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Monotone Relations in Hadamard Spaces

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Abstract. In this paper, the notion of $\mathcal{W}$-property for subsets of $X \times X^{\diamond}$ is introduced and investigated, where X is an Hadamard space and $X^{\diamond}$ is its linear dual space. It is shown that an Hadamard space X is flat if and only if $X \times X^{\diamond}$ has $\mathcal{W}$-property. Moreover, the notion of monotone relation from an Hadamard space to its linear dual space is introduced. A characterization result for monotone relations with $\mathcal{W}$-property (and hence in flat Hadamard spaces) is given. Finally, a type of Debrunner-Flor Lemma concerning extension of monotone relations in Hadamard spaces is proved.

1. Introduction and Preliminaries

Let (X, d) be a metric space. We say that a mapping $c : [0, 1] \to X$ is a *geodesic path* from $x \in X$ to $y \in X$ if $c(0) = x$, $c(1) = y$ and $d(c(t), c(s)) = |t - s| d(x, y)$, for each $t, s \in [0, 1]$. The image of c is said to be a *geodesic segment* joining x and y. A metric space (X, d) is called a *geodesic space* if there is a geodesic path between every two points of X. Also, a geodesic space X is called *uniquely geodesic space* if for each $x, y \in X$ there exists a unique geodesic path from x to y. From now on, in a uniquely geodesic space, we denote the set $c([0, 1])$ by $[x, y]$ and for each $z \in [x, y]$, we write $z = (1 - t)x \oplus ty$, where $t \in [0, 1]$. In this case, we say that z is a *convex combination* of x and y. Hence, $[x, y] = \{(1 - t)x \oplus ty : t \in [0, 1]\}$. More details can be found in [3, 5].

Definition 1.1. [9, Definition 2.2] Let (X, d) be a geodesic space, $v_1, v_2, v_3, \dots, v_n$ be n points in X and $\{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\} \subseteq (0, 1)$ be such that $\sum_{i=1}^{n} \lambda_i = 1$. We define *convex combination* of $\{v_1, v_2, v_3, \dots, v_n\}$ inductively as following:

$$\bigoplus_{i=1}^{n} \lambda_i v_i := (1 - \lambda_n)\Big(\frac{\lambda_1}{1 - \lambda_n} v_1 \oplus \frac{\lambda_2}{1 - \lambda_n} v_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1 - \lambda_n} v_n\Big) \oplus \lambda_n v_n. \tag{1}$$

Note that for every $x \in X$, we have $d\Big(x, \bigoplus_{i=1}^{n} \lambda_i v_i\Big) \le \sum_{i=1}^{n} \lambda_i d(x, v_i)$.

According to [3, Definition 1.2.1], a geodesic space (X, d) is a CAT(0) *space*, if the following condition, so-called *CN-inequality*, holds:

$$d(z, (1 - \lambda)x \oplus \lambda y)^2 \le (1 - \lambda) d(z, x)^2 + \lambda d(z, y)^2 - \lambda(1 - \lambda) d(x, y)^2 \text{ for all } x, y, z \in X, \lambda \in [0, 1]. \tag{2}$$

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One can show that (for instance see [3, Theorem 1.3.3]) CAT(0) spaces are uniquely geodesic spaces. An *Hadamard space* is a complete CAT(0) space.

Let X be an Hadamard space. For each $x, y \in X$, the ordered pair (x, y) is called a *bound vector* and is denoted by $\overrightarrow{xy}$. Indeed, $X^2 = \{\overrightarrow{xy} : x, y \in X\}$. For each $x \in X$, we apply $\mathbf{0}_x := \overrightarrow{xx}$ as *zero bound vector* at x and $-\overrightarrow{xy}$ as the bound vector $\overrightarrow{yx}$. The bound vectors $\overrightarrow{xy}$ and $\overrightarrow{uz}$ are called *admissible* if $y = u$. Therefore the sum of two admissible bound vectors $\overrightarrow{xy}$ and $\overrightarrow{yz}$ is defined by $\overrightarrow{xy} + \overrightarrow{yz} = \overrightarrow{xz}$. Ahmadi Kakavandi and Amini in [2] have introduced the *dual space* of an Hadamard space, by using the concept of quasilinearization of abstract metric spaces presented by Berg and Nikolaev in [4]. The *quasilinearization map* is defined as following:

$$\begin{aligned} &\langle \cdot, \cdot \rangle : X^2 \times X^2 \to \mathbb{R} \\ &\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle := \frac{1}{2}\Big\{d(a,d)^2 + d(b,c)^2 - d(a,c)^2 - d(b,d)^2\Big\};\ a, b, c, d \in X. \end{aligned} \tag{3}$$

Let $x,\ y \in X$, we define the mapping $\varphi_{\overrightarrow{xy}} : X \to \mathbb{R}$ by $\varphi_{\overrightarrow{xy}}(z) = \frac{1}{2}(d(x,z)^2 - d(y,z)^2)$; for each $z \in X$. We will see that $\varphi_{\overrightarrow{xy}}$ possess attractive properties that simplify some calculations. We observe that (3) can be rewritten as following:

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \varphi_{\overrightarrow{cd}}(b) - \varphi_{\overrightarrow{cd}}(a) = \varphi_{\overrightarrow{ab}}(d) - \varphi_{\overrightarrow{ab}}(c).$$

The metric space (X, d) satisfies the *Cauchy-Schwarz inequality* if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \le d(a,b)d(c,d) \text{ for all } a, b, c, d \in X.$$

This inequality characterizes CAT(0) spaces. Indeed, it follows from [4, Corollary 3] that a geodesic space (X, d) is a CAT(0) space if and only if it satisfies in the Cauchy-Schwarz inequality. For an Hadamard space (X, d), consider the mapping

$$\begin{aligned} &\Psi : \mathbb{R} \times X^2 \to C(X, \mathbb{R}) \\ &\quad (t, a, b) \mapsto \Psi(t, a, b)x = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle;\ a, b, x \in X, t \in \mathbb{R}, \end{aligned}$$

where $C(X, \mathbb{R})$ denotes the space of all continuous real-valued functions on X. It follows from Cauchy-Schwarz inequality that $\Psi(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm

$$L(\Psi(t, a, b)) = |t|d(a, b), \text{ for all } a, b \in X, \text{ and all } t \in \mathbb{R}, \tag{4}$$

where the *Lipschitz semi-norm* for any function $\varphi : (X, d) \to \mathbb{R}$ is defined by

$$L(\varphi) = \sup\left\{\frac{\varphi(x) - \varphi(y)}{d(x, y)} : x, y \in X, x \neq y\right\}.$$

A *pseudometric D on* $\mathbb{R} \times X^2$ *induced by the Lipschitz semi-norm* (4), is defined by

$$D((t, a, b), (s, c, d)) = L(\Psi(t, a, b) - \Psi(s, c, d));\ a, b, c, d \in X, t, s \in \mathbb{R}.$$

