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On the Weighted Pseudo Drazin Invertible Elements in Associative Rings and Banach Algebras

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Abstract. In this paper, we introduce and investigate the weighted pseudo Drazin inverse of elements in associative rings and Banach algebras. Some equivalent conditions for the existence of the w-pseudo Drazin inverse of $a + b$ are given. Using the Pierce decomposition, the representations for the w-pseudo Drazin inverse are given in Banach algebras.

1. Introduction

Throughout this paper, $\mathcal{R}$ denotes an associative ring with identity 1. An involution $* : \mathcal{R} \to \mathcal{R}$ is an anti-isomorphism which satisfies

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^*a^*,$$

for all $a, b \in \mathcal{R}$. Let $\mathcal{J}(\mathcal{R})$ and $\mathcal{U}(\mathcal{R})$ be, respectively, the Jacobson radical and the group of units in $\mathcal{R}$. Recall that ([10, Lemma 4.1])

$$\mathcal{J}(\mathcal{R}) = \{a \in \mathcal{R} : 1 - ba \text{ (or } 1 - ab) \text{ is left invertible for any } b \in \mathcal{R}\}. \tag{1.1}$$

Let $\sqrt{\mathcal{J}(\mathcal{R})}$ denote the root of $\mathcal{J}(\mathcal{R})$, which is defined by

$$\sqrt{\mathcal{J}(\mathcal{R})} = \{a \in \mathcal{R} : a^k \in \mathcal{J}(\mathcal{R}) \text{ for some } k \geq 1\}.$$

For any element $a \in \mathcal{R}$, let $comm(a)$ and $comm^2(a)$ be the *commutant* and the *double commutant* (or bi-commutant) of a, which are respectively defined by

$$comm(a) = \{x \in \mathcal{R} : ax = xa\},$$

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$$comm^2(a) = \{x \in \mathcal{R} : xy = yx \text{ for all } y \in comm(a)\}.$$

An element $a \in \mathcal{R}$ is *quasinilpotent* if, for every $x \in comm(a)$, $1 + xa \in \mathcal{U}(\mathcal{R})$ ([7]). Let $\mathcal{R}^{qnil}$ and $\mathcal{R}^{nil}$ be the set of all quasinilpotent elements and the set of all nilpotent elements of $\mathcal{R}$ respectively.

An element $a \in \mathcal{R}$ is said to be *Drazin invertible* if there exists $b \in \mathcal{R}$ such that

$$b \in comm(a), bab = b \text{ and } a^k ba = a^k, \tag{1.2}$$

for some nonnegative integer k [6]. If such b exists then it is unique and will be denoted by $b = a^d$ and is called the *Drazin inverse* of a. If $k = 1$ in (1.2) we say that a is group invertible. The set of all Drazin invertible elements in $\mathcal{R}$ will be denoted by $\mathcal{R}^d$.

The concept of the Drazin inverse was firstly generalized by Koliha for bounded linear operators on Banach spaces and for elements in Banach algebras [8] and then by Koliha and Patrício [9] for elements in a ring. Many properties of such generalized inverses can be found in, for example, [2, 3, 5, 15]. An element $a \in \mathcal{R}$ is said to be *generalized Drazin invertible* if there exists $b \in \mathcal{R}$ such that

$$b \in comm^2(a),\ b = ab^2 \text{ and } a - a^2 b \in \mathcal{R}^{qnil}. \tag{1.3}$$

If such b exists it is unique and denoted by $b = a^{gd}$ and called the *generalized Drazin inverse* of a. The set of all generalized Drazin invertible elements in $\mathcal{R}$ will be denoted by $\mathcal{R}^{gd}$.

Following Wang and Chen [12], an element $a \in \mathcal{R}$ is *pseudo Drazin invertible* if there exists $b \in \mathcal{R}$ such that

$$b \in comm^2(a),\ b = ab^2 \text{ and } a^k - a^{k+1} b \in \mathcal{J}(\mathcal{R}), \text{ for some } k \geq 1. \tag{1.4}$$

If such b exists it is unique and denoted by $b = a^{pd}$. The least positive integer k for which (1.4) hold is called the *pseudo Drazin index* of a and denoted by $i(a)$. The set of all pseudo Drazin invertible elements in $\mathcal{R}$ is denoted by $\mathcal{R}^{pd}$. Then $\mathcal{R}^d \subseteq \mathcal{R}^{pd} \subseteq \mathcal{R}^{gd}$ and the inclusions may be strict. If $\mathcal{A}$ is a Banach algebra, then we replace the double commutator for the commutator in (1.4).

