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(L, M)-Fuzzy Topological-Convex Spaces

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Abstract. In this paper, the notion of (L, M)-fuzzy topological-convex spaces is introduced and some of its characterizations are obtained. Then the notion of (L, M)-fuzzy convex enclosed relation spaces is introduced and its one-to-one correspondence with (L, M)-fuzzy convex space is studied. Based on this, the notion of (L, M)-fuzzy topological-convex enclosed relation spaces is introduced and its categorical isomorphism to (L, M)-fuzzy topological-convex spaces is discussed.

1. Introduction

By convex sets, we traditionally refer to convex sets in Euclidean spaces, where the property 'convexity' was originally inspired by some elementary geometric problems such as shapes of circles and characterizations of polytopes. However, with the increasing fields that convexity involved and the expanding scopes that convexity were applied, many complex convexity problems compelled people to engage in the axiomatic research of convex sets. This leads to the theory of convex spaces, where an abstract convex structure is a set-theoretic structure satisfying several axioms [22]. Its categorical properties has been studied recently [24]. Based on theories of topological spaces and convex spaces, topological-convex space has been introduced and some of its characterizations have been studied [2].

Convex structure has been extended into fuzzy settings by many ways. Maruyama introduced L-convex structure [7] whose characterizations and properties have been discussed [5, 6, 11, 12, 26, 29]. Actually, an L-convex structure is a crisp family of L-fuzzy sets satisfying certain set of axioms that is similar to that an abstract convex structure has. However, from a totally different point of view, Shi and Xiu introduced M-fuzzifying convex structures [18]. Many subsequent studies have been done [23, 27, 28, 31–33]. Further, Shi and Xiu introduced (L, M)-fuzzy convex structure which is a unified form of L-convex structure and M-fuzzifying convex structure [19]. Based on this concept, many characterizations and related theories have been discussed [11, 13, 20, 25, 30].

The aim of this paper is to introduce and characterize (L, M)-fuzzy topological-convex spaces. The arrangement of this paper is as follows. In Section 2, we recall some basic concepts, denotations and results. In Section 3, we define (L, M)-fuzzy topological-convex space and obtain some characterizations.

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In Section 4, we define (L,M)-fuzzy topological-convex enclosed relation space, by which we obtain a characterization of (L,M)-fuzzy topological-convex spaces in a pointview of category aspect.

2. Preliminaries

In this paper, both L and M are completely distributive lattices, and M has an inverse involution $'$. The smallest element and the greatest element of L or M are respectively denoted by $\bot$ and $\top$.

An element $a \in M$ is a prime, if $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$ for all $b, c \in M$. The set of all primes in $M \backslash \{\top\}$ is denoted by $P(M)$. The set of all co-primes in $M \backslash \{\bot\}$ is $J(M) = \{a \in M : a' \in P(M)\}$. A binary relation $\prec$ on M is defined by: for all $a, b \in M$, $a \prec b$ if for all $\varphi \subseteq M$, $b \leq \bigvee \varphi$ implies some $d \in \varphi$ such that $a \leq d$. Clearly, $\beta(\bigvee_{i\in I} a_i) = \bigcup_{i\in I} \beta(a_i)$ for all $\{a_i\}_{i\in I} \subseteq M$, where $\beta(a) = \{b : b \prec a\}$ for all $a \in M$. The opposite relation $\prec^{op}$ of $\prec$ is defined by: $b \prec^{op} a$ if $a' \prec b'$. Clearly, $\alpha(\bigwedge_{i\in I} a_i) = \bigcup_{i\in I} \alpha(a_i)$ for all $\{a_i\}_{i\in I} \subseteq M$, where $\alpha(a) = \{b : a \prec^{op} b\}$ for all $a \in M$. Also, $a = \bigvee \beta(a) = \bigvee \beta^*(a) = \bigwedge \alpha(a) = \bigwedge \alpha^*(a)$ for all $a \in M$, where $\beta^*(a) = \beta(a) \cap J(M)$ and $\alpha^*(a) = \alpha(a) \cap P(M)$. For $p, q \in M$, $p \leq q$ iff $a \prec p$ implies $a \leq q$ for all $a \in \beta^*(\top)$ iff $p \not\leq a$ implies $q \not\leq a$ for all $a \in P(M)$ [18, 30].