For an Hadamard space (X, d), the pseudometric space $(\mathbb{R} \times X^2, D)$ can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions $\mathrm{Lip}(X, \mathbb{R})$. Note that, in view of [2, Lemma 2.1], $D((t, a, b), (s, c, d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$ for all $x, y \in X$. Thus, D induces an equivalence relation on $\mathbb{R} \times X^2$, where the equivalence class of $(t, a, b) \in \mathbb{R} \times X^2$ is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : s \in \mathbb{R}, c, d \in X, D((t, a, b), (s, c, d)) = 0\}.$$

The *dual space* of an Hadamard space (X, d), denoted by X^*, is the set of all equivalence classes $[t\overrightarrow{ab}]$ where $(t, a, b) \in \mathbb{R} \times X^2$, with the metric $D([t\overrightarrow{ab}], [s\overrightarrow{cd}]) := D((t, a, b), (s, c, d))$. Clearly, the definition of equivalence classes implies that $[\overrightarrow{aa}] = [\overrightarrow{bb}]$ for all $a, b \in X$. The *zero element* of X^* is $\mathbf{0} := [t\overrightarrow{aa}]$, where $a \in X$ and $t \in \mathbb{R}$ are arbitrary. It is easy to see that the evaluation $\langle \mathbf{0}, \cdot \rangle$ vanishes for any bound vectors in X^2. Note that in general X^* acts on X^2 by

$$\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle, \text{ where } x^* = [t\overrightarrow{ab}] \in X^* \text{ and } \overrightarrow{xy} \in X^2.$$

The following notation will be used throughout this paper.

$$\Big\langle \sum_{i=1}^{n} \alpha_i x_i^*, \overrightarrow{xy} \Big\rangle := \sum_{i=1}^{n} \alpha_i \langle x_i^*, \overrightarrow{xy} \rangle, \ \alpha_i \in \mathbb{R}, \ x_i^* \in X^*, \ n \in \mathbb{N}, \ x, y \in X.$$

For an Hadamard space (X, d), Chaipunya and Kumam in [7], defined the *linear dual space* of X by

$$X^\diamond = \Big\{ \sum_{i=1}^{n} \alpha_i x_i^* : \alpha_i \in \mathbb{R}, \ x_i^* \in X^*, \ n \in \mathbb{N} \Big\}.$$

Therefore, $X^\diamond = \operatorname{span} X^*$. It is easy to see that $X^\diamond$ is a normed space with the norm $\|x^\diamond\|_\diamond = L(x^\diamond)$ for all $x^\diamond \in X^\diamond$. Indeed:

Lemma 1.2. [14, Proposition 3.5] *Let X be an Hadamard space with linear dual space $X^\diamond$. Then*

$$\|x^\diamond\|_\diamond := \sup \left\{ \frac{\left| \langle x^\diamond, \overrightarrow{ab} \rangle - \langle x^\diamond, \overrightarrow{cd} \rangle \right|}{d(a, b) + d(c, d)} : a, b, c, d \in X, (a, c) \neq (b, d) \right\},$$

is a norm on $X^\diamond$. In particular, $\|[t\overrightarrow{ab}]\|_\diamond = |t| d(a, b)$.

2. Flat Hadamard Spaces and $\mathcal{W}$-property

Let M be a relation from X to $X^\diamond$; i.e., $M \subseteq X \times X^\diamond$. The *domain* and *range* of M are defined, respectively, by

$$\operatorname{Dom}(M) := \Big\{ x \in X : \exists\, x^\diamond \in X^\diamond \text{ such that } (x, x^\diamond) \in M \Big\},$$

and

$$\operatorname{Range}(M) := \Big\{ x^\diamond \in X^\diamond : \exists\, x \in X \text{ such that } (x, x^\diamond) \in M \Big\}.$$

Definition 2.1. Let X be an Hadamard space with linear dual space $X^\diamond$. We say that $M \subseteq X \times X^\diamond$ satisfies *$\mathcal{W}$-property* if there exists $p \in X$ such that the following holds:

$$\Big\langle x^\diamond, \overrightarrow{p((1-\lambda)x_1 \oplus \lambda x_2)} \Big\rangle \leq (1-\lambda)\langle x^\diamond, \overrightarrow{px_1} \rangle + \lambda \langle x^\diamond, \overrightarrow{px_2} \rangle, \ \forall \lambda \in [0, 1], \ \forall x^\diamond \in \operatorname{Range}(M), \ \forall x_1, x_2 \in \operatorname{Dom}(M).$$

Proposition 2.2. *Let X be an Hadamard space with linear dual space $X^\diamond$ and let $M \subseteq X \times X^\diamond$. Then the following statements are equivalent:*

(i) *$M \subseteq X \times X^\diamond$ satisfies the $\mathcal{W}$-property for some $p \in X$.*

(ii) *$M \subseteq X \times X^\diamond$ satisfies the $\mathcal{W}$-property for any $q \in X$.*

(iii) *For any $q \in X$,*

$$\Big\langle x^\diamond, \overrightarrow{q(\oplus_{i=1}^{n} \lambda_i x_i)} \Big\rangle \leq \sum_{i=1}^{n} \lambda_i \langle x^\diamond, \overrightarrow{qx_i} \rangle, \ \textit{for all } x^\diamond \in \operatorname{Range}(M), \{x_i\}_{i=1}^{n} \subseteq \operatorname{Dom}(M), \{\lambda_i\}_{i=1}^{n} \subseteq [0, 1]. \quad (\mathcal{W}_n(q))$$

(iv) *For some $p \in X$, $(\mathcal{W}_n(p))$ holds.*

Proof.

(i) ⇒ (ii): Let $q \in X$ be any arbitrary element of X, $\lambda \in [0,1]$, $x^\diamond \in \text{Range}(M)$, and $x_1, x_2 \in \text{Dom}(M)$. Then

$$\begin{aligned}\left\langle x^\diamond, \overrightarrow{q((1-\lambda)x_1 \oplus \lambda x_2)}\right\rangle &= \left\langle x^\diamond, \overrightarrow{qp} + \overrightarrow{p((1-\lambda)x_1 \oplus \lambda x_2)}\right\rangle \\ &= \langle x^\diamond, \overrightarrow{qp}\rangle + \left\langle x^\diamond, \overrightarrow{p((1-\lambda)x_1 \oplus \lambda x_2)}\right\rangle \\ &\leq (1-\lambda)(\langle x^\diamond, \overrightarrow{qp}\rangle + \langle x^\diamond, \overrightarrow{px_1}\rangle) + \lambda(\langle x^\diamond, \overrightarrow{qp}\rangle + \langle x^\diamond, \overrightarrow{px_2}\rangle) \\ &= (1-\lambda)\langle x^\diamond, \overrightarrow{qp} + \overrightarrow{px_1}\rangle + \lambda\langle x^\diamond, \overrightarrow{qp} + \overrightarrow{px_2}\rangle \\ &= (1-\lambda)\langle x^\diamond, \overrightarrow{qx_1}\rangle + \lambda\langle x^\diamond, \overrightarrow{qx_2}\rangle,\end{aligned}$$

as required.