An element a in $\mathcal{R}$ is pseudo Drazin invertible if and only if a is pseudopolar : a is *pseudopolar* if there exists $e \in \mathcal{R}$ such that

$$e^2 = e \in comm^2(a),\ a + e \in \mathcal{U}(\mathcal{R}) \text{ and } a^k e \in \mathcal{J}(\mathcal{R}) \text{ for some } k \geq 1.$$

The idempotent e is unique and will be denoted $a^{\sqcap}$. In this case, $a^{\sqcap} = 1 - aa^{pd}$, [12].

For $w \in \mathcal{R}$, let $\mathcal{R}_w$ be the ring $\mathcal{R}$ equipped with the w-product

$$a \star b := awb \text{ for all } a, b \in \mathcal{R}.$$

If $w \in \mathcal{U}(\mathcal{R})$, then $1_w = w^{-1}$ is the unit of the ring $\mathcal{R}_w$. For any positive integer n we write $a^{\star n} = a \star \cdots \star a$ (n factors).

Let $w \in \mathcal{U}(\mathcal{R})$. An element $a \in \mathcal{R}$ is said to be *weighted Drazin invertible* or w-Drazin invertible if a is Drazin invertible in $\mathcal{R}_w$. The w-Drazin inverse $a^{d,w}$ of a is defined as the Drazin inverse of a in the ring $\mathcal{R}_w$. The concept of the weighted Drazin inverse was introduced by Cline and Greville [1] for rectangular matrices. In [4], Dajić and Koliha defined and studied the weighted generalized Drazin inverse for bounded linear operators on Banach spaces. In a recent paper [11], Mosić and Djordjević investigated the weighted generalized Drazin inverse for elements in a ring.

The main purpose of this paper is to introduce and to investigate the weighted pseudo Drazin inverse of elements in a ring. In the second section we characterize the weighted pseudo Drazin inverse by means of the weight. Section 3 is devoted to weighted pseudo Drazin invertible elements in a Banach algebra. Using the Pierce decomposition, we give a necessary and sufficient condition for an element to be weighted pseudo Drazin invertible.

2. Weighted pseudo Drazin inverse in associative ring

Definition 2.1. *Let $w \in \mathcal{U}(\mathcal{R})$. An element $a \in \mathcal{R}$ is said weighted pseudo Drazin invertible or w-pseudo Drazin invertible if a is pseudo Drazin invertible in $\mathcal{R}_w$. The w-pseudo Drazin inverse $a^{pd,w}$ of a is defined as the pseudo Drazin inverse of a in the ring $\mathcal{R}_w$. The index $i_w(a)$ is defined as the index of the pseudo Drazin inverse of a in $\mathcal{R}_w$. The set of all weighted pseudo Drazin invertible elements $\mathcal{R}$ is denoted by $\mathcal{R}^{pd,w}$.*

We notice here that the Jacobson radical of $\mathcal{R}_w$ equals to the Jacobson radical of $\mathcal{R}$.

Theorem 2.2. *Let $w \in \mathcal{U}(\mathcal{R})$. For $a \in \mathcal{R}$ the following assertions are equivalent:*

i) $a \in \mathcal{J}(\mathcal{R}_w)$.

ii) $aw \in \mathcal{J}(\mathcal{R})$.

iii) $wa \in \mathcal{J}(\mathcal{R})$.

