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Certain Unified Integral Inequalities in Connection with Quantum Fractional Calculus

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Abstract. Some unified integral inequalities involving quantum fractional calculus are obtained via the functions having classical and relative convexity property.

1. Introduction and Preliminaries

1.1. *Few Classes of Generalized Convex Functions*

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, I is an interval, is said to be convex function on I, if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y), \ \forall x, y \in I, t \in [0,1]. \tag{1.1}$$

If the reversed inequality in (1.1) holds, then f is said to be concave.
Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I and $a, b \in I$ with $a < b$. Then the following double inequality holds:

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx \le \int_a^b f(x)p(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b p(x)dx, \tag{1.2}$$

where $p : [a, b] \to \mathbb{R}$ is non-negative, integrable, and symmetric about $x = \frac{a+b}{2}$. This inequality is known as the Fejér inequality for convex functions (see[3]).
Recently many new generalizations of classical convexity have been proposed in the literature, for example see [2, 14]. In 2008, Noor [10] introduced and studied a new class of convex set and convex function with respect to an arbitrary function, which is called as relative convex sets and relative convex function respectively.

Definition 1.1 ([10]). *Let $g : H \to H$ and K_g be any set in real Hilbert space H. The set K_g is said to be relative convex (g-convex) with respect to the function $g : H \to H$, if*

$$(1-t)x + tg(y) \in K_g, \forall x, y \in H : x, g(y) \in K_g, t \in [0,1].$$

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It has been observed [10] that every convex set is relative convex, but the converse is not true.

Definition 1.2 ([10]). *A function $f : K_g \to H$ is said to be relative convex (g-convex) with respect to the function $g : H \to H$, if*

$$f((1-t)x + tg(y)) \leq (1-t)f(x) + tf(g(y)),$$

for all $x, y \in H : x, g(y) \in K_g$ and $t \in [0,1]$.

Clearly every convex function is relative convex, but the converse is not true.
In [9] Noor established a new refinement of Hermite-Hadamard's type of inequality utilizing relative convex functions as follows:

Theorem 1.3. *Let $f : K_g = [a, g(b)] \to \mathbb{R}$ be a relative convex function. Then, we have*

$$f\left(\frac{a+g(b)}{2}\right) \leq \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \leq \frac{f(a)+f(g(b))}{2}.$$

In [8] Noor et al. introduced and studied the class of relative h-convex functions in connection with Hermite-Hadamard type of inequalities.

Definition 1.4 ([8]). *Let $h : (0,1) \to (0,\infty)$. A function $f : K_g \to H$ is said to be relative h-convex function (g_h-convex) with respect to the function $g : H \to H$, if*

$$f((1-t)x + tg(y)) \leq h(1-t)f(x) + h(t)f(g(y)),$$

for all $x, y \in H : x, g(y) \in K_g, t \in (0,1)$.

Theorem 1.5 ([8]). *Let $f : K_g \to \mathbb{R}$ be a relative h-convex function, such that $h(\frac{1}{2}) \neq 0$, then, we obtain*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+g(b)}{2}\right) \leq \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \leq (f(a)+f(g(b)))\int_0^1 h(t)dt.$$

Recently many researchers have utilized the concepts of fractional and quantum calculus and obtained various new and novel analogues of classical inequalities. For some very useful and interesting details on fractional calculus, see [7]. And for details regarding quantum calculus, see [6, 11].

1.2. Elements of Quantum Calculus

We now recall some previously known concepts on q-calculus which will be used in this paper.
For $q \in (0,1)$ and $a \in \mathbb{C}$, the q-shifted factorials are defined by (see[4])

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n = 1, 2 \ldots. \tag{1.3}$$

$$(a;q)_\infty = \lim_{n\to+\infty}(a;q)_n = \prod_{k=0}^{\infty}(1-aq^k). \tag{1.4}$$

We also denote

$$[a]_q = \frac{1-q^a}{1-q}, \qquad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \qquad n \in \mathbb{N}.$$

The q-derivative $D_q f$ of a function f is given by [6]:

$$(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}, \qquad if \quad x \neq 0, \tag{1.5}$$