(ii) ⇒ (iii): We proceed by induction on n. By Definition 2.1 the claim is true for $n = 2$. Now assume that $(\mathcal{W}_{n-1}(q))$ is true. In view of equation (1),

$$\begin{aligned}\left\langle x^\diamond, \overrightarrow{q(\oplus_{i=1}^n \lambda_i x_i)}\right\rangle &= \left\langle x^\diamond, \overrightarrow{q((1-\lambda_n)\Big(\frac{\lambda_1}{1-\lambda_n}x_1 \oplus \frac{\lambda_2}{1-\lambda_n}x_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1-\lambda_n}x_{n-1}\Big) \oplus \lambda_n x_n)}\right\rangle \\ &\leq (1-\lambda_n)\left\langle x^\diamond, \overrightarrow{q\Big(\frac{\lambda_1}{1-\lambda_n}x_1 \oplus \frac{\lambda_2}{1-\lambda_n}x_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1-\lambda_n}x_{n-1}\Big)}\right\rangle + \lambda_n\langle x^\diamond, \overrightarrow{qx_n}\rangle \\ &\leq (1-\lambda_n)\sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_n}\langle x^\diamond, \overrightarrow{qx_i}\rangle + \lambda_n\langle x^\diamond, \overrightarrow{qx_n}\rangle \\ &= \sum_{i=1}^{n-1} \lambda_i\langle x^\diamond, \overrightarrow{qx_i}\rangle + \lambda_n\langle x^\diamond, \overrightarrow{qx_n}\rangle \\ &= \sum_{i=1}^{n} \lambda_i\langle x^\diamond, \overrightarrow{qx_i}\rangle.\end{aligned}$$

(iii) ⇒ (iv): Clear.

(iv) ⇒ (i): Take $n = 2$ in $(\mathcal{W}_n(p))$.

We are done. □

Remark 2.3. *It should be noticed that Proposition 2.2 implies that $\mathcal{W}$-property is independent of the choice of the element $p \in X$.*

Definition 2.4. [11, Definition 3.1] An Hadamard space (X, d) is said to be *flat* if equality holds in the CN-inequality, i.e., for each $x, y \in X$ and $\lambda \in [0,1]$, the following holds:

$$d(z, (1-\lambda)x \oplus \lambda y)^2 = (1-\lambda)d(z,x)^2 + \lambda d(z,y)^2 - \lambda(1-\lambda)d(x,y)^2, \textit{ for all } z \in X.$$

Proposition 2.5. *Let X be an Hadamard space. The following statements are equivalent:*

(i) *X is a flat Hadamard space.*

(ii) $\langle \overrightarrow{x((1-\lambda)x \oplus \lambda y)}, \overrightarrow{ab}\rangle = \lambda\langle \overrightarrow{xy}, \overrightarrow{ab}\rangle$, *for all $a, b, x, y \in X$ and all $\lambda \in [0,1]$.*

(iii) *$X \times X^\diamond$ has $\mathcal{W}$-property.*

(iv) *Any subset of $X \times X^\diamond$ has $\mathcal{W}$-property.*

(v) *For each $p, z \in X$, the mapping $\varphi_{\overrightarrow{pz}}$ is convex.*

(vi) *For each $p, z \in X$, the mapping $\varphi_{\overrightarrow{pz}}$ is affine, in the sense that:*

$$\varphi_{\overrightarrow{pz}}((1-\lambda)x \oplus \lambda y) = (1-\lambda)\varphi_{\overrightarrow{pz}}(x) + \lambda\varphi_{\overrightarrow{pz}}(y),\ \forall\, x, y \in X, \forall\, \lambda \in [0,1].$$

Proof.

(i) $\Leftrightarrow$ (ii): [11, Theorem 3.2].

(ii)$\Rightarrow$ (iii): Let $x, y \in X$, $\lambda \in [0,1]$ and $(x, x^\diamond) \in X \times X^\diamond$. Then $x^\diamond = \sum_{i=1}^n \alpha_i[t_i\overrightarrow{a_ib_i}] \in X^\diamond$, and hence by using (ii) we get:

$$\begin{aligned}
\Big\langle x^\diamond, \overrightarrow{p\big((1-\lambda)x \oplus \lambda y\big)}\Big\rangle &= \sum_{i=1}^n \alpha_i t_i \Big\langle \overrightarrow{a_ib_i}, \overrightarrow{px} + \overrightarrow{x\big((1-\lambda)x \oplus \lambda y\big)}\Big\rangle \\
&= \sum_{i=1}^n \alpha_i t_i \Big(\Big\langle \overrightarrow{a_ib_i}, \overrightarrow{px}\Big\rangle + \Big\langle \overrightarrow{a_ib_i}, \overrightarrow{x\big((1-\lambda)x \oplus \lambda y\big)}\Big\rangle\Big) \\
&= \sum_{i=1}^n \alpha_i t_i \Big(\Big\langle \overrightarrow{a_ib_i}, \overrightarrow{px}\Big\rangle + \lambda\Big\langle \overrightarrow{a_ib_i}, \overrightarrow{xy}\Big\rangle\Big) \\
&= \sum_{i=1}^n \alpha_i t_i \Big(\Big\langle \overrightarrow{a_ib_i}, \overrightarrow{px}\Big\rangle + \lambda\Big\langle \overrightarrow{a_ib_i}, \overrightarrow{py} - \overrightarrow{px}\Big\rangle\Big) \\
&= \sum_{i=1}^n \alpha_i t_i \Big((1-\lambda)\Big\langle \overrightarrow{a_ib_i}, \overrightarrow{px}\Big\rangle + \lambda\Big\langle \overrightarrow{a_ib_i}, \overrightarrow{py}\Big\rangle\Big) \\
&= (1-\lambda)\sum_{i=1}^n \alpha_i t_i \Big\langle \overrightarrow{a_ib_i}, \overrightarrow{px}\Big\rangle + \lambda \sum_{i=1}^n \alpha_i t_i \Big\langle \overrightarrow{a_ib_i}, \overrightarrow{py}\Big\rangle \\
&= (1-\lambda)\Big\langle \sum_{i=1}^n \alpha_i[t_i\overrightarrow{a_ib_i}], \overrightarrow{px}\Big\rangle + \lambda\Big\langle \sum_{i=1}^n \alpha_i[t_i\overrightarrow{a_ib_i}], \overrightarrow{py}\Big\rangle \\
&= (1-\lambda)\langle x^\diamond, \overrightarrow{px}\rangle + \lambda\langle x^\diamond, \overrightarrow{py}\rangle.
\end{aligned}$$

Therefore $X \times X^\diamond$ has $\mathcal{W}$-property.

(iii) $\Leftrightarrow$ (iv): Straightforward.