Proof. *i)* $\Longrightarrow$ *ii)*: Assume that $a \in \mathcal{J}(\mathcal{R}_w)$. Let $b \in \mathcal{R}$ and set $c = bw^{-1}$. Then by (1.1), $1_w - a \star c = w^{-1} - a \star c$ is left invertible in $\mathcal{R}_w$. Hence ,there exists some $d \in \mathcal{R}_w$ such that $1_w = w^{-1} = d \star (w^{-1} - a \star c)$. Then $w^{-1} = d - dwawc$. Thus, $1 = dw(1 - awb)$ and so $1 - awb$ is left invertible for all $b \in \mathcal{R}$. Therefore, $aw \in \mathcal{J}(\mathcal{R})$ by (1.1).

ii) $\Longrightarrow$ *i)*: Suppose that $aw \in \mathcal{J}(\mathcal{R})$. Let $b \in \mathcal{R}_w$ and set $c = bw$. Then by (1.1), $1 - awc$ is left invertible. Hence, there exists some $d \in \mathcal{R}$ such that $1 = d(1 - awc)$. Thus, $w^{-1} = d(w^{-1} - awcw^{-1}) = dw^{-1} \star (w^{-1} - a \star b)$. Therefore, $w^{-1} - a \star b$ is left invertible for all $b \in \mathcal{R}_w$ and so $a \in \mathcal{J}(\mathcal{R}_w)$.

The equivalence *i)* $\Longleftrightarrow$ *iii)* goes similarly. □

In the following we give the relationship between the weighted pseudo Drazin inverse of an element and its weight.

Theorem 2.3. *Let $w \in \mathcal{U}(\mathcal{R})$. For $a \in \mathcal{R}$, the following assertions are equivalent:*

i) a is w-pseudo Drazin invertible with w-pseudo Drazin inverse $a^{pd,w} = b \in \mathcal{R}$.

ii) aw is pseudo Drazin invertible in $\mathcal{R}$ and $(aw)^{pd} = bw$.

iii) wa is pseudo Drazin invertible in $\mathcal{R}$ with $(wa)^{pd} = wb$.

Moreover, the w-pseudo Drazin inverse $a^{pd,w}$ satisfies

$$a^{pd,w} = ((aw)^{pd})^2 a = a((wa)^{pd})^2. \tag{2.1}$$

Proof. i)$\Longrightarrow$ ii): Assume that a is w-pseudo Drazin invertible with w-pseudo Drazin inverse $a^{pd,w} = b$. Then

$$b \in comm_w^2(a),\ b \star a \star b = b \text{ and } a^{\star k} - a^{\star k+1} \star b \in \mathcal{J}(\mathcal{R}_w).$$

Step 1. We show that $bw \in comm^2(aw)$:
Let $y \in \mathcal{R}$ such that $awy = yaw$. Then, $a \star (yw^{-1}) = (yw^{-1}) \star a$. Hence, $b \star (yw^{-1}) = (yw^{-1}) \star b$. Thus, $bwyw^{-1} = yb$ and then $bwy = ybw$. Therefore, $bw \in comm^2(aw)$.
Step 2. We have $(bw)aw(bw) = bw$:
Since $b \star a \star b = b$, $bwawb = b$ and so $(bw)aw(bw) = bw$.
Step 3. $(aw)^k - (aw)^{k+1}bw \in \mathcal{J}(\mathcal{R})$: Indeed, since $a^{\star k} - a^{\star k+1} \star b = (aw)^{k-1}a - (aw)^{k+1}b \in \mathcal{J}(\mathcal{R}_w)$, it follows from Theorem 2.2 that $((aw)^{k-1}a - (aw)^{k+1}b)w = (aw)^k - (aw)^{k+1}bw \in \mathcal{J}(\mathcal{R})$.

ii) $\Longrightarrow$ i): Suppose that aw is pseudo Drazin invertible with pseudo Drazin inverse $(aw)^{pd} = c \in \mathcal{R}$. Then

$$c \in comm^2(aw),\ c(aw)c = c \text{ and } (aw)^k - (aw)^{k+1}c \in \mathcal{J}(\mathcal{R}).$$

Note that $c^2aw = bw$. Then $b = c^2a$. Next, we prove $a^{pd,w} = b$.

