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Generalized Hirano Inverses in Banach Algebras

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Abstract. Let $\mathcal{A}$ be a Banach algebra. An element $a \in \mathcal{A}$ has generalized Hirano inverse if there exists $b \in \mathcal{A}$ such that

$$b = bab, ab = ba, a^2 - ab \in \mathcal{A}^{qnil}.$$

We prove that $a \in \mathcal{A}$ has generalized Hirano inverse if and only if $a - a^3 \in \mathcal{A}^{qnil}$, if and only if a is the sum of a tripotent and a quasinilpotent that commute. The Cline's formula for generalized Hirano inverses is thereby obtained. Let $a, b \in \mathcal{A}$ have generalized Hirano inverses. If $a^2b = aba$ and $b^2a = bab$, we prove that $a + b$ has generalized Hirano inverse if and only if $1 + a^d b$ has generalized Hirano inverse. The generalized Hirano inverses of operator matrices on Banach spaces are also studied.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra with an identity. The commutant of $a \in \mathcal{A}$ is defined by $comm(a) = \{x \in A \mid xa = ax\}$. The double commutant of $a \in \mathcal{A}$ is defined by $comm^2(a) = \{x \in \mathcal{A} \mid xy = yx \text{ for all } y \in comm(a)\}$. An element $a \in \mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) in case there exists $b \in \mathcal{A}$ such that

$$b = bab, b \in comm(a), a - a^2b \in \mathcal{A}^{qnil}.$$

The preceding b is unique, if it exists, and we denote it by a^d. Here, $\mathcal{A}^{qnil}$ denote the set of all quasinilpotents of the Banach algebra $\mathcal{A}$, i.e.,

$$\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 + ax \in \mathcal{A} \text{ is invertible for all } x \in comm(a)\}.$$

Let $v(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$. We use $\mathcal{A}^{-1}$ to denote the set of all units in $\mathcal{A}$. We note that

$$\begin{aligned} \mathcal{A}^{qnil} &= \{a \in \mathcal{A} \mid 1 + \lambda a \in \mathcal{A}^{-1} \textit{ for all } \lambda \in \mathbb{C}\} \\ &= \{a \in \mathcal{A} \mid \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0, \textit{i.e.}, v(a) = 0\} (\text{see [7]}). \end{aligned}$$

The motivation of this paper is to extend generalized Drazin inverses in Banach algebras to a wider case by means of tripotents p, i.e, $p^3 = p$. An element $a \in \mathcal{A}$ has generalized Hirano inverse if there exists $b \in \mathcal{A}$ such that

$$b = bab, b \in comm(a), a^2 - ab \in \mathcal{A}^{qnil}.$$

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We may replace the double commutator for the commutator in the preceding definition for a Banach algebra (see Proposition 2.9). Many elementary properties of generalized Hirano inverses were investigated in [2].

As it is well known, $a \in \mathcal{A}$ has g-Drazin inverse if and only if there exists an idempotent $e \in comm(a)$ such that $a + e \in \mathcal{A}^{-1}$ and $ae \in \mathcal{A}^{qnil}$. Here, the spectral idempotent e is unique, and it is denoted by a^{π}. In Section 2, we prove that $a \in \mathcal{A}$ has generalized Hirano inverse if and only if $a - a^3 \in \mathcal{A}^{qnil}$, if and only if a is the sum of a tripotent and a quasinilpotent that commute.

Let $a, b \in \mathcal{A}$. Then ab has g-Drazin inverse if and only if ba has g-Drazin inverse and $(ba)^d = b((ab)^d)^2 a$. This was known as Cline's formula for g-Drazin inverses (see [10]). In Section 3, we extend Cline's formula for generalized Hirano inverses.

In Section 4, we are concerned on additive property for generalized Hirano inverses. Let $a, b \in \mathcal{A}$ have generalized Hirano inverses. If $a^2b = aba$ and $b^2a = bab$, we prove that $a + b$ has generalized Hirano inverse if and only if $1 + a^d b$ has generalized Hirano inverse.

Finally, in the last section, we investigate generalized Hirano inverses for operator matrices on Banach spaces.

Throughout the paper, all Banach algebra are complex with identity 1. Let X be an arbitrary complex Banach space and $\mathcal{L}(X)$ be the Banach algebra of all bounded operators on X.

2. Generalized Hirano inverses

The aim of this section is to present new characterizations of generalized Hirano inverses which will be used repeatedly. We begin with

Lemma 2.1. *[19, Lemma 2.10 and Lemma 2.11] Let $\mathcal{A}$ be a Banach algebra, $a, b \in \mathcal{A}, a^2b = aba$ and $b^2a = bab$.*

(1) *If $a, b \in \mathcal{A}^{qnil}$, then $a + b \in \mathcal{A}^{qnil}$.*
(2) *If a or $b \in \mathcal{A}^{qnil}$, then $ab \in \mathcal{A}^{qnil}$.*

Following Mosić (see [12]) an element $a \in \mathcal{A}$ has gs-Drazin inverse if there exists $b \in \mathcal{A}$ such that $b = bab$, $b \in comm(a)$ and $a - ab \in \mathcal{A}^{qnil}$. It was proved that $a \in \mathcal{A}$ has gs-Drazin inverse if and only if there exists an idempotent $e \in comm(a)$ such that $a - e \in \mathcal{A}^{qnil}$ (see [6, Theorem 3.2]. We have

Lemma 2.2. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *a has gs-Drazin inverse.*
(2) *$a - a^2 \in \mathcal{A}^{qnil}$.*

Proof. $\Longrightarrow$ Write $a = e + w$ with $e^2 = e \in comm(a), w \in \mathcal{A}^{qnil}$. Then $a - a^2 = (1 - 2e - w)w \in \mathcal{A}^{qnil}$, by Lemma 2.1.