$(D_q f)(0) = f^{'}(0)$ provided $f^{'}(0)$ exists.
If f is differentiable, then $(D_q f)(x)$ tend to $f^{'}(x)$ as q tends to 1.
The q-Jackson integral on $[0, b]$ is defined by [5] as:

$$\int_0^b f(t)d_qt = (1-q)b\sum_{k=0}^{\infty} f(bq^k)q^k,$$

provided the sum converge absolutely.
The q-Jackson integral in a generic interval $[a, b]$ is given by [5]

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt.$$

A q-analogue of the integration by parts formula is given by

$$\int_a^b g(x)D_qf(x)d_qx = f(b)g(b) - f(a)g(a) - \int_a^b f(qx)D_qg(x)d_qx. \tag{1.6}$$

In [13], the authors presented a Riemann-type q-integral by:

$$\begin{aligned}\int_a^b f(x)d_q^R x &= (1-q)(b-a)\sum_{k=0}^{\infty} f(a+(b-a)q^k)\\ &= (b-a)\int_0^1 f(a+(b-a)t)d_qt.\end{aligned}$$

Definition 1.6 ([12]). *Let $f \in L[a, b]$. The Riemann-Liouville q-integrals $J^{\alpha}_{q,a}f$ and $J^{\alpha}_{q,b}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{q,a}f(x) = \frac{1}{\Gamma_q(\alpha)}\int_a^x (x-qt)_q^{(\alpha-1)}f(t)d_q^R t,\ x > a \tag{1.7}$$

and

$$J^{\alpha}_{q,b}f(x) = \frac{1}{\Gamma_q(\alpha)}\int_x^b (t-x)_q^{(\alpha-1)}f(t)d_q^R t,\ b > x, \tag{1.8}$$

where $\Gamma_q(\alpha) = \frac{1}{1-q}\int_0^1 \left(\frac{u}{1-q}\right)^{\alpha-1} e_q(qu)d_qu,\quad e_q(t) = \prod_{k=0}^{\infty}(1-q^kt)$ *and* $\Gamma_q(\alpha+1) = [\alpha]_q\Gamma_q(\alpha)$.

2. Main Results

In this section, we derive our main results.

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function and q-integrable, then*

$$\begin{aligned}&J^{\alpha}_{q,a}f(b) + J^{\alpha}_{q,b}f(a)\\ &\leq \frac{1}{\Gamma_q(\alpha+1)}\left(((b-a)^{\alpha} - (a-aq)_q^{(\alpha)})f(b) + (b-aq)_q^{(\alpha)}f(a)\right)\\ &+ \frac{q(f(b)-f(a))}{(b-a)\Gamma_q(\alpha+2)}\left(-\left(b-qa-\frac{b-a}{q}\right)_q^{(\alpha+1)} + (b-qa)_q^{(\alpha+1)} - \frac{(b-a)^{\alpha+1}}{q}\right).\end{aligned}$$

Proof.

$$\begin{aligned}
&J^{\alpha}_{q,a}f(b)+J^{\alpha}_{q,b}f(a)\\
&=\frac{1}{\Gamma_q(\alpha)}\left(\int_a^b (b-qt)_q^{(\alpha-1)}f(t)d_q^R t+\int_a^b (t-a)_q^{(\alpha-1)}f(t)d_q^R t\right)\\
&=\frac{b-a}{\Gamma_q(\alpha)}\int_0^1\left((b-q(a+(b-a)t))_q^{(\alpha-1)}+(a+(b-a)t-a)_q^{(\alpha-1)}\right)f(a+(b-a)t)d_qt\\
&=\frac{(b-a)^{\alpha}}{\Gamma_q(\alpha)}\left(\int_0^1\left((\frac{b-qa}{b-a}-qt)_q^{(\alpha-1)}+(t)_q^{(\alpha-1)}\right)f(a+(b-a)t)d_qt\right).
\end{aligned}$$