Step 1. $b \in comm_w^2(a)$:
Let $y \in \mathcal{R}_w$ such that $y \star a = a \star y$. Then, $ywa = awy$ and so $(yw)aw = ywaw = awyw$. Hence, $yw \in comm(aw)$ and then $ywc = cyw$. Now $b \star y = bwy = c^2awy = c^2ywa = ywc^2a = ywb = y \star b$. Then, $b \in comm_w^2(a)$.
Step 2. $b \star a \star b = b$:
we have $b \star a \star b = bwawb = c^2awawc^2a = cawc^2a = c^2a = b$.
Step 3. $a^{\star k} - a^{\star k+1} \star b \in \mathcal{J}(\mathcal{R}_w)$:
Since $(aw)^k - (aw)^{k+1}c = (aw)^k - (aw)^{k+1}c^2aw = ((aw)^{k-1}a - (aw)^kawc^2a)w \in \mathcal{J}(\mathcal{R})$, by Theorem 2.2, $(aw)^{k-1}a - (aw)^kawc^2a = a^{\star k} - a^{\star k+1} \star b \in \mathcal{J}(\mathcal{R}_w)$.

The equivalence i) $\Longleftrightarrow$ iii) goes similarly.

Now assume that a is w-pseudo Drazin invertible. Then, $((aw)^{pd})^2a = a^{pd,w}$ from the proof of ii) $\Longrightarrow$ i). By the same way, we get $a((wa)^{pd})^2 = a^{pd,w}$. □

Remark 2.4. *From the proof of Theorem 2.3 we deduce that if $a \in \mathcal{R}$ is w-pseudo Drazin invertible, then the pseudo Drazin indices $i_w(a)$, $i(aw)$ and $i(wa)$ satisfy*

$$max\{i(aw), i(wa)\} \le i_w(a) \le min\{i(aw), i(wa)\}.$$

Therefore, $i_w(a) = i(aw) = i(wa)$.

In following theorem ii) and iii) were presented in a Banach algebra in [13]. Here we prove that it is still true in an associative ring.

Theorem 2.5. *Let $a \in \mathcal{R}$ be pseudo Drazin invertible. Then the following are true:*

i) $a = a^{pd}$ *if and only if* $a^3 = a$.

ii) $(a^{pd})^{pd} = a^2a^{pd}$.

iii) $a^{pd}(a^{pd})^{pd} = aa^{pd}$.

Proof. i) Assume that $a = a^{pd}$. Then, $a^3 = a(a^{pd})^2 = a^{pd} = a$. Conversely, if $a^3 = a$, then for $b = a$ we have $b \in comm^2(a)$, $bab = a$ and $a - a^2b = 0 \in \mathcal{J}(\mathcal{R})$. Thus, a is pseudo Drazin inverse and $a^{pd} = b = a$.

ii) Since $a^{pd}a^2a^{pd} = a^2a^{pd}a^{pd}$, $a^{pd}a^2a^{pd}a^{pd} = a^{pd}$ and $a^2a^{pd}a^{pd}a^2a^{pd} = a^2a^{pd}$, we have $(a^{pd})^\# = a^2a^{pd}$. Which implies that $(a^{pd})^{pd} = a^2a^{pd}$.

iii) From ii) we have $a^{pd}(a^{pd})^{pd} = a^{pd}a^2a^{pd} = aa^{pd}$. □

Corollary 2.6. *Let $a \in \mathcal{R}$ be pseudo Drazin invertible. Then $(a^{pd})^{pd} = a$ if and only if a is group invertible in $\mathcal{R}$.*

Proof. Since $(a^{pd})^{pd} = a^2a^{pd}$, we have $a = a^2a^{pd}$, therefore, it is easy to verify that a^{pd} is the group inverse of a. □

Theorem 2.7. *Let $w \in \mathcal{U}(\mathcal{R})$. Assume that $a \in \mathcal{R}$ is w-pseudo Drazin invertible. Then $a^{pd,w}$ is w- pseudo Drazin invertible and the following are true:*

i) $a^{pd,w} = a$ *if and only if* $a = a^{\star 3} = awawa$.

ii) $(a^{pd,w})^{pd,w} = aw(aw)^{pd}a = awa(wa)^{pd}$.

iii) $a^{pd,w} \star (a^{pd,w})^{pd,w} = awa^{pd,w} = (aw)^{pd}a$.

iv) $((a^{pd,w})^{pd,w})^{pd,w} = a^{pd,w}$.