$\Longleftarrow$ Let $q := a - a^2$. Then $4q(4q - 1)^{-1} \in \mathcal{A}^{qnil}$. Consider the infinite series

$$-\frac{1}{2}\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}\Big(4q(4q-1)^{-1}\Big)^k,$$

where the coefficients are binomial coefficients. Clearly, $\upsilon\Big(4q(4q-1)^{-1}\Big) = 0$. Hence the series converges absolutely to an element z. Moreover, we have the formal relation

$$1 - \sqrt{1 - 4q(4q-1)^{-1}} = 2z.$$

This implies that

$$z^2 - z = q(4q-1)^{-1}.$$

Here, z commutes with every element of $\mathcal{A}$ which commutes with $q(4q - 1)^{-1}$. That is, there exists $z \in \mathcal{A}$ such that $z^2 - z = q(4q - 1)^{-1}$ and $z \in comm^2(q(4q - 1)^{-1})$ (see [14, Lemma 2.3.8]).

Let $0 \neq \lambda \in \mathbb{C}$. Set $y = 2\lambda z$. Then $y^2 - 2\lambda y = 4\lambda^2 q(4q-1)^{-1} \in \mathcal{A}^{qnil}$. Hence,

$$\begin{aligned}(y-\lambda)^2 &= y^2 - 2\lambda y + \lambda^2 \\ &= \lambda^2 + 4\lambda^2 q(4q-1)^{-1} \\ &\in \mathcal{A}^{-1}.\end{aligned}$$

It follows that $y - \lambda \in \mathcal{A}^{-1}$, and so $y \in \mathcal{A}^{qnil}$. This implies that $z = \frac{1}{2}\lambda^{-1} y \in \mathcal{A}^{qnil}$.

Let $f = a - (2a-1)z$. Then $a - f = (2a-1)z \in \mathcal{A}^{qnil}$. Since $a \in comm(a)$, we see that $a \in comm(q(4q-1)^{-1})$; hence, $a \in comm(z)$, and so $az = za$. We easily see that $(1-2a)^2(1-4q)^{-1} = 1$; hence,

$$\begin{aligned}& (a-(2a-1)z)(1-a+(2a-1)z) \\ =\ & (a-a^2) + (-(2a-1)(1-a) + a(2a-1))z - (2a-1)^2 z^2 \\ =\ & (a-a^2) + (2a-1)^2(z-z^2) \\ =\ & q - (2a-1)^2 q(1-4q)^{-1} \\ =\ & 0,\end{aligned}$$

and so $(a-(2a-1)z)^2 = a - (2a-1)z$. That is, $f^2 = f \in comm(a)$, as desired. □

Lemma 2.3. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *a has generalized Hirano inverse.*
(2) *$a^2 \in \mathcal{A}$ has gs-Drazin inverse.*
(3) *There exists $b \in comm(a)$ such that*

$$b = (ab)^2, a^2 - a^2 b \in \mathcal{A}^{qnil}.$$

Proof. (1) ⇒ (3) By hypothesis, there exists $c \in comm(a)$ such that $c = c^2 a$ and $a^2 - ac \in \mathcal{A}^{qnil}$. Let $b = c^2$. Then $b \in comm(a), b = c^4 a^2 = b^2 a^2 = (ab)^2$. Moreover, we have $a^2 - a^2 b = a^2 - ac \in \mathcal{A}^{qnil}$, as desired.

(3) ⇒ (2) By assumption, we have $b \in comm(a)$ such that $b = (ab)^2, a^2 - a^2 b \in \mathcal{A}^{qnil}$. Hence $b \in comm(a^2), b = ba^2 b$. Therefore $a^2 \in \mathcal{A}$ has gs-Drazin inverse.

(2) ⇒ (1) Since $a^2 \in \mathcal{A}$ has gs-Drazin inverse, then there exists $c \in comm^2(a^2)$ such that $c = c^2 a^2$ and $a^2 - a^2 c \in \mathcal{A}^{qnil}$ (see [6, Remark 2.2]). Set $b = ac$. Since $a \in comm(a^2)$, we see that $ca = ac$; hence, $ab = ba$. Moreover, $b = b^2 a$ and $a^2 - ab = a^2 - a^2 c \in \mathcal{A}^{qnil}$. Therefore a has the generalized Hirano inverse, as asserted. □

Theorem 2.4. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *a has generalized Hirano inverse.*
(2) *$a - a^3 \in \mathcal{A}^{qnil}$.*

Proof. $\Longrightarrow$ In view of Lemma 2.3, $a^2 \in \mathcal{A}$ has gs-Drazin inverse. It follows by Lemma 2.2 that, $a(a-a^3) = a^2 - a^4 \in \mathcal{A}^{qnil}$, and so $(a-a^3)^2 = a(a-a^3)(1-a^2) \in \mathcal{A}^{qnil}$ by Lemma 2.1. If $x \in comm(a-a^3)$, then $x^2 \in comm(a-a^3)^2$; and so $1 - (a-a^3)^2 x^2 \in \mathcal{A}^{-1}$. Thus, $1 - (a-a^3)x \in \mathcal{A}^{-1}$. We infer that $a - a^3 \in \mathcal{A}^{qnil}$, as required.