By convexity of f, we get

$$\begin{aligned}
&J^{\alpha}_{q,a}f(b)+J^{\alpha}_{q,b}f(a)\\
&\leq\frac{(b-a)^{\alpha}}{\Gamma_q(\alpha)}\int_0^1\left(\left(\frac{b-qa}{b-a}-qt\right)_q^{(\alpha-1)}+(t)_q^{(\alpha-1)}\right)\left((1-t)f(a)+tf(b)\right)d_qt\\
&\leq\frac{(b-a)^{\alpha}}{\Gamma_q(\alpha+1)}\int_0^1\left(-D_q\left(\frac{b-qa}{b-a}-t\right)_q^{(\alpha)}+D_q(t)_q^{(\alpha)}\right)\left((1-t)f(a)+tf(b)\right)d_qt.
\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}
&J^{\alpha}_{q,a}f(b)+J^{\alpha}_{q,b}f(a)\\
&\leq\frac{(b-a)^{\alpha}}{\Gamma_q(\alpha+1)}\int_0^1\left(-D_q\left(\frac{b-qa}{b-a}-t\right)_q^{(\alpha)}+D_q(t)_q^{(\alpha)}\right)((1-t)f(a)+tf(b))d_qt\\
&\leq\frac{(b-a)^{\alpha}}{\Gamma_q(\alpha+1)}\left(\left(1-\left(\frac{a-aq}{b-a}\right)_q^{(\alpha)}\right)f(b)+(\frac{b-aq}{b-a})_q^{(\alpha)}f(a)\right)\\
&+\frac{(b-a)^{\alpha}}{\Gamma_q(\alpha+1)}(f(b)-f(a))\int_0^1\left(\left(\frac{b-qa}{b-a}-t\right)_q^{(\alpha)}-(t)_q^{(\alpha)}\right)d_qt\\
&=\frac{1}{\Gamma_q(\alpha+1)}\left(((b-a)^{\alpha}-(a-aq)_q^{(\alpha)})f(b)+(b-aq)_q^{(\alpha)}f(a)\right)\\
&+\frac{q(f(b)-f(a))}{(b-a)\Gamma_q(\alpha+2)}\left(-\left(b-qa-\frac{b-a}{q}\right)_q^{(\alpha+1)}+(b-qa)_q^{(\alpha+1)}-\frac{(b-a)^{\alpha+1}}{q}\right).
\end{aligned}$$

This completes the proof. □

Lemma 2.2. *Let $p:[a,b]\subset\mathbb{R}_+\to\mathbb{R}$ be a nonnegative, q-integrable and symmetric about $\frac{a+b}{2}$, then we have*

$$J^{\alpha}_{q,b}p(a)=\frac{1}{\Gamma_q(\alpha)}\int_a^b (b-t)_q^{(\alpha-1)}p(t)d_q^R t.$$

Proof.

$$\begin{aligned}
J^{\alpha}_{q,b}p(a)&=\frac{1}{\Gamma_q(\alpha)}\int_a^b (t-a)_q^{(\alpha-1)}p(t)d_q^R t\\
&=\frac{1}{\Gamma_q(\alpha)}\int_a^b (t-a)_q^{(\alpha-1)}p(a+b-t)d_q^R t\\
&=\frac{1}{\Gamma_q(\alpha)}\int_a^b (b-t)_q^{(\alpha-1)}p(t)d_q^R t.
\end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a convex function q-integrable and $p : [a, b] \to \mathbb{R}$ be a non- negative, q-integrable and symmetric about $x = \frac{a+b}{2}$. Then we have*

$$f\left(\frac{a+b}{2}\right) J^{\alpha}_{q,b} p(a) \leq 2 J^{\alpha}_{q,b}(fp)(a) \leq (f(a) + f(b)) J^{\alpha}_{q,b}(p)(a) \tag{2.1}$$

Proof. Since f is convex and p is nonnegative, q-integrable and symmetric about $x = \frac{a+b}{2}$, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)(t)_q^{(\alpha-1)} p(a + (b-a)t) \\ \leq \;& (t)_q^{(\alpha-1)} f(a + (b-a)t) p(a + (b-a)t) + (t)_q^{(\alpha-1)} f(b + (a-b)t) p(a + (b-a)t) \\ = \;& (t)_q^{(\alpha-1)} f(a + (b-a)t) p(a + (b-a)t) + (t)_q^{(\alpha-1)} f(b + (a-b)t) p(a + b - a - (b-a)t). \end{aligned}$$