Proof. Since a is w-pseudo Drazin invertible, from Theorem 2.3, we have aw is pseudo Drazin invertible and $(aw)^{pd} = a^{pd,w}w$. Then by Theorem 2.5, we have $a^{pd,w}w = (aw)^{pd}$ is pseudo Drazin invertible. Therefore, $a^{pd,w}$ is w-pseudo Drazin invertible by Theorem 2.3.

i) Since a is w-pseudo Drazin invertible and $a^{pd,w} = a$, we have aw is pseudo Drazin invertible and $(aw)^{pd} = aw$ by Theorem 2.3. Then From Theorem 2.5 i), we obtain $(aw)^{pd} = aw$ if and only if $a = a^{*3} = awawa$.

ii) $(a^{pd,w})^{pd,w} = ((a^{pd,w}w)^{pd})^2 a^{pd,w} = (((aw)^{pd})^{pd})^2 a^{pd,w} = ((aw)^2(aw)^{pd})^2$
$((aw)^{pd})^2 a = aw(aw)^{pd}a$. Similarly, we have $(a^{pd,w})^{pd,w} = awa(wa)^{pd}$.

iii) $a^{pd,w} \star (a^{pd,w})^{pd,w} = (aw)^{pd}aw(aw)^{pd}a = (aw)^{pd}a$.

iv) From ii), Theorem 2.3 and Theorem 2.5 ii), we have $((a^{pd,w})^{pd,w})^{pd,w} = (a^{pd,w}w)(a^{pd,w}w)^{pd}a^{pd,w} = (aw)^{pd}((aw)^{pd})^{pd}a^{pd,w} = (aw)^{pd}(aw)^2(aw)^{pd}((aw)^{pd})^2 a = ((aw)^{pd})^2 a = a^{pd,w}$. □

Proposition 2.8. *Let $\mathcal{R}$ be a ring with involution, $w \in \mathcal{U}(\mathcal{R})$. Then $a \in \mathcal{R}$ is w- pseudo Drazin invertible if and only if a^* is w^*-pseudo Drazin invertible. In this case,*

$$(a^*)^{pd,w^*} = (a^{pd,w})^* \text{ and } i_{w^*}(a^*) = i_w(a).$$

Proof. By [12, Proposition 1.5 and Theorem 3.2] aw is pseudo Drazin invertible if and only if $(aw)^* = w^*a^*$ is pseudo Drazin invertible and $i(aw) = i((aw)^*)$. The result follows from Theorem 2.3. □

3. Weighted pseudo Drazin inverse in a Banach algebra

In this section, let $\mathcal{A}$ be a Banach algebra with unit. For an element $w \in \mathcal{U}(\mathcal{A})$, let $\mathcal{A}_w$ be the Banach algebra equipped with the w-product $a \star b = awb$ for all $a, b \in \mathcal{A}$.

Lemma 3.1. *[16] Let $a \in \mathcal{A}$. Then the following assertions are equivalent:*

i) a is pseudo Drazin invertible.

ii) a^n is pseudo Drazin invertible for any $n \in \mathbb{N}$.

iii) a^n is pseudo Drazin invertible for some $n \in \mathbb{N}$.

Theorem 3.2. *Let $w \in \mathcal{U}(\mathcal{A})$, $a \in \mathcal{A}$. If $aw = wa$, then the following assertions are equivalent:*

i) a is w-pseudo Drazin invertible.

ii) a^n is w^n-pseudo Drazin invertible for any $n \in \mathbb{N}$.

iii) a^n is w^n-pseudo Drazin invertible for some $n \in \mathbb{N}$.

In this case, $(a^n)^{pd,w^n} = (a^{pd,w})^n$.

Proof. i)⟹ii): Since a is w-pseudo Drazin invertible, aw is pseudo Drazin invertible by Theorem 2.3. It follows from Lemma 3.1 that $(aw)^n = a^n w^n$ is pseudo Drazin invertible, thus, a^n is w^n-pseudo Drazin invertible for any $n \in \mathbb{N}$.

ii)⟹ iii): It is clear.

iii)⟹ i): By assumption $(aw)^n = a^n w^n$ is pseudo Drazin invertible for some $n \in \mathbb{N}$. Thus, aw is pseudo Drazin invertible by Lemma 3.1. Therefore, a is w-pseudo Drazin invertible.