$\Longleftarrow$ Set $b = \frac{a^2+a}{2}$ and $c = \frac{a^2-a}{2}$. Then we check that

$$\begin{aligned}b^2 - b &= \tfrac{1}{4}(a^4 + 2a^3 - a^2 - 2a) = \tfrac{1}{4}(a+2)(a^3-a); \\ c^2 - c &= \tfrac{1}{4}(a^4 - 2a^3 - a^2 + 2a) = \tfrac{1}{4}(a-2)(a^3-a).\end{aligned}$$

Hence $b^2 - b, c^2 - c \in \mathcal{A}^{qnil}$. Clearly, $a^2 = \frac{a^2+a}{2} + \frac{a^2-a}{2} = b + c$, and so $a^2 - a^4 = (b+c) - (b+c)^2 = (b-b^2) + (c-c^2) - 2bc$. On the other hand, $bc = \frac{a^4-a^2}{4}$, and so

$$\frac{1}{2}(a^2 - a^4) = (b - b^2) + (c - c^2) \in \mathcal{A}^{qnil}.$$

In light of Lemma 2.2, $a^2 \in \mathcal{A}$ has gs-Drazin inverse. This completes the proof by Lemma 2.3. □

Corollary 2.5. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. If $a \in \mathcal{A}$ has generalized Hirano inverse, then $a^n \in \mathcal{A}$ has generalized Hirano inverse for any $n \in \mathbb{N}$.*

Proof. In view of Theorem 2.4, $a - a^3 \in \mathcal{A}^{qnil}$. Then $a^n - (a^n)^3 = a^n - (a^3)^n = (a - a^3)f(a) \in \mathcal{A}^{qnil}$ for some polynomial $f(t)$ with integral coefficients. According to Theorem 2.4, $a^n \in \mathcal{A}$ has generalized Hirano inverse, as asserted. □

Lemma 2.6. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then a has generalized Hirano inverse if and only if $\frac{a^2+a}{2}$ and $\frac{a^2-a}{2}$ have gs-Drazin inverses.*

Proof. $\Longrightarrow$ Set $b := \frac{a^2+a}{2}$. Then

$$\begin{array}{rcl} b^2 - b & = & \frac{1}{4}(a+2)(a^3-a) \\ & \in & \mathcal{A}^{qnil}. \end{array}$$

In light of Lemma 2.2, $b \in \mathcal{A}$ has gs-Drazin inverse. Likewise, $\frac{a^2-a}{2}$ has gs-Drazin inverse, as desired.

$\Longleftarrow$ Set $b = \frac{a^2+a}{2}$ and $c = \frac{a^2-a}{2}$. Then $a^2 = b + c$. In view of Lemma 2.2, $b^2 - b, c^2 - c \in \mathcal{A}^{qnil}$. Since $bc = cb$, as in the proof of Theorem 2.4,

$$\frac{1}{2}(a^2 - a^4) = (b - b^2) + (c - c^2) \in \mathcal{A}^{qnil}.$$

Hence $a^2 \in \mathcal{A}$ has gs-Drazin inverse. This completes the proof by Lemma 2.3. □

We have accumulated all the information necessary to prove the following.

Theorem 2.7. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *$a \in \mathcal{A}$ has generalized Hirano inverse.*
(2) *There exists $e^3 = e \in comm(a)$ such that $a - e \in \mathcal{A}^{qnil}$.*

Proof. $\Longrightarrow$ Let $b = \frac{a^2+a}{2}$ and $c = \frac{a^2-a}{2}$. In view of Lemma 2.6, b and c have gs-Drazin inverses. According to [6, Theorem 3.2], for a Banach algebra $\mathcal{A}$, we indeed have $f^2 = f \in comm^2(b)$ and $g^2 = g \in comm^2(c)$ such that

$$b - f, c - g \in \mathcal{A}^{qnil}.$$

As $ab = ba$ and $ac = ca$, we see that $fa = af$ and $ga = ag$. Hence $gb = bg$ and $fc = cf$. This implies that $fg = gf$. Therefore $a = b - c = (f - g) + (b - f) - (c - g)$. Clearly, $(b - f)(c - g) = (c - g)(b - f)$. In light of Lemma 2.1, $(b - f) - (c - g) \in \mathcal{A}^{qnil}$. Moreover, we check that $(f - g)^3 = f - g$. Set $e = f - g$. Then $a - e \in \mathcal{A}^{qnil}$, as required.