(iv) $\Rightarrow$ (v): Let $x, y \in X$ and $\lambda \in [0,1]$, then

$$\begin{aligned}
(1-\lambda)\varphi_{\overrightarrow{pz}}(x) + \lambda\varphi_{\overrightarrow{pz}}(y) - \varphi_{\overrightarrow{pz}}((1-\lambda)x \oplus \lambda y) &= \lambda(\varphi_{\overrightarrow{pz}}(y) - \varphi_{\overrightarrow{pz}}(x)) + \varphi_{\overrightarrow{pz}}(x) - \varphi_{\overrightarrow{pz}}((1-\lambda)x \oplus \lambda y) \\
&= \lambda\langle \overrightarrow{pz}, \overrightarrow{xy}\rangle + \langle \overrightarrow{pz}, \overrightarrow{((1-\lambda)x \oplus \lambda y)x}\rangle \\
&= \lambda\langle \overrightarrow{pz}, \overrightarrow{py} - \overrightarrow{px}\rangle + \langle \overrightarrow{pz}, \overrightarrow{px} - \overrightarrow{p((1-\lambda)x \oplus \lambda y)}\rangle \\
&= \lambda\langle \overrightarrow{pz}, \overrightarrow{py}\rangle + (1-\lambda)\langle \overrightarrow{pz}, \overrightarrow{px}\rangle - \langle \overrightarrow{pz}, \overrightarrow{p((1-\lambda)x \oplus \lambda y)}\rangle \\
&\geq 0.
\end{aligned}$$

Therefore, $\varphi_{\overrightarrow{pz}}$ is convex.

(v) $\Rightarrow$ (vi): It is easy.

(vi) ⇒ (iii): Let $x, y, p \in X$, $\lambda \in [0,1]$ and $x^\diamond = \sum_{i=1}^n \alpha_i[t_i\overrightarrow{p_iz_i}] \in X^\diamond$ be given. Then

$$\begin{aligned}
\Big\langle x^\diamond, \overrightarrow{p\big((1-\lambda)x \oplus \lambda y\big)}\Big\rangle &= \Big\langle \sum_{i=1}^n \alpha_i x_i^*, \overrightarrow{p\big((1-\lambda)x \oplus \lambda y\big)}\Big\rangle \\
&= \sum_{i=1}^n \alpha_i t_i \Big\langle \overrightarrow{p_iz_i}, \overrightarrow{p\big((1-\lambda)x \oplus \lambda y\big)}\Big\rangle \\
&= \sum_{i=1}^n \alpha_i t_i \Big(\varphi_{\overrightarrow{p_iz_i}}((1-\lambda)x \oplus \lambda y) - \varphi_{\overrightarrow{p_iz_i}}(p)\Big) \\
&= \sum_{i=1}^n \alpha_i t_i \Big((1-\lambda)\varphi_{\overrightarrow{p_iz_i}}(x) + \lambda\varphi_{\overrightarrow{p_iz_i}}(y) - \varphi_{\overrightarrow{p_iz_i}}(p)\Big) \\
&= \sum_{i=1}^n \alpha_i t_i \Big((1-\lambda)(\varphi_{\overrightarrow{p_iz_i}}(x) - \varphi_{\overrightarrow{p_iz_i}}(p)) + \lambda(\varphi_{\overrightarrow{p_iz_i}}(y) - \varphi_{\overrightarrow{p_iz_i}}(p))\Big) \\
&= \sum_{i=1}^n \alpha_i t_i \Big((1-\lambda)\Big\langle \overrightarrow{p_iz_i}, \overrightarrow{px}\Big\rangle + \lambda\Big\langle \overrightarrow{p_iz_i}, \overrightarrow{py}\Big\rangle\Big) \\
&= (1-\lambda)\sum_{i=1}^n \alpha_i t_i \Big\langle \overrightarrow{p_iz_i}, \overrightarrow{px}\Big\rangle + \lambda \sum_{i=1}^n \alpha_i t_i \Big\langle \overrightarrow{p_iz_i}, \overrightarrow{py}\Big\rangle \\
&= (1-\lambda)\Big\langle \sum_{i=1}^n \alpha_i [t_i\overrightarrow{p_iz_i}], \overrightarrow{px}\Big\rangle + \lambda\Big\langle \sum_{i=1}^n \alpha_i [t_i\overrightarrow{p_iz_i}], \overrightarrow{py}\Big\rangle \\
&= (1-\lambda)\langle x^\diamond, \overrightarrow{px}\rangle + \lambda\langle x^\diamond, \overrightarrow{py}\rangle;
\end{aligned}$$

i.e., $X \times X^\diamond$ has $\mathcal{W}$-property.

(iii)⇒ (ii): For $a, b, x, y \in X$ and $\lambda \in [0,1]$, we have:

$$\begin{aligned}
\lambda\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle - \langle \overrightarrow{ab}, \overrightarrow{x((1-\lambda)x \oplus \lambda y)}\rangle &= \lambda\Big(\langle \overrightarrow{ab}, \overrightarrow{py} - \overrightarrow{px}\rangle\Big) - \langle \overrightarrow{ab}, \overrightarrow{p((1-\lambda)x \oplus \lambda y)} - \overrightarrow{px}\rangle \\
&= \lambda\Big(\langle \overrightarrow{ab}, \overrightarrow{py}\rangle - \langle \overrightarrow{ab}, \overrightarrow{px}\rangle\Big) - \langle \overrightarrow{ab}, \overrightarrow{p((1-\lambda)x \oplus \lambda y)}\rangle + \langle \overrightarrow{ab}, \overrightarrow{px}\rangle \\
&= (1-\lambda)\langle \overrightarrow{ab}, \overrightarrow{px}\rangle + \lambda\langle \overrightarrow{ab}, \overrightarrow{py}\rangle - \langle \overrightarrow{ab}, \overrightarrow{p((1-\lambda)x \oplus \lambda y)}\rangle \\
&= (1-\lambda)\langle x^\diamond, \overrightarrow{px}\rangle + \lambda\langle x^\diamond, \overrightarrow{py}\rangle\rangle - \langle x^\diamond, \overrightarrow{p((1-\lambda)x \oplus \lambda y)},
\end{aligned}$$

where $x^\diamond = [\overrightarrow{ab}] \in X^\diamond$. Since $X \times X^\diamond$ has $\mathcal{W}$-property, one can deduce that:

$$\lambda\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle \geq \langle \overrightarrow{ab}, \overrightarrow{x((1-\lambda)x \oplus \lambda y)}\rangle. \tag{5}$$

Hence, by interchanging the role of a and b in (5), we obtain:

$$\langle \overrightarrow{ab}, \overrightarrow{x((1-\lambda)x \oplus \lambda y)}\rangle \geq \lambda\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle. \tag{6}$$

Finally, (5) and (6) yield:

$$\langle \overrightarrow{ab}, \overrightarrow{p((1-\lambda)x \oplus \lambda xy)}\rangle = \lambda\Big(\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle\Big).$$

We are done. □

The next example shows that there exists a relation $M \subseteq X \times X^\diamond$ in the non-flat Hadamard spaces which doesn't have the $\mathcal{W}$-property.