Integrating with respect to t on $[0, 1]$, we get

$$\begin{aligned} & \frac{1}{(b-a)^{\alpha}} f\left(\frac{a+b}{2}\right) \int_a^b (x-a)_q^{(\alpha-1)} p(x) d_q^R x \\ \leq \;& \frac{1}{(b-a)^{\alpha}} \int_a^b (x-a)_q^{(\alpha-1)} f(x) p(x) d_q^R x + \frac{1}{(b-a)^{\alpha}} \int_a^b (b-x)_q^{(\alpha-1)} f(x) p(x) d_q^R x \end{aligned}$$

Therefore, by Lemma 2.2, we have

$$\begin{aligned} & \frac{1}{(b-a)^{\alpha}} f\left(\frac{a+b}{2}\right) \int_a^b (x-a)_q^{(\alpha-1)} p(x) d_q^R x \\ \leq \;& \frac{2}{(b-a)^{\alpha}} \int_a^b (x-a)_q^{(\alpha-1)} f(x) p(x) d_q^R x \end{aligned}$$

the first inequality is proved.
For the proof of the second inequality in (2.1) we first note that if f is a convex function, then, for all $t \in [0, 1]$, it yields

$$f(ta + (1-t)b) + f(tb + (1-t)a) \leq f(a) + f(b) \tag{2.2}$$

Then multiplying both sides of (2.2) by $\frac{(b-a)^{\alpha}}{\Gamma_q(\alpha)}(t)_q^{\alpha} p(a + (b-a)t)$ and q-integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned} & \frac{(b-a)^{\alpha}}{\Gamma_q(\alpha)} \int_0^1 (t)_q^{(\alpha-1)} f(a + (b-a)t) p(a + (b-a)t) d_q t \\ & + \frac{(b-a)^{\alpha}}{\Gamma_q(\alpha)} \int_0^1 (t)_q^{(\alpha-1)} f(b + (a-b)t) p(a + (b-a)t) d_q t \\ \leq \;& \frac{(b-a)^{\alpha}}{\Gamma_q(\alpha)} (f(a) + f(b)) \int_0^1 (t)_q^{(\alpha-1)} p(a + (b-a)t) d_q t \end{aligned}$$

Then

$$\begin{aligned} & \frac{1}{\Gamma_q(\alpha)} \int_a^b (x-a)_q^{(\alpha-1)} f(x) p(x) d_q^R x + \frac{1}{\Gamma_q(\alpha)} \int_a^b (b-x)_q^{(\alpha-1)} f(x) p(x) d_q^R x \\ \leq \;& (f(a) + f(b)) \frac{1}{\Gamma_q(\alpha)} \int_a^b (x-a)_q^{(\alpha-1)} p(x) d_q^R x. \end{aligned}$$

The proof is completed. □

Theorem 2.4. *Let $f : K_g = [a, g(b)] \subset \mathbb{R}_+ \to \mathbb{R}$ be a relative convex with respect to $g : H \to H$ and q-integrable, then*

$$\begin{aligned}
&J_{q,a}^{\alpha} f(g(b)) + J_{q,g(b)}^{\alpha} f(a)\\
&\leq \frac{(g(b)-a)^{\alpha}}{\Gamma_q(\alpha+1)}\left(\left(-\left(\frac{a-qa}{g(b)-a}\right)_q^{(\alpha)}+1\right)f(g(b))+\left(\frac{g(b)-qa}{g(b)-a}\right)_q^{(\alpha)}f(a)\right)\\
&+\frac{(g(b)-a)^{\alpha}}{\Gamma_q(\alpha+2)}\left(-q\left(\frac{g(b)-qa}{g(b)-a}-\frac{1}{q}\right)_q^{(\alpha+1)}+q\left(\frac{g(b)-qa}{g(b)-a}\right)_q^{(\alpha+1)}-1\right)(-f(a)+f(g(b))).
\end{aligned}$$