$\Longleftarrow$ By hypothesis, there exists $e^3 = e \in comm(a)$ such that $w := a - e \in \mathcal{A}^{qnil}$. Hence, $a = e + w$, and so $a^2 = e^2 + (2e + w)w$. Then $a^2 - e^2 = (2e + w)w \in \mathcal{A}^{qnil}$. In light of [6, Theorem 3.2], $a^2 \in \mathcal{A}$ has gs-Drazin inverse. Therefore we complete the proof, by Lemma 2.3. □

Corollary 2.8. *Let $\mathbb{C}$ be the field of complex numbers, and let $A \in M_n(\mathbb{C})$. Then the following are equivalent:*

(1) *A has generalized Hirano inverse.*
(2) *A is the sum of a tripotent and a nilpotent matrices that commutate.*
(3) *The eigenvalues of A are only $-1, 0$ or 1.*
(4) *A is similar to $diag(J_1, \cdots, J_r)$, where*

$$J_i = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}, \ \lambda = -1, 0 \text{ or } 1.$$

Proof. In view of Lemma 2.3, $A \in M_n(\mathbb{C})$ has generalized Hirano inverse if and only if $A^2 \in M_n(\mathbb{C})$ has gs-Drazin inverse. We note that $M_n(\mathbb{C})^{qnil}$ is just the set of all $n \times n$ complex nilpotent matrices over $\mathbb{C}$. Therefore we are done by [15, Example 2.5]. □

We close this section with a characterization of a generalized Hirano inverse in terms of its double commutant.

Proposition 2.9. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *a has generalized Hirano inverse.*
(2) *There exists $e^3 = e \in comm^2(a)$ such that $a - e \in \mathcal{A}^{qnil}$.*
(3) *There exists $b \in comm^2(a)$ such that*
$$b = (ab)^2, a^2 - a^2b \in \mathcal{A}^{qnil}.$$

Proof. (1) ⇒ (2) In view of Lemma 2.6, $\frac{a^2+a}{2}$ and $\frac{a^2-a}{2}$ have gs-Drazin inverses. As in the proof of Theorem 2.7, we can find idempotents $f, g \in comm^2(a)$ such that $a - (f - g) \in \mathcal{A}^{qnil}$. Let $e = f - g$. Then $e^3 = e \in comm^2(a)$, as required.

(2) ⇒ (3) Set $b = (a^2 + 1 - e^2)^{-1}e^2$, as in the proof of Theorem 2.7, we have $b = (ab)^2, a^2 - a^2b \in \mathcal{A}^{qnil}$. Since $e \in comm^2(a)$, we check that $b \in comm^2(a)$, as desired.

(3) ⇒ (1) This is obvious by Lemma 2.3. □

3. Multiplicative property

Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$. In [10, Lemma 2.2], it was proved that $ab \in \mathcal{A}^{qnil}$ if and only if $ba \in \mathcal{A}^{qnil}$. We generalized this fact as follows.

Lemma 3.1. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$. If*
$$(ac)^2a = (db)^2a,$$
$$(ac)^2d = (db)^2d,$$
then the following are equivalent:

(1) $(ac)^2 \in \mathcal{A}^{qnil}$.
(2) $(bd)^2 \in \mathcal{A}^{qnil}$.

Proof. As $(ac)^2a = (db)^2a$, we have $acacaca = dbdbaca$. Let $aca = a', c = c', dbd = d'$ and $b = b'$. Then we have $a'c'a' = d'b'a'$. Also by $(ac)^2d = (db)^2d$ we have $acacdbd = dbdbdbd$ which implies $a'c'd' = d'b'd'$. Let $(ac)^2 \in \mathcal{A}^{qnil}$, then $acac \in \mathcal{A}^{qnil}$ which implies that $a'c' \in \mathcal{A}^{qnil}$. By applying [11, Lemma 3.1] we conclude that $d'b' \in \mathcal{A}^{qnil}$ and so $(bd)^2 \in \mathcal{A}^{qnil}$. The converse follows by a similar way. □

Under the hypothesis of Lemma 3.1, we note that $ac \in \mathcal{A}^{qnil}$ and $bd \in \mathcal{A}^{qnil}$ are equivalent. Also we easily prove that $ac \in \mathcal{A}^d$ if and only if $bd \in \mathcal{A}^d$. We come now to the main result of this section.

Theorem 3.2. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$. If*
$$(ac)^2a = (db)^2a,$$
$$(ac)^2d = (db)^2d,$$
then the following are equivalent:

(1) *$ac \in \mathcal{A}$ has generalized Hirano inverse.*
(2) *$bd \in \mathcal{A}$ has generalized Hirano inverse.*

Proof. (1) ⇒ (2) In view of Theorem 2.4, $ac - (ac)^3 \in \mathcal{A}^{qnil}$. By Lemma 2.1, $ac(ac - (ac)^3) \in \mathcal{A}^{qnil}$ which implies that $(ac)^2 - (ac)^4 \in \mathcal{A}^{qnil}$. Thus we have, $((1 - acac)ac)^2 = ((ac)^2 - (ac)^4)(1 - acac) \in \mathcal{A}^{qnil}$. Let $a' = (1 - acac)a, c' = c, b' = b$ and $d' = (1 - dbdb)d$. Then $(a'c')^2 \in \mathcal{A}^{qnil}$. Also

$$\begin{aligned} (a'c')^2a' &= ((1 - acac)ac)^2((1 - acac)a) \\ &= (ac)^2 - 2(ac)^4 + (ac)^6)(a - (ac)^2a) \\ &= (ac)^2a - 3(ac)^4a + 3(ac)^6a - (ac)^8a \\ &= (db)^2a - 3(db)^4a + 3(db)^6a - (db)^8a \\ &= ((1 - dbdb)db)^2a' \\ &= (d'b')^2a'. \end{aligned}$$

By the same way we can prove that $(a'c')^2d' = (d'b')^2d'$. Then by Lemma 3.1, $(b'd')^2 \in \mathcal{A}^{qnil}$ which implies that $(bd - (bd)^3)^2 \in \mathcal{A}^{qnil}$ and so $bd - (bd)^3 \in \mathcal{A}^{qnil}$. Then by Theorem 2.4, bd has generalized Hirano inverse.