Example 2.6. *Consider the following equivalence relation on* $\mathbb{N} \times [0,1]$*:*

$$(n,t) \sim (m,s) \Leftrightarrow t = s = 0 \text{ or } (n,t) = (m,s).$$

Set $X := \frac{\mathbb{N}\times[0,1]}{\sim}$ *and let* $d : X \times X \to \mathbb{R}$ *be defined by*

$$d([(n,t)],[(m,s)]) = \begin{cases} |t-s| & n = m, \\ t+s & n \neq m. \end{cases}$$

The geodesic joining $x = [(n,t)]$ *to* $y = [(m,s)]$ *is defined as follows:*

$$(1-\lambda)x \oplus \lambda y := \begin{cases} [(n,(1-\lambda)t - \lambda s)] & 0 \leq \lambda \leq \frac{t}{t+s}, \\ [(m,(\lambda-1)t + \lambda s)] & \frac{t}{t+s} \leq \lambda \leq 1, \end{cases}$$

whenever $x \neq y$ *and vacuously* $(1-\lambda)x \oplus \lambda x := x$*. It is known that (see [1, Example 4.7])* (X,d) *is an* $\mathbb{R}$*-tree space. It follows from [3, Example 1.2.10], that any* $\mathbb{R}$*-tree space is an Hadamard space. Let* $x = [(2,\frac{1}{2})]$*,* $y = [(1,\frac{1}{2})]$*,* $a = [(3,\frac{1}{3})]$*,* $b = [(2,\frac{1}{2})]$ *and* $\lambda = \frac{1}{5}$*. Then* $\frac{4}{5}x \oplus \frac{1}{5}y = [(2,\frac{3}{10})]$ *and*

$$\left\langle \overrightarrow{x(\frac{4}{5}x \oplus \frac{1}{5}y)}, \overrightarrow{ab} \right\rangle = \frac{-1}{6} \neq \frac{-1}{10} = \frac{1}{5}\left\langle \overrightarrow{xy}, \overrightarrow{ab} \right\rangle.$$

Now, Proposition 2.5(ii) implies that (X,d) *is not a flat Hadamard space. For each* $n \in \mathbb{N}$*, set* $x_n := [(n,\frac{1}{2})]$ *and* $y_n := [(n,\frac{1}{n})]$*. Now, we define*

$$M := \left\{ (x_n, [\overrightarrow{y_{n+1}y_n}]) : n \in \mathbb{N} \right\} \subseteq X \times X^\diamond.$$

Take $p = [(1,1)] \in X$*,* $[\overrightarrow{y_5y_4}] \in \text{Range}(M)$ *and* $\lambda = \frac{1}{3}$*. Clearly,* $\tilde{x} := (1-\lambda)x_1 \oplus \lambda x_3 = [(1,\frac{1}{6})]$ *and* $\left\langle [\overrightarrow{y_5y_4}], \overrightarrow{p\tilde{x}} \right\rangle = \frac{1}{24}$*, while,*

$$\frac{2}{3}\langle [\overrightarrow{y_5y_4}], \overrightarrow{px_1} \rangle + \frac{1}{3}\langle [\overrightarrow{y_5y_4}], \overrightarrow{px_3} \rangle = \frac{1}{40}.$$

Therefore, M *doesn't have the* $\mathcal{W}$*-property.*

3. Monotone Relations

Ahmadi Kakavandi and Amini [2] introduced the notion of monotone operators in Hadamard spaces. In [10], Khatibzadeh and Ranjbar, investigated some properties of monotone operators and their resolvents and also proximal point algorithm in Hadamard spaces. Chaipunya and Kumam [7] studied the general proximal point method for finding a zero point of a maximal monotone set-valued vector field defined on Hadamard spaces. They proved the relation between the maximality and Minty's surjectivity condition. Zamani Eskandani and Raeisi [14], by using products of finitely many resolvents of monotone operators, proposed an iterative algorithm for finding a common zero of a finite family of monotone operators and a common fixed point of an infinitely countable family of non-expansive mappings in Hadamard spaces. In this section, we will characterize the notation of monotone relations in Hadamard spaces based on characterization of monotone sets in Banach spaces [8, 12, 13].

Definition 3.1. Let X be an Hadamard space with linear dual space $X^\diamond$. The set $M \subseteq X \times X^\diamond$ is called *monotone* if $\langle x^\diamond - y^\diamond, \overrightarrow{yx} \rangle \geq 0$, for all $(x,x^\diamond)$,$(y,y^\diamond)$ in M.

Example 3.2. *Let x_n, y_n and M be the same as in Example 2.6. Let $(u,u^\diamond),(v,v^\diamond)\in M$. There exists $m,n\in\mathbb{N}$ such that $u=x_n$, $u^\diamond:=[\overrightarrow{y_{n+1}y_n}]$, $v=x_m$ and $v^\diamond:=[\overrightarrow{y_{m+1}y_m}]$. Then*

$$\begin{aligned}\langle u^\diamond-v^\diamond,\overrightarrow{vu}\rangle=\langle u^\diamond,\overrightarrow{vu}\rangle-\langle v^\diamond,\overrightarrow{vu}\rangle&=\Big\langle\Big[\overrightarrow{\big[(n+1,\frac{1}{n+1})\big]\big[(n,\frac{1}{n})\big]}\Big],\overrightarrow{\big[(m,\frac{1}{2})\big]\big[(n,\frac{1}{2})\big]}\Big\rangle\\&\quad-\Big\langle\Big[\overrightarrow{\big[(m+1,\frac{1}{m+1})\big]\big[(m,\frac{1}{m})\big]}\Big],\overrightarrow{\big[(m,\frac{1}{2})\big]\big[(n,\frac{1}{2})\big]}\Big\rangle\\&=\begin{cases}0, & n=m,\\ \frac{1}{m+1}+\frac{1}{n}+\frac{1}{m}, & n=m+1,\\ \frac{1}{n+1}+\frac{1}{n}+\frac{1}{m}, & n=m-1,\\ \frac{1}{n}+\frac{1}{m}, & n\notin\{m-1,m,m+1\}.\end{cases}\end{aligned}$$

Therefore, $\langle u^\diamond-v^\diamond,\overrightarrow{vu}\rangle\geq 0$ which shows that, M is a monotone relation.

