}\sum_{k=0}^{n-1}\|S^kv_n-p\|^2-\beta_n(1-\beta_n)\|u_n-\frac{1}{n}\sum_{k=0}^{n-1}S^ku_n\|^2\\
&\leq (1-\beta_n)\|u_n-p\|^2+\beta_n\|u_n-p\|^2-\beta_n(1-\beta_n)\|u_n-\frac{1}{n}\sum_{k=0}^{n-1}S^ku_n\|^2\\
&\leq \|u_n-p\|^2-\beta_n(1-\beta_n)\|u_n-\frac{1}{n}\sum_{k=0}^{n-1}S^ku_n\|^2. \qquad (19)
\end{aligned}
$$

So, we get from (19) and (5)

$$
\begin{aligned}
\|y_n-p\|^2 &= \|(1-\alpha_n)u_n+\frac{\alpha_n}{n}\sum_{k=0}^{n-1}S^kv_n-p\|^2\\
&= \|(1-\alpha_n)(u_n-p)+\alpha_n(\frac{1}{n}\sum_{k=0}^{n-1}S^kv_n-p)\|^2\\
&= (1-\alpha_n)\|u_n-p\|^2+\alpha_n\|\frac{1}{n}\sum_{k=0}^{n-1}(S^kv_n-p)\|^2-\alpha_n(1-\alpha_n)\|u_n-\frac{1}{n}\sum_{k=0}^{n-1}S^kv_n\|^2\\
&\leq (1-\alpha_n)\|u_n-p\|^2+\alpha_n\frac{1}{n}\sum_{k=0}^{n-1}\|S^kv_n-p\|^2-\alpha_n(1-\alpha_n)\|u_n-\frac{1}{n}\sum_{k=0}^{n-1}S^kv_n\|^2\\
&\leq (1-\alpha_n)\|u_n-p\|^2+\alpha_n\|v_n-p\|^2-\alpha_n(1-\alpha_n)\|u_n-\frac{1}{n}\sum_{k=0}^{n-1}S^kv_n\|^2\\
&\leq (1-\alpha_n)\|u_n-p\|^2+\alpha_n\|v_n-p\|^2\\
&\leq (1-\alpha_n)\|u_n-p\|^2+\alpha_n[\|u_n-p\|^2-\beta_n(1-\beta_n)\|u_n-\frac{1}{n}\sum_{k=0}^{n-1}S^ku_n\|^2]\\
&\leq \|u_n-p\|^2-\alpha_n\beta_n(1-\beta_n)\|u_n-\frac{1}{n}\sum_{k=0}^{n-1}S^ku_n\|^2\\
&\leq \|x_n-p\|^2-\alpha_n\beta_n(1-\beta_n)\|u_n-\frac{1}{n}\sum_{k=0}^{n-1}S^ku_n\|^2, \qquad (20)
\end{aligned}
$$

which implies from (20) and $0<\alpha<\alpha_n<\beta<1$ that

$$
\begin{aligned}
\alpha\beta_n(1-\beta_n)\|u_n-\frac{1}{n}\sum_{k=0}^{n-1}S^ku_n\|^2 &\leq \alpha_n\beta_n(1-\beta_n)\|u_n-\frac{1}{n}\sum_{k=0}^{n-1}S^ku_n\|^2\\
&\leq \|x_n-p\|^2-\|y_n-p\|^2\\
&\leq (\|x_n-p\|+\|y_n-p\|)(\|x_n-y_n\|). \qquad (21)
\end{aligned}
$$

In virtue of $0<\alpha<\beta_n<\beta<1$, (9), (11) and (21), we get

$$
\lim_{n\to\infty}\|u_n-\frac{1}{n}\sum_{k=0}^{n-1}S^ku_n\|=0. \qquad (22)
$$

Step (IV): $x^*\in\Theta=F(S)\cap\Omega$, where x^* is the limit in (3.5).
To do so, we firstly show that $x^*\in\Omega$.