(2) ⇒ (1) This is similar. □

Corollary 3.3. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$. If*

$$aca = dba,$$
$$dbd = acd,$$

then the following are equivalent:

(1) *$ac \in \mathcal{A}$ has generalized Hirano inverse.*
(2) *$bd \in \mathcal{A}$ has generalized Hirano inverse.*

Proof. Let $aca = dba$ and $dbd = acd$. Then $(ac)^2a = (db)^2a$ and $(ac)^2d = (db)^2d$. So the result follows from Theorem 3.2. □

Corollary 3.4. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c \in \mathcal{A}$. If $aba = aca$, then the following are equivalent:*

(1) *$ac \in \mathcal{A}$ has generalized Hirano inverse.*
(2) *$ba \in \mathcal{A}$ has generalized Hirano inverse.*

Proof. Let $d = a$. It is easy to show that $(ac)a = (db)a$ and $(ac)d = (db)d$. So the result follows from Theorem 3.2. □

In particular, $ab \in \mathcal{A}$ has generalized Hirano inverse if and only if $ba \in \mathcal{A}$ has generalized Hirano inverse. Corollary 3.3 and Corollary 3.4 are just special cases of Theorem 3.3 in [13].

Corollary 3.5. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$. If $acac = dbdb$, then the following are equivalent:*

(1) *$ac \in \mathcal{A}$ has generalized Hirano inverse.*
(2) *$bd \in \mathcal{A}$ has generalized Hirano inverse.*

Proof. It is easy to show that $(ac)^2a = (db)^2a$ and $(ac)^2d = (db)^2d$. So the proof is true by Theorem 3.2. □

We note that if $aca = dba, dbd = acd$, then $(ac)^2a = (db)^2a, (ac)^2d = (db)^2d$. But the converse is not true.

Example 3.6. *Let σ be an operator, acting on separable Hilbert space $l_2(\mathbb{N})$, defined by*

$$\sigma(x_1, x_2, x_3, x_4, \cdots) = (0, x_1, x_2, 0, 0, \cdots),$$

and let $\mathcal{A} = M_2\Big(\mathcal{L}(l_2(\mathbb{N}))\Big)$. Choose

$$a = \begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, d = a.$$

Then $(ac)^2a = (db)^2a, (ac)^2d = (db)^2d$, but $aca \neq dba$. In this case, $ac \in \mathcal{A}$ has generalized Hirano inverse.

4. Additive property

Now we are concerned on additive property of generalized Hirano inverses in a Banach algebra $\mathcal{A}$. Since every generalized Hirano invertible element in a Banach algebra has g-Drazin inverse, we now derive

Lemma 4.1. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ have generalized Hirano inverses. If $a^2b = aba$ and $b^2a = bab$, then ab has generalized Hirano inverse.*

Proof. One easily checks that $ab-(ab)^3 = ab-(aba)bab = ab-a^2(b^2a)b = ab-a^2(bab)b = ab-a(aba)b^2 = ab-a^3b^3$. Set $x = (a-a^3)b$ and $y = a^3(b-b^3)$. Then $ab-(ab)^3 = x+y$.

Let $c = a-a^3$. Then

$$\begin{aligned} c^2b &= (a-a^3)^2b \\ &= (a^2-2a^4+a^6)b \\ &= a^2b-2a^4b+a^6b \\ &= (a-a^3)b(a-a^3) \\ &= cbc. \end{aligned}$$

Likewise, we have $b^2c = bcb$. In light of Theorem 2.4, $a-a^3 \in \mathcal{A}^{qnil}$. It follows by Lemma 2.1, that $x \in \mathcal{A}^{qnil}$. Similarly, $y \in \mathcal{A}^{qnil}$. The reader could check $x^2y = xyx$ and $y^2x = yxy$. By using Lemma 2.1 again, $x+y \in \mathcal{A}^{qnil}$. Therefore $ab-(ab)^3 = x+y \in \mathcal{A}^{qnil}$. This completes the proof by Theorem 2.4. □

Lemma 4.2. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ have generalized Hirano inverses and $ab = ba$. If $1+a^db$ has generalized Hirano inverse, then $a+b$ has generalized Hirano inverse.*