In this case, $(a^n)^{pd,w^n} = ((a^n w^n)^{pd})^2 a^n = (((aw)^n)^{pd})^2 a^n = (((aw)^{pd})^2)^n a^n = (((aw)^{pd})^2 a)^n = (a^{pd,w})^n$. □

Remark 3.3. *If $aw \neq wa$, then the formula $(a^n)^{pd,w^n} = (a^{pd,w})^n$ in Theorem 3.2 does not hold in general. For example, let*

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \; w = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

in $\mathbb{C}^{2\times 2}$, then $aw \neq wa$, and we can calculate that

$$(a^2w^2)^{pd} = \begin{pmatrix} \frac{1}{3} & \frac{2}{9} \\ 0 & 0 \end{pmatrix}, \; (aw)^{pd} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & 0 \end{pmatrix},$$

thus,

$$(a^2)^{pd,w^2} = \begin{pmatrix} \frac{1}{9} & \frac{1}{9} \\ 0 & 0 \end{pmatrix}, \; (a^{pd,w})^2 = \begin{pmatrix} \frac{1}{16} & \frac{1}{16} \\ 0 & 0 \end{pmatrix},$$

therefore, $(a^2)^{pd,w^2} \neq (a^{pd,w})^2$.

Theorem 3.4. *Let $w \in \mathcal{U}(\mathcal{A})$. Assume that a and $b \in \mathcal{A}$ are w-pseudo Drazin invertible. If $awb = bwa = 0$, then $a + b$ is w-pseudo Drazin invertible and $(a + b)^{pd,w} = a^{pd,w} + b^{pd,w}$.*

Proof. By assumption aw and bw are pseudo Drazin invertible and $awbw = bwaw = 0$. Then, it follows from [13, Theorem 2.5] that $aw + bw$ is pseudo Drazin invertible and $(aw + bw)^{pd} = (aw)^{pd} + (bw)^{pd}$, thus, $(a + b)^{pd,w} = a^{pd,w} + b^{pd,w}$ by Theorem 2.3. □

Corollary 3.5. *Let $w \in \mathcal{U}(\mathcal{A})$. Assume that $a_1, a_2, \cdots, a_n \in \mathcal{A}$ are w-pseudo Drazin invertible. If $a_i w a_j = 0$ $(i, j = 1, \cdots, n, i \neq j)$, then $a_1 + a_2 + \cdots + a_n$ is w-pseudo Drazin invertible and*

$$(a_1 + a_2 + \cdots + a_n)^{pd,w} = a_1^{pd,w} + \cdots + a_n^{pd,w}.$$

Lemma 3.6. *[13] Let $a, b \in \mathcal{A}$ be pseudo Drazin invertible, $ab = \lambda ba\,(\lambda \neq 0)$. Then*

i) $a^{pd} b = \lambda^{-1} b a^{pd}$.

ii) $a b^{pd} = \lambda^{-1} b^{pd} a$.

iii) $(ab)^{pd} = b^{pd} a^{pd} = \lambda^{-1} a^{pd} b^{pd}$.

Theorem 3.7. *Let $w \in \mathcal{U}(\mathcal{A})$. Assume that a and $b \in \mathcal{A}$ are w-pseudo Drazin invertible. If $awb = \lambda bwa\,(\lambda \neq 0)$, then awb is w-pseudo Drazin invertible, and*

i) $a^{pd,w} wb = \lambda^{-1} bwa^{pd,w}$.

ii) $awb^{pd,w} = \lambda^{-1} b^{pd,w} wa$.

iii) $(awb)^{pd,w} = b^{pd,w} wa^{pd,w} = \lambda^{-1} a^{pd,w} wb^{pd,w}$.

Proof. i) By assumption aw and bw are pseudo Drazin invertible and $awbw = \lambda bwaw$. Then, by Lemma 3.6, $(aw)^{pd} bw = \lambda^{-1} bw (aw)^{pd}$, and thus, $a^{pd,w} wb = \lambda^{-1} bwa^{pd,w}$.

The proof of ii) and iii) are similar to the proof of i). □

Let $\lambda = 1$ in Theorem 3.7, we have following corollary.