By the boundedness of A and (9), we get $Ax_n \to Ax^*$. Then we have from Lemma 2.4 and (17)

$$||T_{r_n}^{F_2}Ax_n - T_r^{F_2}Ax_n|| \leq |1 - \frac{r}{r_n}|||T_{r_n}^{F_2}Ax_n - Ax_n|| \to 0, \quad as\ n \to \infty$$

and

$$||T_r^{F_2}Ax_n - Ax_n|| \leq ||T_r^{F_2}Ax_n - T_{r_n}^{F_2}Ax_n|| + ||T_{r_n}^{F_2}Ax_n - Ax_n|| \to 0, \quad as\ n \to \infty.$$

Since $T_r^{F_2}$ is nonexpansive, we easily get from Lemma 2.2 and Lemma 2.4

$$T_r^{F_2}Ax^* = Ax^*, \ \ i.e, \ Ax^* \in F(T_r^{F_2}) = EP(F_2).$$

Let $w_n = (I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n$. By (16) we have

$$||w_n - x_n|| = ||\gamma A^*(I - T_{r_n}^{F_2})Ax_n|| \to 0, \quad as\ n \to \infty.$$

We also have from (16)

$$||T_{r_n}^{F_1}w_n - T_r^{F_1}w_n|| \leq |1 - \frac{r}{r_n}|||T_{r_n}^{F_1}w_n - w_n|| \to 0, \quad as\ n \to \infty.$$

Therefore,

$$||T_r^{F_1}w_n - w_n|| \leq ||T_r^{F_1}w_n - T_{r_n}^{F_1}w_n|| + ||T_{r_n}^{F_1}w_n - w_n|| \to 0, \quad as\ n \to \infty.$$

Since $T_r^{F_1}$ is nonexpansive, we get from Lemma 2.2 and Lemma 2.4

$$T_r^{F_1}x^* = x^*, \ \ i.e, \ x^* \in F(T_r^{F_1}) = EP(F_1).$$

Therefore, $x^* \in \Omega$.

Now, we prove that $x^* \in F(S)$.

By means of (18) and (22), we easily get from $x_n \to x^*$

$$\frac{1}{n}\sum_{k=0}^{n-1} S^k u_n \to x^*, \quad as\ \ n \to \infty. \tag{23}$$

Since S is a 2-generalized hybrid mapping, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ such that for all $x, y \in C_1$

$$\begin{aligned}\alpha_1||S^2x - Sy||^2 + \alpha_2||Sx - Sy||^2 \quad + \quad & (1 - \alpha_1 - \alpha_2)||x - Sy||^2 \leq \beta_1||S^2x - y||^2 \\ & +\beta_2||Sx - y||^2 + (1 - \beta_1 - \beta_2)||x - y||^2.\end{aligned}$$

Since $F(S) \neq \emptyset$, then S is quasi-nonexpansive. So $||S^n u_n - p|| \leq ||u_n - p|| \leq ||x_n - p||$,which implies that $\{S^n u_n\}$ is bounded. Since S is a 2-generalized hybrid mapping, we have for all $y \in C_1$ and $k = 0, 2, 3, ..., n - 1$

$$\begin{aligned}
0 \quad \leq \quad & \beta_1||S^{k+2}x_n - y||^2 + \beta_2||S^{k+1}x_n - y||^2 + (1 - \beta_1 - \beta_2)||S^k x_n - y||^2 \\
& -\alpha_1||S^{k+2}x_n - Sy||^2 - \alpha_2||S^{k+1}x_n - Sy||^2 - (1 - \alpha_1 - \alpha_2)||S^k x_n - Sy||^2 \\
= \quad & \beta_1\{||S^{k+2}x_n - Sy||^2 + 2\langle S^{k+2}x_n - Sy, Sy - y\rangle + ||Sy - y||^2\} + \beta_2\{||S^{k+1}x_n - Sy||^2 \\
& +2\langle S^{k+1}x_n - Sy, Sy - y\rangle + ||Sy - y||^2\} + (1 - \beta_1 - \beta_2)\{||S^k x_n - Sy||^2 \\
& +2\langle S^k x_n - Sy, Sy - y\rangle + ||Sy - y||^2\} - \alpha_1||S^{k+2}x_n - Sy||^2 - \alpha_2||S^{k+1}x_n - Sy||^2 \\
& -(1 - \alpha_1 - \alpha_2)||S^k x_n - Sy||^2 \\
= \quad & ||Sy - y||^2 + 2\langle \beta_1 S^{k+2}x_n + \beta_2 S^{k+1}x_n + (1 - \beta_1 - \beta_2)S^k x_n - Sy, Sy - y\rangle \\
& +(\beta_1 - \alpha_1)\{||S^{k+2}x_n - Sy||^2 - ||S^k x_n - Sy||^2\} + (\beta_2 - \alpha_2)\{||S^{k+1}x_n - Sy||^2 - ||S^k x_n - Sy||^2\} \\
= \quad & ||Sy - y||^2 + 2\langle S^k x_n - Sy, Sy - y\rangle + 2\langle \beta_1(S^{k+2}x_n - S^k x_n) + \beta_2(S^{k+1}x_n - S^k x_n), Sy - y\rangle \\
& +(\beta_1 - \alpha_1)\{||S^{k+2}x_n - Sy||^2 - ||S^k x_n - Sy||^2\} + (\beta_2 - \alpha_2)\{||S^{k+1}x_n - Sy||^2 - ||S^k x_n - Sy||^2\}.
\end{aligned}$$