X, Y are nonempty. 2^X is the power set of X and 2^X_{fin} is the set of all finite subsets of X. L^X is the set of all L-fuzzy sets on X, whose greatest (resp. smallest) element is $\underline{\top}$ (resp. $\underline{\bot}$). An L-fuzzy set with the constant value $\lambda \in L$ is denoted by $\underline{\lambda}$. A subset $\varphi \subseteq L^X$ is said to be up-directed, denoted by $\varphi \overset{dir}{\subseteq} L^X$, if for all $A_i, A_j \in \varphi$ there is $A_k \in \varphi$ such that $A_i, A_j \leq A_k$. In this case, we denote $\bigvee \varphi$ by $\bigvee^{dir} \varphi$. For any $x \in X$ and any $\lambda \in L$, the L-fuzzy set $x_\lambda \in L^X$ is called an L-fuzzy point which is defined by $x_\lambda(x) = \lambda$ and $x_\lambda(y) = \bot$ for any $y \in X \backslash \{x\}$. In particular, x_λ is called a molecular in L^X if $\lambda \in J(L)$. The set of all moleculars in L^X is denoted by $J(L^X)$. For any $A \in L^X$, we denote $\beta(A) = \{x_\lambda \in L^X : \lambda \in \beta(A(x))\}$ and $\beta^*(A) = \beta(A) \cap J(L^X)$ [15]. Further, for any $A \in L^X$, we also denote $\mathfrak{F}(A) = \{F \in L^X : \exists \varphi \in 2^{\beta^*(A)}_{fin}, F = \bigvee \varphi\}$ [30]. In particular, $\mathfrak{F}(\underline{\top})$ is written by $\mathfrak{F}(L^X)$. It is proved that (1) $B \leq A$ iff $\mathfrak{F}(B) \subseteq \mathfrak{F}(A)$; (2) $\beta^*(A) \subseteq \mathfrak{F}(A) \overset{dir}{\subseteq} L^X$ and $\bigvee \mathfrak{F}(A) = A$; (3) $\mathfrak{F}(\bigvee^{dir}_{i\in I} A_i) = \bigcup_{i\in I} \mathfrak{F}(A_i)$ [30].

Denotations not mentioned here can be seen in [15, 16, 30]. Next, we recall some definitions and results of (L,M)-fuzzy closure structures, (L,M)-fuzzy topologies and (L,M)-fuzzy convex structures.

Definition 2.1. ([1, 16]) A mapping $\mathcal{T} : L^X \to M$ is called an (L,M)-fuzzy closure structure and the pair $(X, \mathcal{T})$ is called an (L,M)-fuzzy closure space, if

(LMT1) $\mathcal{T}(\underline{\bot}) = \mathcal{T}(\underline{\top}) = \top$;

(LMT2) $\mathcal{T}(\bigwedge\limits_{i\in I} A_i) \geq \bigwedge\limits_{i\in I} \mathcal{T}(A_i)$ for $\{A_i\}_{i\in I} \subseteq L^X$.

Theorem 2.2. ([16]) *The (L,M)-fuzzy closure operator $cl_{\mathcal{T}} : L^X \to M^{J(L^X)}$ (briefly, cl) of an (L,M)-fuzzy closure space $(X, \mathcal{T})$ is defined by:*

$$\forall A \in L^X, x_\lambda \in J(L^X),\ cl(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq A} [\mathcal{T}(B)]'.$$

Then cl satisfies:

(LMCL0) $cl(A)(x_\lambda) = \bigwedge_{\mu \prec \lambda} cl(A)(x_\mu)$;

(LMCL1) $cl(\underline{\bot})(x_\lambda) = \bot$;

(LMCL2) $cl(A)(x_\lambda) = \top$ *whenever* $x_\lambda \leq A$;

(LMCL3) $cl(A) \leq cl(B)$ *whenever* $A \leq B$;

(LMCL4) $cl(A)(x_\lambda) = \bigwedge\limits_{x_\lambda \not\leq B \geq A} \bigvee\limits_{y_\mu \not\leq B} cl(B)(y_\mu)$.