Corollary 3.8. *Let $w \in \mathcal{U}(\mathcal{A})$. Assume that a and $b \in \mathcal{A}$ are w-pseudo Drazin invertible. If $awb = bwa$, then awb is w-pseudo Drazin invertible and*

$$(awb)^{pd,w} = a^{pd,w} wb^{pd,w}.$$

Proposition 3.9. *Let $w \in \mathcal{U}(\mathcal{A})$, a and $b \in \mathcal{A}$. If awb is w-pseudo Drazin invertible, then so is bwa and*

$$(bwa)^{pd,w} = bw((awb)^{pd,w} w)^2 a.$$

Proof. It follows from [12, Theorem 3.6] and Theorem 2.3. □

Proposition 3.10. *Let $a, b \in \mathcal{A}$ be w-pseudo Drazin invertible, $w \in \mathcal{U}(\mathcal{A})$. If $awawb = awbwa$ and $bwbwa = bwawb$, then awb is w-pseudo Drazin invertible, and*

$$(awb)^{pd,w} = a^{pd,w} wb^{pd,w}.$$

Proof. It follows from [14, Theorem 2.8] and Theorem 2.3. □

Proposition 3.11. *Let $a, b \in \mathcal{A}$ be w-pseudo Drazin invertible, $w \in \mathcal{U}(\mathcal{A})$. If $awawb = awbwa$ and $bwbwa = bwawb$, then $a + b$ is w-pseudo Drazin invertible if and only if $w^{-1} + a^{pd,w} wb$ is w-pseudo Drazin invertible.*

Proof. It follows from [14, Theorem 2.10] and Theorem 2.3. □

Let $e, f \in \mathcal{A}$ be idempotents. Then for any $a \in \mathcal{A}$, we have

$$a = 1 \cdot a \cdot 1 = (e + 1 - e)a(f + 1 - f) = eaf + ea(1 - f) + (1 - e)af + (1 - e)a(1 - f).$$

we may write a as follows

$$a = \begin{pmatrix} eaf & ea(1-f) \\ (1-e)af & (1-e)a(1-f) \end{pmatrix}_{e,f}. \tag{3.1}$$

This matrix representation of a is called the Pierce decomposition of a. The usual algebraic operations $a + b$ and ab in $\mathcal{A}$ can be interpreted as simple operations between appropriate matrices over $\mathcal{A}$.

Theorem 3.12. *[16] Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}$ is pseudo Drazin invertible if and only if there exists an idempotent $e \in \mathcal{A}$ such that*

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_{e,e}$$

where $a_1 \in \mathcal{U}(e\mathcal{A}e)$ and $a_2 \in \sqrt{\mathcal{J}((1-e)\mathcal{A}(1-e))}$. In this case, the pseudo Drazin inverse of a is given by

$$a^{pd} = \begin{pmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{e,e}.$$

Using the Pierce decomposition, we give a necessary and sufficient condition for an element in Banach algebra to be weighted pseudo Drazin invertible.

Theorem 3.13. *Let $w \in \mathcal{U}(\mathcal{A})$. An element $a \in \mathcal{A}$ is w-pseudo Drazin invertible if and only if there exist two idempotents e and f such that*

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_{e,f}, \; w = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}_{f,e};$$

where $a_1w_1 \in \mathcal{U}(e\mathcal{A}e)$, $w_1a_1 \in \mathcal{U}(f\mathcal{A}f)$, $a_2w_2 \in \sqrt{\mathcal{J}((1-e)\mathcal{A}(1-e))}$ and $w_2a_2 \in \sqrt{\mathcal{J}((1-f)\mathcal{A}(1-f))}$.

In this case, the w-pseudo Drazin inverse of a satisfies

$$a^{pd,w} = \begin{pmatrix} a_1((w_1a_1)^{-1})^2 & 0 \\ 0 & 0 \end{pmatrix}_{e,f} = \begin{pmatrix} ((a_1w_1)^{-1})^2a_1 & 0 \\ 0 & 0 \end{pmatrix}_{e,f}.$$

Proof. ⇒) Assume that a is w-pseudo Drazin invertible. Then aw and wa are pseudo Drazin invertible elements by Theorem 2.3. It follows from Theorem 3.12 that

$$aw = \begin{pmatrix} (aw)_1 & 0 \\ 0 & (aw)_2 \end{pmatrix}_{e,e}, \; wa = \begin{pmatrix} (wa)_1 & 0 \\ 0 & (wa)_2 \end{pmatrix}_{f,f},$$

where $e = aw(aw)^{pd}$, $f = wa(wa)^{pd}$ and $(aw)_1 \in \mathcal{U}(e\mathcal{A}e)$, $(wa)_1 \in \mathcal{U}(f\mathcal{A}f)$, $(aw)_2 \in \sqrt{\mathcal{J}((1-e)\mathcal{A}(1-e))}$, $(wa)_2 \in \sqrt{\mathcal{J}((1-f)\mathcal{A}(1-f))}$.