Summing these inequalities from $k = 0, 1, ..., n-1$ and diving by n, we have by denoting $z_n = \frac{1}{n}\sum_{k=0}^{n-1} S^k u_n$,for all $n \geq 1$

$$\begin{aligned} 0 \quad \leq \quad & \|Sy - y\|^2 + 2\langle z_n - Sy, Sy - y\rangle + 2\frac{1}{n}\langle \beta_1(S^{n+1}x_n - S^n x_n - Sx_n - x_n) + \beta_2(S^n x_n - x_n), Sy - y\rangle \\ & +(\beta_1 - \alpha_1)\frac{1}{n}\{\|S^{n+1}x_n - Sy\|^2 + \|S^n x_n - Sy\|^2 - \|Sx_n - Sy\|^2 - \|x_n - Sy\|^2\} \\ & +(\beta_2 - \alpha_2)\frac{1}{n}\{\|S^n x_n - Sy\|^2 - \|x_n - Sy\|^2\}. \end{aligned}$$

From (23) and the boundedness of $\{S^n u_n\}$, we have

$$0 \leq \|Sy - y\|^2 + 2\langle x^* - Sy, Sy - y\rangle.$$

Denote $y = x^*$, we have

$$0 \leq \|Sx^* - x^*\|^2 + 2\langle x^* - Sx^*, Sx^* - x^*\rangle = -\|Sx^* - x^*\|^2.$$

Hence $x^* \in F(S)$. This shows that $x^* \in \Theta$. □ □

4. Numerical Example

In this section, a numerical example will be illustrated to verify the validity of the proposed algorithm in Section 3.

Example 4.1. *Consider the following split equilibrium problem driven by* 2*-generalized hybrid mapping S: find* $x \in \mathbb{R}$ *such that*

$$\begin{cases} F_1(x, \widetilde{x}) \geq 0, \ \forall \widetilde{x} \in C_1, \\ y = Ax \in C_2, \\ F_2(y, \widetilde{y}) \geq 0, \ \forall \widetilde{y} \in C_2, \\ x \in F(S), \end{cases} \tag{24}$$

where

$$\begin{aligned} & H_1 = H_2 = \mathbb{R}, \\ & C_1 = [-3, 0], \\ & C_2 = [0, +\infty), \\ & F_1(u, v) = (u - 1)(v - u), \forall u, v \in C_1, \\ & F_2(x, y) = (x + 15)(y - x), \forall x, y \in C_2, \\ & Ax = 3x, \ \forall x \in \mathbb{R}, \\ & Sx = \frac{1}{3}x, \ \forall x \in C_1. \end{aligned}$$

By choosing

$$\begin{aligned} & \alpha_n = \frac{1}{2} + \frac{1}{3n}, \\ & \beta_n = \frac{1}{3} - \frac{1}{6n}, \\ & r = 4, \\ & \gamma = \frac{1}{9}. \end{aligned}$$

It is easy to check that F_1 and F_2 satisfy all conditions in Lemma 3.1, i.e. Assumption 2.3. Analogously to the Theorem 3.2, we abide by the following processes to obtain the solution of (24).

$$\begin{cases} u_n = \frac{1}{25}x_n, \\ v_n = (\frac{2}{3} + \frac{1}{6n})u_n + \frac{1}{n}(\frac{1}{3} - \frac{1}{6n})\sum_{k=0}^{n-1}\frac{1}{3^k}u_n, \\ y_n = (\frac{1}{2} - \frac{1}{3n})u_n + \frac{1}{n}(\frac{1}{2} + \frac{1}{3n})\sum_{k=0}^{n-1}\frac{1}{3^k}v_n. \end{cases}$$

It is easy to get that $0 \in F(0) \bigcap \Omega$.