In the sequel, we need the following notations. Let X be an Hadamard space and $Y\subseteq X$. Put

$$\varsigma_Y:=\Big\{\eta:Y\to[0,+\infty[\ \Big|\ \text{supp}\,\eta\ \textit{is finite}\ \text{and}\ \sum_{x\in Y}\eta(x)=1\Big\}$$

where $\text{supp}\,\eta=\{y\in Y:\eta(y)\neq 0\}$. Clearly, for each $\emptyset\neq A\subset Y$, $\varsigma_A=\{\eta\in\varsigma_Y:\text{supp}\,\eta\subseteq A\}$. It is obvious that ς_A is a convex subset of $\mathbb{R}^Y$. Moreover, if $\emptyset\neq A\subseteq B$, then $\varsigma_A\subseteq\varsigma_B$. Suppose $u\in Y$ be fixed. Define $\delta_u\in\varsigma_Y$ by

$$\delta_u(x)=\begin{cases}1 & x=u,\\ 0 & x\neq u.\end{cases}$$

Let $M\subseteq X\times X^\diamond$ and $\eta\in\varsigma_A$. Then $\text{supp}\eta=\{\lambda_1,\dots,\lambda_n\}$ where $\lambda_i=\eta(x_i,x_i^\diamond)$, for each $1\leq i\leq n$. Let $p\in X$ be fixed. Define $\alpha:\varsigma_{X\times X^\diamond}\to X$ (resp. $\beta:\varsigma_{X\times X^\diamond}\to X^\diamond$ and $\theta_p:\varsigma_{X\times X^\diamond}\to\mathbb{R}$) by

$$\alpha(\eta)=\bigoplus_{i=1}^{n}\lambda_ix_i,\ \Big(\text{resp. }\beta(\eta)=\sum_{i=1}^{n}\lambda_ix_i^\diamond\ \text{and}\ \theta_p(\eta)=\sum_{i=1}^{n}\lambda_i\langle x_i^\diamond,\overrightarrow{px_i}\rangle\Big).$$

Proposition 3.3. *Let X be an Hadamard space, $M\subseteq X\times X^\diamond$ and $p\in X$. Set*

$$\Theta_{p,M}:=\Big\{\eta\in\varsigma_M:\theta_p(\eta)\geq\langle\beta(\eta),\overrightarrow{p\alpha(\eta)}\rangle\Big\}.\tag{7}$$

Then $\Theta_{p,M}=\Theta_{q,M}$ for any $q\in X$.

Proof. It is enough to show that $\Theta_{p,M}\subseteq\Theta_{q,M}$. Let $\eta\in\Theta_{p,M}$ be such that $\text{supp}\eta=\{\lambda_1,\dots,\lambda_n\}$ where $\lambda_i=\eta(x_i,x_i^\diamond)$, for each $1\leq i\leq n$. Then

$$\begin{aligned}\theta_q(\eta)=\sum_{i=1}^{n}\lambda_i\langle x_i^\diamond,\overrightarrow{qx_i}\rangle&=\sum_{i=1}^{n}\lambda_i\langle x_i^\diamond,\overrightarrow{qp}\rangle+\sum_{i=1}^{n}\lambda_i\langle x_i^\diamond,\overrightarrow{px_i}\rangle\\&=\langle\sum_{i=1}^{n}\lambda_ix_i^\diamond,\overrightarrow{qp}\rangle+\theta_p(\eta)=\langle\beta(\eta),\overrightarrow{qp}\rangle+\theta_p(\eta)\\&\geq\langle\beta(\eta),\overrightarrow{qp}\rangle+\langle\beta(\eta),\overrightarrow{p\alpha(\eta)}\rangle\\&=\langle\beta(\eta),\overrightarrow{q\alpha(\eta)}\rangle.\end{aligned}$$

Therefore, $\eta\in\Theta_{q,M}$, i.e., $\Theta_{p,M}\subseteq\Theta_{q,M}$. $\square$

According to Proposition 3.3, for each $M \subseteq X \times X^\diamond$, the set $\Theta_{p,M}$ is independent of the choice of the element $p \in X$ and hence we denote the set $\Theta_{p,M}$ by Θ_M.

Theorem 3.4. *Let X be an Hadamard space and $M \subseteq X \times X^\diamond$ satisfies the $\mathcal{W}$-property. Then M is a monotone set if and only if $\Theta_M = \varsigma_M$.*

Proof. Let M be a monotone set. In view of (7), it is enough to show that $\varsigma_M \subseteq \Theta_M$. Let $\eta \in \varsigma_M$ be such that supp$\eta = \{\lambda_1, \dots, \lambda_n\}$ where $\lambda_i = \eta(x_i, x_i^\diamond)$, for each $1 \le i \le n$. By using Proposition 2.2, we obtain:

$$\begin{aligned}
\theta_p(\eta) - \langle \beta(\eta), \overrightarrow{p\alpha(\eta)} \rangle &= \sum_{i=1}^{n} \lambda_i \langle x_i^\diamond, \overrightarrow{px_i} \rangle - \Big\langle \sum_{j=1}^{n} \lambda_j x_j^\diamond, \overrightarrow{p\big(\oplus_{i=1}^{n} \lambda_i x_i\big)} \Big\rangle \\
&= \sum_{i=1}^{n} \lambda_i \langle x_i^\diamond, \overrightarrow{px_i} \rangle - \sum_{j=1}^{n} \lambda_j \Big\langle x_j^\diamond, \overrightarrow{p\big(\oplus_{i=1}^{n} \lambda_i x_i\big)} \Big\rangle \\
&\ge \sum_{i=1}^{n} \lambda_i \langle x_i^\diamond, \overrightarrow{px_i} \rangle - \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i \lambda_j \langle x_j^\diamond, \overrightarrow{px_i} \rangle \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i \lambda_j \langle x_i^\diamond, \overrightarrow{px_i} \rangle - \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i \lambda_j \langle x_j^\diamond, \overrightarrow{px_i} \rangle \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i \lambda_j \langle x_i^\diamond - x_j^\diamond, \overrightarrow{px_i} \rangle \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i \lambda_j \langle x_j^\diamond - x_i^\diamond, \overrightarrow{px_j} \rangle \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \langle x_i^\diamond - x_j^\diamond, \overrightarrow{px_i} - \overrightarrow{px_j} \rangle \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \langle x_i^\diamond - x_j^\diamond, \overrightarrow{x_j x_i} \rangle \ge 0.
\end{aligned}$$

Then $\varsigma_M \subseteq \Theta_M$ and hence $\varsigma_M = \Theta_M$. For the converse, let $(x, x^\diamond), (y, y^\diamond) \in M$ and set $\eta := \frac{1}{2}\delta_{(x,x^\diamond)} + \frac{1}{2}\delta_{(y,y^\diamond)} \in \varsigma_M$. By using $\mathcal{W}$-property, we get:

$$\begin{aligned}
\frac{1}{4} \langle x^\diamond - y^\diamond, \overrightarrow{yx} \rangle &= \frac{1}{4} \langle x^\diamond - y^\diamond, \overrightarrow{px} - \overrightarrow{py} \rangle \\
&= \frac{1}{4} (\langle x^\diamond - y^\diamond, \overrightarrow{px} \rangle - \langle x^\diamond - y^\diamond, \overrightarrow{py} \rangle) \\
&= \frac{1}{4} \langle x^\diamond, \overrightarrow{px} \rangle + \frac{1}{4} \langle y^\diamond, \overrightarrow{py} \rangle - \frac{1}{4} \langle y^\diamond, \overrightarrow{px} \rangle - \frac{1}{4} \langle x^\diamond, \overrightarrow{py} \rangle \\
&= \frac{1}{2} \langle x^\diamond, \overrightarrow{px} \rangle + \frac{1}{2} \langle y^\diamond, \overrightarrow{py} \rangle - \frac{1}{4} \langle x^\diamond, \overrightarrow{px} \rangle - \frac{1}{4} \langle x^\diamond, \overrightarrow{py} \rangle - \frac{1}{4} \langle y^\diamond, \overrightarrow{px} \rangle - \frac{1}{4} \langle y^\diamond, \overrightarrow{py} \rangle \\
&\ge \frac{1}{2} \langle x^\diamond, \overrightarrow{px} \rangle + \frac{1}{2} \Big\langle y^\diamond, \overrightarrow{py} \Big\rangle - \Big\langle \frac{1}{2} x^\diamond + \frac{1}{2} y^\diamond, \overrightarrow{p(\frac{1}{2} x \oplus \frac{1}{2} y)} \Big\rangle \\
&= \frac{1}{2} \langle x^\diamond, \overrightarrow{px} \rangle + \frac{1}{2} \langle y^\diamond, \overrightarrow{py} \rangle - \frac{1}{2} \Big\langle x^\diamond, \overrightarrow{p(\frac{1}{2} x \oplus \frac{1}{2} y)} \Big\rangle - \frac{1}{2} \Big\langle y^\diamond, \overrightarrow{p(\frac{1}{2} x \oplus \frac{1}{2} y)} \Big\rangle \\
&= \theta_p(\eta) - \langle \beta(\eta), \overrightarrow{p\alpha(\eta)} \rangle \ge 0.
\end{aligned}$$