We have

$$\begin{aligned} ea &= (aw)(aw)^{pd}a \\ &= (aw)(aw)((aw)^{pd})^2a \\ &= (aw)(aw)a((wa)^{pd})^2 \quad \text{by (2.1)} \\ &= a(wa)(wa)^{pd} \\ &= af. \end{aligned}$$

Also,

$$\begin{array}{rcl} we &=& w(aw)(aw)^{pd} \\ &=& w(aw)(aw)((aw)^{pd})^2 \\ &=& w(aw)((aw)^{pd})^2 aw \\ &=& w(aw)a((wa)^{pd})^2 w \ \text{ by (2.1)} \\ &=& (wa)(wa)^{pd} w \\ &=& fw. \end{array}$$

Then $ea = af$ and $we = fw$ imply that $ea(1-f) = (1-e)af = fw(1-e) = (1-f)we = 0$.

Now the Pierce decompositions of a and w are

$$a = \begin{pmatrix} eaf & 0 \\ 0 & (1-e)a(1-f) \end{pmatrix}_{e,f} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_{e,f}$$

and

$$w = \begin{pmatrix} fwe & 0 \\ 0 & (1-f)w(1-e) \end{pmatrix}_{f,e} = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}_{f,e}.$$

Hence,

$$aw = \begin{pmatrix} a_1w_1 & 0 \\ 0 & a_2w_2 \end{pmatrix}_{e,e}, \ wa = \begin{pmatrix} w_1a_1 & 0 \\ 0 & w_2a_2 \end{pmatrix}_{f,f},$$

and $a_1w_1 = (aw)_1 \in \mathcal{U}(e\mathcal{A}e)$, $w_1a_1 = (wa)_1 \in \mathcal{U}(f\mathcal{A}f)$, $a_2w_2 = (aw)_2 \in \sqrt{\mathcal{J}((1-e)\mathcal{A}(1-e))}$, $w_2a_2 = (wa)_2 \in \sqrt{\mathcal{J}((1-f)\mathcal{A}(1-f))}$.

Finally, by (2.1) the pseudo Drazin inverse of a is

$$\begin{array}{rcl} a^{pd,w} &=& a((wa)^{pd})^2 \\ &=& \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_{e,f} \begin{pmatrix} (w_1a_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}^2_{f,f} \\ &=& \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_{e,f} \begin{pmatrix} ((w_1a_1)^{-1})^2 & 0 \\ 0 & 0 \end{pmatrix}_{f,f} \\ &=& \begin{pmatrix} a_1((w_1a_1)^{-1})^2 & 0 \\ 0 & 0 \end{pmatrix}_{e,f}. \end{array}$$

By the same way we get that $a^{pd,w} = \begin{pmatrix} ((a_1w_1)^{-1})^2a_1 & 0 \\ 0 & 0 \end{pmatrix}_{e,f}$.

$\Leftarrow$) Assume that there exist two idempotents e and f such that

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_{e,f}, \ w = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}_{f,e},$$

where $a_1w_1 \in \mathcal{U}(e\mathcal{A}e)$, $w_1a_1 \in \mathcal{U}(f\mathcal{A}f)$, $a_2w_2 \in \sqrt{\mathcal{J}((1-e)\mathcal{A}(1-e))}$ and $w_2a_2 \in \sqrt{\mathcal{J}((1-f)\mathcal{A}(1-f))}$. Then

$$aw = \begin{pmatrix} a_1w_1 & 0 \\ 0 & a_2w_2 \end{pmatrix}_{e,e}$$

and $a_1w_1 \in (e\mathcal{A}e)^{-1}$, $a_2w_2 \in \sqrt{\mathcal{J}((1-e)\mathcal{A}(1-e)}$. Therefore, aw is pseudo Drazin invertible by Theorem 3.12 and so a is w-pseudo Drazin invertible by Theorem 2.3. $\square$

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