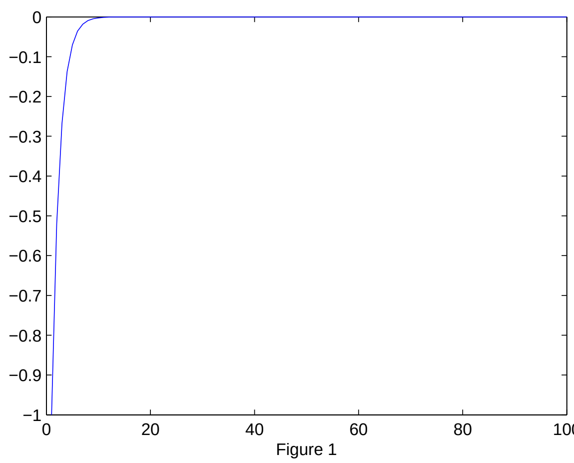
Moreover, numerical results in Table 1 for $\{x_n\}$ also is demonstrated as follows

Table 1:

Numerical results for $x_0 = -2$ *and* $x_0 = -1$			
n	x_n	n	x_n
1	-2.0000	1	1.0000
2	-1.0400	2	-5.2000×10^{-1}
3	-5.3541	3	-2.6770×10^{-1}
4	-2.7456×10^{-1}	4	-1.3728×10^{-1}
5	-1.4054×10^{-1}	5	-7.0270×10^{-2}
6	-7.1863×10^{-2}	6	-3.5932×10^{-2}
⋮	⋮		⋮
98	-9.1955×10^{-29}	98	-4.5978×10^{-29}
99	-4.6901×10^{-29}	99	-2.3450×10^{-29}
100	-2.3921×10^{-29}	100	-1.1960×10^{-29}

See table 1 for the values $x_0 = -2$ or $x_0 = -1$, we obtain $x_n \to 0$, as $n \to \infty$.

See Figure 1 and Figure 2 for the values $x_0 = x_1 = -1$ and $x_0 = x_1 = -2$. The computations associated with example were performed using MATLAB software.