Conversely, if $cl : L^X \to M^{J(L^X)}$ *satisfies (LMCl1) –(LMCl4), then* $\mathcal{T}_{cl} : L^X \to M$*, defined by:*

$$\forall A \in L^X,\ \mathcal{T}_{cl}(A) = \bigwedge_{x_\lambda \not\leq A} [cl(A)(x_\lambda)]',$$

is an (L,M)-fuzzy closure structure with $cl_{\mathcal{T}_{cl}} = cl$.

If an operator cl satisfies (LMCL0)–(LMCL3), then (LMCL4) is equivalent to (LMCL4*) [15, 16].
(LMCL4*) $cl(\bigvee cl(A)_{[a]})_{[a]} \subseteq cl(A)_{[a]}$ for all $a \in M$.

Definition 2.3. ([21]) An (L, M)-fuzzy closure structure $\mathcal{T} : L^X \to M$ is called an (L, M)-cotopology and the pair $(X, \mathcal{T})$ is called an (L, M)-fuzzy cotopological space, if $\mathcal{T}$ further satisfies
(LMT3) $\mathcal{T}(A \vee B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$ for $A, B \in L^X$.

Theorem 2.4. ([15]) *If $(X, \mathcal{T})$ is an (L, M)-fuzzy topological space, then $cl_{\mathcal{T}} : L^X \to M^{J(L^X)}$ (briefly, cl) satisfies (LMCL0)–(LMCL5), where*
(LMCL5) $cl(A \vee B)(x_\lambda) = cl(A)(x_\lambda) \vee cl(B)(x_\lambda)$.
Conversely, if $cl : L^X \to M^{J(L^X)}$ satisfies (LMCL1) –(LMCL5), then $\mathcal{T}_{cl}$ is an (L, M)-fuzzy topology.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be (L, M)-fuzzy topological spaces. A mapping $f : X \to Y$ is called an (L, M)-fuzzy continuous mapping, if $cl_{\mathcal{T}_X}(A)(x_\lambda) \leq cl_{\mathcal{T}_Y}(f_L^{\to}(A))(f_L^{\to}(x_\lambda))$ for $A \in L^X$ and $x_\lambda \in J(L^X)$ [15, 21].

Definition 2.5. ([19]) An (L, M)-fuzzy closure structure $\mathcal{C} : L^X \to M$ is called an (L, M)-fuzzy convex structure and the pair $(X, \mathcal{C})$ is called an (L, M)-fuzzy convex space, if $\mathcal{C}$ further satisfies

(LMC3) $\mathcal{C}(\bigvee_{i \in I}^{dir} A_i) \geq \bigwedge_{i \in I} \mathcal{C}(A_i)$ for any up-directed set $\{A_i\}_{i \in I} \overset{dir}{\subseteq} L^X$.

Theorem 2.6. ([30]) *The (L, M)-fuzzy closure operator of an (L, M)-fuzzy convex space $(X, \mathcal{C})$ is also called the (L, M)-fuzzy hull operator which is denoted by $co_{\mathcal{C}}$ or co. It satisfies (LMCL0)–(LMCL4) and*
(LMDF) $co(A)(x_\lambda) = \bigwedge\limits_{\mu \in \beta^*(\lambda)} \bigvee\limits_{F \in \mathfrak{F}(A)} co(F)(x_\mu)$.
Conversely, for an operator co satisfying (LMCL1)–(LMCL4) and (LMDF), $\mathcal{C}_{co} : L^X \to M$, defined by:

$$\forall A \in L^X, \;\; \mathcal{C}_{co}(A) = \bigwedge_{x_\lambda \nleq A} [co(A)(x_\lambda)]',$$

is an (L, M)-fuzzy convex structure with $co_{\mathcal{C}_{co}} = co$.

Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be (L, M)-fuzzy convex spaces. A mapping $f : X \to Y$ is called an (L, M)-fuzzy convexity preserving (briefly, (L, M)-fuzzy CP) mapping, if $co_{\mathcal{C}_X}(A)(x_\lambda) \leq co_{\mathcal{C}_Y}(f_L^{\to}(A))(f_L^{\to}(x_\lambda))$ for $A \in L^X$ and $x_\lambda \in J(L^X)$ [19].

Definition 2.7. ([19]) Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex space. A mapping $\mathcal{B} : L^X \to M$ is called an (L, M)-fuzzy convex base of $\mathcal{C}$, if

$$\forall A \in L^X, \;\; \mathcal{C}(A) = \bigvee_{\bigvee_{i \in I}^{dir} A_i = A} \; \bigwedge_{i \in I} \mathcal{B}(A_i).$$

If $\mathcal{B}$ is an (L, M)-fuzzy convex base of $\mathcal{C}$, then $\mathcal{C}$ is denoted by $\mathcal{C}_{\mathcal{B}}$. Any (L, M)-fuzzy closure structure $\mathcal{T}$ is an (L, M)-fuzzy convex base [30].

Definition 2.8. ([20]) A binary relation $\mathcal{E} : L^X \times L^X \to M$ is called an (L, M)-fuzzy topological enclosed relation and the pair $(X, \mathcal{E})$ is called an (L, M)-fuzzy topological enclosed ralation space, if
(LMTER1) $\mathcal{E}(\underline{\bot}, \underline{\bot}) = \top$;
(LMTER2) $\mathcal{E}(A, B) > \bot$ implies $A \leq B$;
(LMTER3) $\mathcal{E}(A, \bigwedge_{i \in I} B_i) = \bigwedge_{i \in I} \mathcal{E}(A, B_i)$;
(LMTER4) $\mathcal{E}(A, B) \leq \bigvee_{C \in L^X} \mathcal{E}(A, C) \wedge \mathcal{E}(C, B)$;
(LMTER5) $\mathcal{E}(A \vee B, C) = \mathcal{E}(A, C) \wedge \mathcal{E}(B, C)$.

Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be (L, M)-fuzzy topological enclosed relation spaces. A mapping $f : X \to Y$ is called an (L, M)-fuzzy topological enclosed relation preserving (briefly, (L, M)-fuzzy TERP) mapping if

$$\mathcal{E}_Y(f^{\to}(U), V) \leq \mathcal{E}_X(U, f_L^{\leftarrow}(V))$$

for any $U \in L^X$ and any $V \in L^Y$.

In [20], it is proved that there is a one-to-one correspondence between (L, M)-fuzzy topological enclosed relations and (L, M)-fuzzy cotopologies.

3. (L, M)-Fuzzy Topological-Convex Spaces

In [22], a topological-convex space consists a cotopology and a convex structure compatible with each other. Based on such space, many combining properties such as continuities of convex hull operators, uniformities of convex spaces and topological-convex separations can be studied intensively. Thus topological-convex is a key link in combining Topological Theory and Convex Theory.

Recall that if X is a set equipped with a cotopology $\mathcal{T}$ and a convex structure C, then the triple $(X, \mathcal{T}, C)$ is called a topological-convex space provided that $\mathcal{T}$ is compatible with C, that is, all polytopes are closed ($co_C(F) \in \mathcal{T}$ for any $F \in 2^X_{fin}$) [22]. In this section, we extend this concept into (L, M)-fuzzy settings and obtain some of its characterizations. Before this, we give a brief observation of topological-convex spaces.