Proof. Since $a \in \mathcal{A}$ has generalized Hirano inverse, it has generalized Drazin inverse. It follows from $ab = ba$ that $a^db = ba^d$, and then $1+a^db-(1+a^db)^3 = a^db-(a^db)^3-3a^db(1+a^db) \in \mathcal{A}^{qnil}$. By virtue of Theorem 2.4 and Lemma 2.1, we have

$$a^d-(a^d)^3 = (a^d)^4a^3-(a^d)^4a = (a^d)^4(a^3-a) \in \mathcal{A}^{qnil}.$$

Hence $a^d \in \mathcal{A}$ has generalized Hirano inverse. In light of Lemma 4.1, $a^db \in \mathcal{A}$ has generalized Hirano inverse, and so $a^db-(a^db)^3 \in \mathcal{A}^{qnil}$. In view of Lemma 2.1, we have $3a^db(1+a^db) \in \mathcal{A}^{qnil}$, and then

$$\begin{aligned} 3ab(a+b) &= 3(a-a^2a^d)b(a+b)+3a^2b(aa^d+a^db) \\ &= 3(a-a^2a^d)b(a+b)+3a^2b(aa^d+a(a^d)^2b) \\ &= 3(a-a^2a^d)b(a+b)+3a^3a^db(1+a^db) \\ &\in \mathcal{A}^{qnil}. \end{aligned}$$

Consequently, $(a+b)-(a+b)^3 = (a-a^3)+(b-b^3)-3ab(a+b) \in \mathcal{A}^{qnil}$. Accordingly, $a+b$ has generalized Hirano inverse, by Theorem 2.4. □

Let $p \in \mathcal{A}$ be an idempotent, and let $x \in \mathcal{A}$. Then we write

$$x = pxp+px(1-p)+(1-p)xp+(1-p)x(1-p),$$

and induce a representation given by the matrix

$$x = \begin{pmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-p) \end{pmatrix}_p,$$

and so we may regard such matrix as an element in $\mathcal{A}$. For any idempotent e in $\mathcal{A}$, $(eAe)^{qnil} \subseteq \mathcal{A}^{qnil}$.

We now ready to prove the following.

Theorem 4.3. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ have generalized Hirano inverses. If $a^2b = aba$ and $b^2a = bab$, then $a+b$ has generalized Hirano inverse if and only if $1+a^db$ has generalized Hirano inverse.*

Proof. $\Longrightarrow$ Write $1 + a^d b = x + y$ where $x = 1 - aa^d$ and $y = a^d(a + b)$. Then $x^2 = x \in \mathcal{A}$ has generalized Hirano inverse and $xy = 0$. Since $a(ab) = a^2 b = aba$ and $a^d \in comm^2(a)$, we see that $a^d(ab) = (ab)a^d$. Hence $yx = a^d(a + b)(1 - aa^d) = a^d(aa^d)b(1 - aa^d) = a^d(ab)a^d(1 - aa^d) = 0$.

In light of [19, Lemma 2.5], one checks that $(a^d)^2 b = (a^d)(a^d b) = (a^d b)a^d$, and so $(a^d)^2(a + b) = a^d(a + b)a^d$. Moreover, we have $aba^d = aa^d b$ and $b^2 a^d = ba^d b$. Thus $(a + b)^2 a^d = (a + b)a^d(a + b)$. As in the proof of Lemma 4.2, a^d has generalized Hirano inverse. In light of Lemma 4.1, $a^d(a + b) \in \mathcal{A}$ has generalized Hirano inverse. Since $1 + x^d y = 1 + xy = 1$, we see that, $1 + a^d b = x + y \in \mathcal{A}$ has generalized Hirano inverse, by Lemma 4.2.

$\Longleftarrow$ Choose $p = aa^d$. In view of [19, Lemma 2.5], $aa^d b(1 - aa^d) = aba^d(1 - aa^d) = 0$. Then

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & 0 \\ * & b_2 \end{pmatrix}_p,$$

where $a_1 = pap$, $a_2 = (1 - p)a(1 - p)$, $b_1 = pbp$ and $b_2 = (1 - p)b(1 - p)$. Hence,

$$a + b = \begin{pmatrix} a_1 + b_1 & 0 \\ * & a_2 + b_2 \end{pmatrix}_p.$$

Step 1. By using [19, Lemma 2.5], we have

$$\begin{array}{c} (aa^d)^2 b = a(aa^d)a^d b = a(a^d b)aa^d = (aa^d)b(aa^d), \\ b^2(aa^d) = b(ba^d)a = (ba^d)(ba) = b(aa^d)b. \end{array}$$

It follows by Lemma 4.1 that $(aa^d)b$ has generalized Hirano inverse. Clearly, we have $1 + (a^2 a^d)^d aa^d b = 1 + a^d b \in \mathcal{A}$ has generalized Hirano inverse. Since $(a^2 a^d)(aa^d b) = (aa^d b)(a^2 a^d)$, by Lemma 4.2, we have $a^2 a^d + aa^d b = aa^d(a + b) \in \mathcal{A}$ has generalized Hirano inverse. In view of Corollary 3.4, we see that $a_1 + b_1 = (aa^d)(a + b)(aa^d) \in \mathcal{A}$ has generalized Hirano inverse.