Therefore, M is monotone. □

Corollary 3.5. *Let X be a flat Hadamard space and $M \subseteq X \times X^\diamond$. Then M is a monotone set if and only if $\Theta_M = \varsigma_M$.*

Proof. Since X is flat, Proposition 2.5 implies that $M \subseteq X \times X^\diamond$ satisfies the $\mathcal{W}$-property. Then the conclusion follows immediately from Theorem 3.4. □

A fundamental result concerning monotone operators is the extension theorem of Debrunner-Flor (for a proof see [6, Theorem 4.3.1] or [15, Proposition 2.17]). In the sequel, we prove a type of this result for monotone relations from an Hadamard space to its linear dual space. First, we recall some notions and results.

Definition 3.6. [2, Definition 2.4] Let $\{x_n\}$ be a sequence in an Hadamard space X. The sequence $\{x_n\}$ is said to be *weakly convergent* to $x \in X$, denoted by $x_n \xrightarrow{w} x$, if $\lim_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle = 0$, for all $y \in X$.

One can easily see that convergence in the metric implies weak convergence.

Lemma 3.7. [14, Proposition 3.6] *Let $\{x_n\}$ be a bounded sequence in an Hadamard space (X, d) with linear dual space $X^\diamond$ and let $\{x_n^\diamond\}$ be a sequence in $X^\diamond$. If $\{x_n\}$ is weakly convergent to x and $x_n^\diamond \xrightarrow{\|\cdot\|_\diamond} x^\diamond$, then $\langle x_n^\diamond, \overrightarrow{x_n z}\rangle \to \langle x^\diamond, \overrightarrow{xz}\rangle$, for all $z \in X$.*

Theorem 3.8. *Let X be an Hadamard space and $M \subseteq X \times X^\diamond$ be a monotone relation satisfies the $\mathcal{W}$-property. Let $C \subseteq X^\diamond$ be a compact and convex set, and $\varphi : C \to X$ be a continuous function. Then there exists $z^\diamond \in C$ such that $\{(\varphi(z^\diamond), z^\diamond)\} \cup M$ is monotone.*

Proof. Let $x \in X$, $u^\diamond, v^\diamond \in X^\diamond$ be arbitrary and fixed element. Consider the function $\tau : C \to \mathbb{R}$ defined by

$$\tau(x^\diamond) = \langle x^\diamond - v^\diamond, \overrightarrow{x\varphi(u^\diamond)}\rangle, \ x^\diamond \in C.$$

Let $\{x_n^\diamond\} \subseteq C$ be such that $x_n^\diamond \xrightarrow{\|\cdot\|_\diamond} x^\diamond$, for some $x^\diamond \in C$. By Lemma 3.7,

$$\langle x_n^\diamond - v^\diamond, \overrightarrow{x\varphi(u^\diamond)}\rangle \to \langle x^\diamond - v^\diamond, \overrightarrow{x\varphi(u^\diamond)}\rangle.$$

Thus $\tau(x_n^\diamond) \to \tau(x^\diamond)$. Hence τ is continuous. For every $(y, y^\diamond) \in M$, set

$$U(y, y^\diamond) := \{u^\diamond \in C : \langle u^\diamond - y^\diamond, \overrightarrow{y\varphi(u^\diamond)}\rangle < 0\}.$$

Continuity of τ implies that $U(y, y^\diamond)$ is an open subset of C. Suppose that the conclusion fails. Then for each $u^\diamond \in C$ there exists $(y, y^\diamond) \in M$ such that $u^\diamond \in U(y, y^\diamond)$. This means that the family of open sets $\{U(y, y^\diamond)\}_{(y,y^\diamond)\in M}$ is an open cover of C. Using the compactness of C, we obtain that $C = \bigcup_{i=1}^n U(y_i, y_i^\diamond)$. In addition, [15, Page 756] implies that there exists a partition of unity associated with this finite subcover. Hence, there are continuous functions $\psi_i : X^\diamond \to \mathbb{R}$ $(1 \le i \le n)$ satisfying

(i) $\sum_{i=1}^n \psi_i(x^\diamond) = 1$, for all $x^\diamond \in C$.
(ii) $\psi_i(x^\diamond) \ge 0$, for all $x^\diamond \in C$ and all $i \in \{1, \dots, n\}$.
(iii) $\{x^\diamond \in C : \psi_i(x^\diamond) > 0\} \subseteq U_i := U(y_i, y_i^\diamond)$ for all $i \in \{1, \dots, n\}$.

Set $K := \mathrm{co}(\{y_1^\diamond, \dots, y_n^\diamond\}) \subseteq C$ and define

$$\begin{aligned} \iota : & K \to K \\ & u^\diamond \mapsto \sum_{i=1}^n \psi_i(u^\diamond) y_i^\diamond. \end{aligned}$$

Let $\{u_m^\diamond\} \subseteq K$ be such that $u_m^\diamond \to u^\diamond$,

$$\left\| \iota(u_m^\diamond) - \iota(u^\diamond) \right\|_\diamond = \left\| \sum_{i=1}^n \psi_i(u_m^\diamond) y_i^\diamond - \sum_{i=1}^n \psi_i(u^\diamond) y_i^\diamond \right\|_\diamond$$

$$
\begin{aligned}
&= \Big\| \sum_{i=1}^{n} (\psi_i(u_m^\diamond) - \psi_i(u^\diamond)) y_i^\diamond \Big\|_\diamond \\
&\le \sum_{i=1}^{n} \Big\| (\psi_i(u_m^\diamond) - \psi_i(u^\diamond)) y_i^\diamond \Big\|_\diamond \\
&\le \sum_{i=1}^{n} \Big| \psi_i(u_m^\diamond) - \psi_i(u^\diamond) \Big| \|y_i^\diamond\|_\diamond.
\end{aligned}
$$