Proof. Now

$$\begin{aligned}
&J_{q,a}^{\alpha} f(g(b)) + J_{q,g(b)}^{\alpha} f(a)\\
&=\frac{1}{\Gamma_q(\alpha)}\left(\int_a^{g(b)}(g(b)-qt)_q^{(\alpha-1)}f(t)d_q^R t+\int_a^{g(b)}(t-a)_q^{(\alpha-1)}f(t)d_q^R t\right)\\
&=\frac{g(b)-a}{\Gamma_q(\alpha)}\left(\int_0^1(g(b)-q(a+(g(b)-a)t))_q^{(\alpha-1)}+((g(b)-a)t)_q^{(\alpha-1)}\right)f(a+(g(b)-a)t)d_qt\\
&=\frac{(g(b)-a)^{\alpha}}{\Gamma_q(\alpha)}\left(\int_0^1\left(\frac{g(b)-qa}{g(b)-a}-qt\right)_q^{(\alpha-1)}+(t)_q^{(\alpha-1)}\right)f(a+(g(b)-a)t)d_qt.
\end{aligned}$$

Since f is relative convex, we obtain

$$\begin{aligned}
&J_{q,a}^{\alpha} f(g(b)) + J_{q,g(b)}^{\alpha} f(a)\\
&\leq\frac{(g(b)-a)^{\alpha}}{\Gamma_q(\alpha)}\left(\int_0^1\left(\frac{g(b)-qa}{g(b)-a}-qt\right)_q^{(\alpha-1)}+(t)_q^{(\alpha-1)}\right)((1-t)f(a)+tf(g(b)))\,d_qt.
\end{aligned}$$

Using q-integration by parts, we have

$$\begin{aligned}
&J_{q,a}^{\alpha} f(g(b)) + J_{q,g(b)}^{\alpha} f(a)\\
&\leq\frac{(g(b)-a)^{\alpha}}{\Gamma_q(\alpha+1)}\int_0^1\left(-D_q\left(\frac{g(b)-qa}{g(b)-a}-t\right)_q^{(\alpha)}+D_q(t)_q^{(\alpha)}\right)((1-t)f(a)+tf(g(b)))\,d_qt\\
&=\frac{(g(b)-a)^{\alpha}}{\Gamma_q(\alpha+1)}\left(\left(-\left(\frac{g(b)-qa}{g(b)-a}-1\right)_q^{(\alpha)}+(1)_q^{(\alpha)}\right)f(g(b))+\left(\frac{g(b)-qa}{g(b)-a}\right)_q^{(\alpha)}f(a)\right)\\
&+\frac{(g(b)-a)^{\alpha}}{\Gamma_q(\alpha+1)}\int_0^1\left(\left(\frac{g(b)-qa}{g(b)-a}-t\right)_q^{(\alpha)}-(t)_q^{(\alpha)}\right)(-f(a)+f(g(b)))\,d_qt\\
&=\frac{(g(b)-a)^{\alpha}}{\Gamma_q(\alpha+1)}\left(\left(-\left(\frac{a-qa}{g(b)-a}\right)_q^{(\alpha)}+1\right)f(g(b))+\left(\frac{g(b)-qa}{g(b)-a}\right)_q^{(\alpha)}f(a)\right)\\
&+\frac{(g(b)-a)^{\alpha}}{\Gamma_q(\alpha+2)}\left(-q\left(\frac{g(b)-qa}{g(b)-a}-\frac{1}{q}\right)_q^{(\alpha+1)}+q\left(\frac{g(b)-qa}{g(b)-a}\right)_q^{(\alpha+1)}-1\right)(-f(a)+f(g(b))).
\end{aligned}$$

This completes the proof. □

Theorem 2.5. *Let $f : K_g = [a, g(b)] \subset \mathbb{R}_+ \to \mathbb{R}$ be a relative h-convex with respect to two functions $h : (0, 1) \to$*