Step 2. $b \in \mathcal{A}^{qnil}$. Clearly, $a_2 = a - a^2 a^d \in \mathcal{A}^{qnil}$. In view of [19, Lemma 2.5], we compute

$$\begin{array}{rcl} \big(b(1 - aa^d)\big)^2 & = & (b - baa^d)(b - baa^d) \\ & = & b^2 - b^2 aa^d - baa^d b + ba(a^d b)aa^d \\ & = & b^2 - b^2 aa^d - baa^d b + baaa^d(a^d b) \\ & = & b^2(1 - aa^d). \end{array}$$

By induction, we have $\big(b(1 - aa^d)\big)^n = b^n(1 - aa^d)$ for any $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \|b^n\|^{\frac{1}{n}} = 0$, we easily check that

$$\lim_{n\to\infty} \|\big(b(1 - aa^d)\big)^n\|^{\frac{1}{n}} = 0.$$

Hence $b(1 - aa^d) \in \mathcal{A}^{qnil}$. Then $b_2 = (1 - aa^d)b(1 - aa^d) \in \mathcal{A}^{qnil}$. We easily verify that $a_2^2 b_2 = a_2 b_2 a_2$ and $b_2^2 a_2 = b_2 a_2 b_2$. In light of [19, Lemma 2.10], $a_2 + b_2 \in \mathcal{A}^{qnil}$. In light of [2, Lemma 5.1], $a + b \in \mathcal{A}$ has generalized Hirano inverse.

Step 3. $b \notin \mathcal{A}^{qnil}$. Since $b - b^3 \in \mathcal{A}^{qnil}$, by the argument in Section 2, we have $(b - b^3)(1 - aa^d) \in \mathcal{A}^{qnil}$. Then we check that

$$(1 - aa^d)b - \big((1 - aa^d)b(1 - aa^d)\big)^3 = (1 - aa^d)(b - b^3)(1 - aa^d) \in \mathcal{A}^{qnil}.$$

By virtue of Theorem 2.4, $(1 - aa^d)b \in \mathcal{A}$ has generalized Hirano inverse. It follows by Corollary 3.4 that $b_2 = (1 - aa^d)b(1 - aa^d) \in \mathcal{A}$ has generalized Hirano inverse. Clearly, $a_2 = a - a^2 a^d \in \mathcal{A}^{qnil}$. We easily verify that $a_2^2 b_2 = a_2 b_2 a_2$ and $b_2^2 a_2 = b_2 a_2 b_2$. By Step 2, $a_2 + b_2 \in \mathcal{A}$ has generalized Hirano inverse.

Accordingly, $a + b \in \mathcal{A}$ has generalized Hirano inverse by [2, Lemma 5.1]. $\square$

Corollary 4.4. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ have generalized Hirano inverses. If $ab = ba$, then $a + b$ has generalized Hirano inverse if and only if $1 + a^d b$ has generalized Hirano inverse.*

Proof. This is obvious, by Theorem 4.3. □

For further use, we record the following.

Proposition 4.5. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$. If a, b have generalized Hirano inverses and $ab = 0$, then $a + b$ has generalized Hirano inverse.*

Proof. In view of Theorem 2.4, $a - a^3, b - b^3 \in \mathcal{A}^{qnil}$. It follows by [3, Lemma 2.1] that $a + b - (a + b)^3 = \big((a - a^3) - b(a + b)a\big) + b - b^3 \in \mathcal{A}^{qnil}$. By using Theorem 2.4 again, $a + b$ has generalized Hirano inverses. □

5. Splitting approach

We are now concerned on the generalized Hirano inverse for a operator matrix M. Here,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1}$$

where $A, D \in \mathcal{L}(X)$ have generalized Hirano inverses and X is a complex Banach space. Then M is a bounded linear operator on $X \oplus X$. Here, $\mathcal{L}(X)$ denotes the Banach algebra of bounded linear operators on X. Using different splitting of the operator matrix M as $P + Q$, we will apply preceding results to obtain various conditions for the existence of the generalized Hirano inverse of M.

Lemma 5.1. *Let $A, D \in \mathcal{L}(X)$ have generalized Hirano inverses and $B \in \mathcal{L}(X)$. Then $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in M_2(\mathcal{L}(X))$ has generalized Hirano inverse.*

Proof. In view of Theorem 2.4, $A - A^3, D - D^3 \in \mathcal{L}(X)^{qnil}$. As in a Banach algebra $\mathcal{A}$, $a \in \mathcal{A}^{qnil}$ if and only if for any $\lambda \in \mathbb{C}$, $1 - \lambda a \in \mathcal{A}^{-1}$, we easily see that

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} - \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^3 = \begin{pmatrix} A - A^3 & * \\ 0 & D - D^3 \end{pmatrix} \in M_2(\mathcal{L}(X))^{qnil}.$$

According to Theorem 2.4, we obtain the result. □

Lemma 5.2. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$ have generalized Hirano inverse. If $e^2 = e \in comm(a)$, then $ea \in \mathcal{A}$ has generalized Hirano inverse.*

Proof. Since $a \in \mathcal{A}$ has generalized Hirano inverse, we have $a - a^3 \in \mathcal{A}^{qnil}$, and so $ea - (ea)^3 = e(a - a^3) \in \mathcal{A}^{qnil}$, by Lemma 2.1, This completes the proof by Theorem 2.4. □

Theorem 5.3. *Let $A, D \in \mathcal{L}(X)$ have generalized Hirano inverse and M be given by (5.1). If $BC = CB = 0$, $CA(I - A^\pi) = D^\pi DC$ and $A^\pi AB = BD(I - D^\pi)$, then $M \in M_2(\mathcal{L}(X))$ has generalized Hirano inverse.*