By continuity of ψ_i $(1 \le i \le n)$, letting $m \to +\infty$, then $\psi_i(u_m^\diamond) \to \psi_i(u^\diamond)$ and this implies that $\iota(u_m^\diamond) \to \iota(u^\diamond)$ and so ι is continuous. One can identify K with a finite-dimensional convex and compact set. By using Brouwer fixed point theorem [15, Proposition 2.6], there exists $w^\diamond \in K$ such that $\iota(w^\diamond) = w^\diamond$. Moreover, by using Proposition 2.2 we get:

$$
\begin{aligned}
0 &= \Big\langle \iota(w^\diamond) - w^\diamond, \overrightarrow{\varphi(w^\diamond)(\oplus_j \psi_j(w^\diamond) y_j)} \Big\rangle \\
&= \Big\langle \sum_i \psi_i(w^\diamond)(y_i^\diamond - w^\diamond), \overrightarrow{\varphi(w^\diamond)(\oplus_j \psi_j(w^\diamond) y_j)} \Big\rangle \\
&= \Big\langle \sum_i \psi_i(w^\diamond)(y_i^\diamond - w^\diamond), \overrightarrow{p(\oplus_j \psi_j(w^\diamond) y_j)} \Big\rangle - \Big\langle \sum_i \psi_i(w^\diamond)(y_i^\diamond - w^\diamond), \overrightarrow{p\varphi(w^\diamond)} \Big\rangle \qquad (p \in X) \\
&\le \sum_j \psi_j(w^\diamond) \Big\langle \sum_i \psi_i(w^\diamond)(y_i^\diamond - w^\diamond), \overrightarrow{py_j} \Big\rangle - \Big\langle \sum_i \psi_i(w^\diamond)(y_i^\diamond - w^\diamond), \overrightarrow{p\varphi(w^\diamond)} \Big\rangle \\
&= \sum_j \psi_j(w^\diamond) \Big\langle \sum_i \psi_i(w^\diamond)(y_i^\diamond - w^\diamond), \overrightarrow{py_j} \Big\rangle - \sum_j \psi_j(w^\diamond) \Big\langle \sum_i \psi_i(w^\diamond)(y_i^\diamond - w^\diamond), \overrightarrow{p\varphi(w^\diamond)} \Big\rangle \\
&= \sum_j \psi_j(w^\diamond) \Big\langle \sum_i \psi_i(w^\diamond)(y_i^\diamond - w^\diamond), \overrightarrow{py_j} - \overrightarrow{p\varphi(w^\diamond)} \Big\rangle \\
&= \sum_j \psi_j(w^\diamond) \Big\langle \sum_i \psi_i(w^\diamond)(y_i^\diamond - w^\diamond), \overrightarrow{\varphi(w^\diamond) y_j} \Big\rangle \\
&= \sum_j \psi_j(w^\diamond) \sum_i \psi_i(w^\diamond) \Big\langle y_i^\diamond - w^\diamond, \overrightarrow{\varphi(w^\diamond) y_j} \Big\rangle \\
&= \sum_j \sum_i \psi_j(w^\diamond) \psi_i(w^\diamond) \Big\langle y_i^\diamond - w^\diamond, \overrightarrow{\varphi(w^\diamond) y_j} \Big\rangle \\
&= \sum_{i,j} \psi_i(w^\diamond) \psi_j(w^\diamond) \Big\langle y_i^\diamond - w^\diamond, \overrightarrow{\varphi(w^\diamond) y_j} \Big\rangle. \qquad (8)
\end{aligned}
$$

Set $a_{ij} = \langle y_i^\diamond - w^\diamond, \overrightarrow{\varphi(w^\diamond) y_j} \rangle$. It follows from monotonicity of M that

$$
\begin{aligned}
a_{ii} + a_{jj} - a_{ij} - a_{ji} &= \langle y_i^\diamond - w^\diamond, \overrightarrow{\varphi(w^\diamond) y_i} \rangle + \langle y_j^\diamond - w^\diamond, \overrightarrow{\varphi(w^\diamond) y_j} \rangle \\
&\quad - \langle y_i^\diamond - w^\diamond, \overrightarrow{\varphi(w^\diamond) y_j} \rangle - \langle y_j^\diamond - w^\diamond, \overrightarrow{\varphi(w^\diamond) y_i} \rangle \\
&= \langle y_i^\diamond - y_j^\diamond, \overrightarrow{\varphi(w^\diamond) y_i} - \overrightarrow{\varphi(w^\diamond) y_j} \rangle \\
&= \langle y_i^\diamond - y_j^\diamond, \overrightarrow{y_j y_i} \rangle \ge 0;
\end{aligned}
$$

i.e.,

$$
a_{ii} + a_{jj} \ge a_{ij} + a_{ji}. \qquad (9)
$$

Applying (8) and (9), we obtain:

$$
\begin{aligned}
0 &\le \sum_{i,j}^{n} \psi_i(w^\diamond)\psi_j(w^\diamond)a_{ij} \\
&= \sum_{i<j}^{n} \psi_i(w^\diamond)\psi_j(w^\diamond)a_{ij} + \sum_{i=1}^{n} \psi_i(w^\diamond)^2 a_{ii} + \sum_{i>j}^{n} \psi_i(w^\diamond)\psi_j(w^\diamond)a_{ij} \\
&= \sum_{i=1}^{n} \psi_i(w^\diamond)^2 a_{ii} + \sum_{i<j}^{n} \psi_i(w^\diamond)\psi_j(w^\diamond)(a_{ij} + a_{ji}) \qquad (10) \\
&\le \sum_{i=1}^{n} \psi_i(w^\diamond)^2 a_{ii} + \sum_{i<j}^{n} \psi_i(w^\diamond)\psi_j(w^\diamond)(a_{ii} + a_{jj}). \qquad (11)
\end{aligned}
$$

Set $I(w^\diamond) := \left\{ i \in \{1, \ldots, n\} : w^\diamond \in U_i \right\}$. Applying property (iii) of the partition of unity in (11) we get:

$$
0 \le \sum_{i \in I(w^\diamond)} \psi_i(w^\diamond)^2 a_{ii} + \sum_{\substack{i<j \\ i,j \in I(w^\diamond)}} \psi_i(w^\diamond)\psi_j(w^\diamond)(a_{ii} + a_{jj}). \qquad (12)
$$

By using property (iii) of the partition of unity and the definition of U_i, one deduce that all terms in the right-hand side of (12) are nonpositive. So all of $\psi_i(w^\diamond)$'s must be vanish, which contradicts with (i). □

Corollary 3.9. *Let X be a flat Hadamard space and $M \subseteq X \times X^\diamond$ be a monotone set. Let $C \subseteq X^\diamond$ be a compact and convex set, and $\varphi : C \to X$ be a continuous function. Then there exists $z^\diamond \in C$ such that $\{(\varphi(z^\diamond), z^\diamond)\} \cup M$ is monotone.*

Proof. Since X is flat, it follows from Proposition 2.5 that $M \subseteq X \times X^\diamond$ has $\mathcal{W}$-property. The inclusion follows from Theorem 3.8. □

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