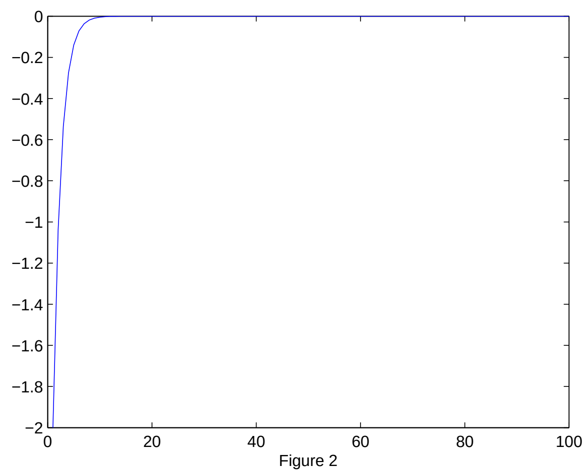


Figure 1: A plot of x_n, $n = 0, 1, 2, \cdots, 100$, for Example 4.1

References

[1] L. C. Ceng and J. C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, J. Comput. Appl. Math. 214(2008), 186-201.
[2] Y. Censor and T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numerical Algorithms, 8(1994), 221-239.
[3] L. Feng, P.Zhuang, F. Liu, I. Turner, V. Anh, J. Li, A fast second-order accurate method for a two-sided space-fractional diffusion equation with variable coefficients, Computers and Mathematics with Applications, 73(2017), 1155-1171.
[4] A. Hamdi, Y. Liou, Y. Yao and C. Luo, The common solutions of the split feasibility problems and fixed point problems, J. Inequal. Appl. 2015, 2015:385.
[5] X. Li, Y. Li, Z. Liu and J. Li, Sensitivity analysis for optimal control problems described by nonlinear fractional evolution inclusions, Fract. Calc. Appl. Anal. 21 (2018), 1439-1470.
[6] J. Li, J.Z. Liu, T. Korakianitis, P.H. Wen, Finite block method in fracture analysis with functionally graded materials, Engineering Analysis with Boundary Elements, 82(2017), 57-67.
[7] J. Li, F. Liu, L. Feng, I. Turner, A novel finite volume method for the Riesz space distributed-order diffusion equation, Computers and Mathematics with Applications, 74(2017),772-783.
[8] J. W. Peng, Y. C. Liou and J. C. Yao, An iterative algorithm combining viscosity method with parallel method for a generalized equilibrium problem and strict pseudocontractions, Fixed Point Theory Appl. 2009(2009), Article ID 794178.
[9] Z. Jouymandi and F. Moradlou, Extragradient methods for solving equilibrium problems, variational inequalities and fixed point problems, Numer. Funct. Anal. Optim., 38 (2017), 1391-1409.
[10] Z. Jouymandi and F. Moradlou, Extragradient methods for split feasibility problems and generalized equilibrium problems in Banach spaces, Math. Meth. Appl. Sci., 41 (2018), 826-838.
[11] M. Hojo, W. Takahashi and I. Termwuttipong, Strong convergence theorem for 2-generalized hybrid mapping in Hilbert spaces, Nonlinear Anal.75(2012), 2166-2176.
[12] H. H. Bauschke and P.L. Combettes, A weak-to-strong convergence principle for Fejer-monotone methods in Hilbert spaces, Math. Oper. Res. 26 (2001) 248-264.
[13] A. Jabbari and R. Keshavarzi, Fixed points of generalized hybrid mappings on L2-embedded sets in Banach spaces, Fixed Point Theory, 20(2019), 203-210.
[14] J. Li, B.L. Guo, Divergent Solution to the Nonlinear Schrödinger Equation with the Combined Power-Type Nonlinearities, Journal of Applied Analysis and Computation, 7(2017), 249-263.
[15] J. Li,T. Huang,J.H. Yue, C. Shi, P.H. Wen, Anti-plane fundamental solutions of functionally graded materials and applications to fracture mechanics, Journal of Strain Analysis for Engineering Design, 52(2017),1-12.
[16] Z.H. Liu, S. Migorski, S.D. Zeng, Partial differential variational inequalities involving nonlocal boundary conditions in Banach spaces, Journal of Differential Equations, 263(7)(2017),3989-4006.
[17] M.V. Solodov and B.F. Svaiter, Forcing strong convergence of proximal point iterations in a Hilbert space, Math. Program. 87 (2000) 189-202.
[18] W. Takahashi, Y. Takeuchi and R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (2008) 276-286.
[19] W. Takahashi and J.-C. Yao, Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces, Taiwanese J. Math. 15 (2011) 457-472.
[20] P. Kocourek, W. Takahashi and J.-C. Yao, Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces, Taiwanese J. Math. 14 (2010) 2497-2511.
[21] S. Alizadeh and F. Moradlou, Weak convergence theorems for 2-generalized hybrid mapping and equilibrium problems,Commun Korean Math.31(2016) 765-777.
[22] S. Alizadeh and F. Moradlou, A Strong Convergence Theorem for Equilibrium problems and Generalized Hybrid mappings, Mediterr. J. Math. 13 (2016) 379-390.
[23] S. Alizadeh and F. Moradlou, A weak convergto ence theorem for 2-generalized hybrid mappings, ROMAI J.,11(2015) 131-138.
[24] K.R. Kazmi and S.H. Rizvi, Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem, J. Egypt. Math. Soc. 21(2013), 44-51.
[25] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63(1994), 123-145.
[26] S. Suantai, P. Cholamjiak, Y.J. Cho and W. Cholamjiak, On solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert spaces, Fixed Point Theory Appl. 2016(2016), Article ID 35.
[27] P.L. Combettes and S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6(2005), 117-136.
[28] W. Takahashi, Introduction to Nonlinear and Convex Analysis, Yokohoma Publishers, Yokohoma, 2009.