Proof. Let

$$P = \begin{pmatrix} A(I - A^\pi) & B \\ 0 & DD^\pi \end{pmatrix}, \ Q = \begin{pmatrix} AA^\pi & 0 \\ C & D(I - D^\pi) \end{pmatrix}.$$

Then $M = P + Q$. Since $A(I - A^\pi) = A(AA^d)$, it follows by Lemma 5.2 that $A(I - A^\pi)$ has generalized Hirano inverse. On the other hand, $DD^\pi = D - D^2D^d$ is quasinilpotent, and so DD^π has generalized Hirano inverse. In light of Lemma 5.1, $P \in M_2(\mathcal{L}(X))$ has generalized Hirano inverse. Likewise, $Q \in M_2(\mathcal{L}(X))$ has generalized Hirano inverse. It is easy to verify that

$$\begin{aligned} PQ &= \begin{pmatrix} 0 & BD(I - D^\pi) \\ DD^\pi C & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & AA^\pi B \\ CA(I - A^\pi) & 0 \end{pmatrix} \\ &= QP. \end{aligned}$$

Also we have

$$P^d = \begin{pmatrix} (A(I-A^{\pi}))^d & X \\ 0 & D^dD^{\pi} \end{pmatrix} = \begin{pmatrix} A^d & X \\ 0 & 0 \end{pmatrix}$$

where $X = (A^d)^2 \sum_{n=0}^{\infty}(A^d)^n B(DD^{\pi})^n$. Hence,

$$\begin{array}{rcl} P^dQ & = & \begin{pmatrix} A^d & X \\ 0 & 0 \end{pmatrix}\begin{pmatrix} AA^{\pi} & 0 \\ C & D(I-D^{\pi}) \end{pmatrix} \\ & = & \begin{pmatrix} XC & XD(I-D^{\pi}) \\ 0 & 0 \end{pmatrix}, \end{array}$$

where $XC = (A^d)^2(B + \sum_{n=1}^{\infty}(A^d)^n B(DD^{\pi})^n)C = 0$ as $BC = 0$, $B(DD^{\pi})^nC = 0$. Moreover, we have

$$\begin{array}{rl} & XD(I-D^{\pi}) \\ = & (A^d)^2(B + \sum_{n=1}^{\infty}(A^d)^n B(DD^{\pi})^n)D(I-D^{\pi}) \\ = & (A^d)^2BD(I-D^{\pi}) + (A^d)^2 \sum_{n=1}^{\infty}(A^d)^n BD^{n+2}D^{\pi}D^d \\ = & (A^d)^2A^{\pi}AB \\ = & 0, \end{array}$$

and so $P^dQ = 0$. Thus, $I_2 + P^dQ \in M_2(\mathcal{L}(X))$ has generalized Hirano inverse. Therefore we complete the proof by Corollary 4.4. □

In the proof of Theorem 5.3, we choose

$$P = \begin{pmatrix} A(I-A^{\pi}) & B \\ 0 & D^2D^d \end{pmatrix},\ Q = \begin{pmatrix} AA^{\pi} & 0 \\ C & DD^{\pi} \end{pmatrix}.$$

Analogously, we can derive

Proposition 5.4. *Let $A, D \in \mathcal{L}(X)$ have generalized Hirano inverses and M be given by* (5.1). *If $BC = CB = 0$, $CA(I-A^{\pi}) = (I-D^{\pi})DC$ and $A^{\pi}AB = BDD^{\pi}$, then $M \in M_2(\mathcal{L}(X))$ has generalized Hirano inverse.*

We now turn to the operator matrix M with trivial generalized Schur complement, i.e., $D = CA^dB$ (see [4, Theorem 5.2.1]). We have

Theorem 5.5. *Let $A \in \mathcal{L}(X)$ have generalized Hirano inverse, $D \in \mathcal{L}(X)$ and M be given by* (5.1). *Let $W = AA^d + A^dBCA^d$. If AW has generalized Hirano inverse,*

$$A^{\pi}BC = BCA^{\pi} = AA^{\pi}B = 0, D = CA^dB,$$

then M has generalized Hirano inverse.

Proof. We easily see that

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix}, Q = \begin{pmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{pmatrix}.$$

By assumption, we verify that $QP = 0$. Clearly, Q is nilpotent, and so it has generalized Hirano inverse. Furthermore, we have

$$P = P_1 + P_2,\ P_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix},\ P_2 = \begin{pmatrix} AA^{\pi} & 0 \\ CA^{\pi} & 0 \end{pmatrix}$$

and $P_2P_1 = 0$. Obviously, P_2 has generalized Hirano inverse. Moreover, we have

$$P_1 = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} \begin{pmatrix} A & AA^dB \end{pmatrix}.$$

By hypothesis, we see that

$$\begin{pmatrix} A & AA^dB \end{pmatrix} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} = AW$$

has generalized Hirano inverse. In light of [2, Corollary 4.2], P_1 has generalized Hirano inverse. Thus, by Proposition 4.5, P has generalized Hirano inverse. By using Proposition 4.5 again, M has generalized Hirano inverse, as asserted. □

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