$(0,\infty)$ *and* $g : H \to H$ *and q-integrable, then*

$$
\begin{aligned}
&J^{\alpha}_{q,a} f(g(b)) + J^{\alpha}_{q,g(b)} f(a)\\
&\leq \left(-\left(\frac{g(b)-qa}{g(b)-a}-1\right)^{(\alpha)}_q + 1\right)(h(0)R_\alpha(f(a)) + h(1)R_\alpha(f(g(b))))\\
&+\left(\frac{g(b)-qa}{g(b)-a}\right)^{(\alpha)}_q (h(1)R_\alpha(f(a)) + h(0)R_\alpha(f(g(b))))\\
&+ S_\alpha(f(a)) \int_a^{g(b)} ((1-q)g(b) - aq + qt)^{(\alpha)}_q D_q h\left(\frac{t-a}{g(b)-a}\right) d^R_q t\\
&- R_\alpha(f(a)) \int_O^1 (1-t)^{(\alpha)}_q D_q h(t) d^R_q t + S_\alpha(f(g(b))) \int_a^{g(b)} (g(b)-qt)^{(\alpha)}_q D_q h\left(\frac{t-a}{g(b)-a}\right) d^R_q t\\
&- R_\alpha(f(g(b))) \int_O^1 (t)^{(\alpha)}_q D_q h(t) d^R_q t
\end{aligned}
$$

where $S_\alpha(f(a)) = \frac{f(a)}{(g(b)-a)\Gamma_q(\alpha+1)}$ *and* $R_\alpha(f(a)) = \frac{q^\alpha f(a)(g(b)-a)^\alpha}{\Gamma_q(\alpha+1)}$.

Proof. Since f is relative h-convex with respect to two functions h and g, then we have

$$
\begin{aligned}
&J^{\alpha}_{q,a} f(g(b)) + J^{\alpha}_{q,g(b)} f(a)\\
&\leq \frac{(g(b)-a)^\alpha}{\Gamma_q(\alpha+1)} \int_0^1 \left(-D_q(\frac{g(b)-qa}{g(b)-a} - t)^{(\alpha)}_q + D_q(t)^{(\alpha)}_q\right)(h(1-t)f(a) + h(t)f(g(b)))\, d_q t\\
&\leq \frac{(g(b)-a)^\alpha}{\Gamma_q(\alpha+1)} \left(-\left(\frac{g(b)-qa}{g(b)-a}-1\right)^{(\alpha)}_q + 1\right)(h(0)f(a) + h(1)f(g(b)))\\
&+ \frac{(g(b)-a)^\alpha}{\Gamma_q(\alpha+1)} \left(\frac{g(b)-qa}{g(b)-a}\right)^{(\alpha)}_q (h(1)f(a) + h(0)f(g(b)))\\
&+ \frac{(g(b)-a)^\alpha}{\Gamma_q(\alpha+1)} \int_0^1 \left((\frac{g(b)-qa}{g(b)-a} - qt)^{(\alpha)}_q - (qt)^{(\alpha)}_q\right)\Big(D_q(h(1-t))f(a) + D_q h(t) f(g(b))\Big)\, d_q t\\
&\leq \left(-\left(\frac{g(b)-qa}{g(b)-a}-1\right)^{(\alpha)}_q + 1\right)(h(0)R_\alpha(f(a)) + h(1)R_\alpha(f(g(b))))\\
&+\left(\frac{g(b)-qa}{g(b)-a}\right)^{(\alpha)}_q (h(1)R_\alpha(f(a)) + h(0)R_\alpha(f(g(b))))\\
&+ S_\alpha(f(a)) \int_a^{g(b)} ((1-q)g(b) - aq + qt)^{(\alpha)}_q D_q h\left(\frac{t-a}{g(b)-a}\right) d^R_q t\\
&- R_\alpha(f(a)) \int_O^1 (1-t)^{(\alpha)}_q D_q h(t) d^R_q t + S_\alpha(f(g(b))) \int_a^{g(b)} (g(b)-qt)^{(\alpha)}_q D_q h\left(\frac{t-a}{g(b)-a}\right) d^R_q t\\
&- R_\alpha(f(g(b))) \int_O^1 (t)^{(\alpha)}_q D_q h(t) d^R_q t
\end{aligned}
$$

The proof is completed. □

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