Remark 3.1. Let X be a set equipped with a cotopology $\mathcal{T}$ and a convex structure C. Then $\mathcal{T} \cap C$ is a closure structure whose closure operator is denoted by $cl_{\mathcal{T}\cap C}$.

(1) A cotopology $\mathcal{T}$ is compatible with a convex structure C iff $co_C(F) = cl_{\mathcal{T}\cap C}(F)$ for all $F \in 2^X_{fin}$.

Indeed, the sufficiency is clear. For the necessity, if $\mathcal{T}$ is compatible with C, then $co_C(F) = cl_{\mathcal{T}}(co_C(F))$ for any $F \in 2^X_{fin}$. Thus $co_C(F) \in \mathcal{T} \cap C$ showing that

$$co_C(F) \subseteq cl_{\mathcal{T}\cap C}(F) \subseteq cl_{\mathcal{T}}(co_C(F)) = co_C(F).$$

(2) In a convex space (X, C), a subset $\mathcal{B} \subseteq C$ is called a base if there is an up-directed subset $\mathcal{B}_1 \subseteq \mathcal{B}$ such that $A = \bigcup \mathcal{B}$ for each $A \in C$. In addition, a subset $\mathcal{B} \subseteq C$ is a convex base of C iff $\mathcal{B}$ contains all polytopes [22]. Thus, $\mathcal{T}$ is compatible with C iff C has a closed base (i.e., C has a convex base $\mathcal{B} \subseteq \mathcal{T} \cap C$).

Based on Remark 3.1 and Definition 2.7, we present (L, M)-fuzzy topological-convex spaces as follows.

Definition 3.2. Let X be a set equipped with an (L, M)-fuzzy cotopology $\mathcal{T}$ and an (L, M)-fuzzy convex structure C. The triple $(X, \mathcal{T}, C)$ is called an (L, M)-fuzzy topological-convex space if $\mathcal{T}$ is compatible with C. That is, C has an (L, M)-fuzzy convex base $\mathcal{B} : L^X \to M$ such that $\mathcal{B} \leq \mathcal{T} \wedge C$.

Let $(X, \mathcal{T}_X, C_X)$ and $(Y, \mathcal{T}_Y, \mathcal{T}_Y)$ be (L, M)-fuzzy topological-convex spaces. A mapping $f : X \to Y$ is called an (L, M)-fuzzy topology-convexity structure preserving (or, (L, M)-fuzzy TCP) mapping, if $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is an (L, M)-fuzzy continuous mapping and $f : (X, C_X) \to (Y, C_Y)$ is an (L, M)-fuzzy CP mapping.

The category of (L, M)-fuzzy topological-convex spaces and (L, M)-fuzzy TCP mappings is denoted by (L, M)-**TCS**.

Clearly, an (L, M)-fuzzy cotopology $\mathcal{T}$ is compatible with an (L, M)-fuzzy convex structure C iff $\mathcal{T} \wedge C$ is an (L, M)-fuzzy convex base of C. Next, we characterize (L, M)-fuzzy topological-convex spaces.

Theorem 3.3. *If X is equipped with an (L, M)-fuzzy cotopology $\mathcal{T}$ and an (L, M)-fuzzy convex structure C, then the following conditions are equivalent:*

(1) $\mathcal{T}$ is compatible with C;

(2) For any $A \in L^X$ and any $x_\lambda \in J(L^X)$,

$$co_C(A)(x_\lambda) = \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(A)} \bigwedge_{x_\mu \not\leq B \geq G} [(\mathcal{T} \wedge C)(B)]';$$

(3) For any $F \in \mathfrak{F}(L^X)$ and any $x_\lambda \in J(L^X)$,

$$co_C(F)(x_\lambda) = \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} \bigwedge_{x_\mu \not\leq B \geq G} [(\mathcal{T} \wedge C)(B)]'.$$

Proof. (1) ⇒ (2). Let $\mathcal{B} \leq \mathcal{T} \wedge C$ be an (L, M)-fuzzy convex base of C. By (LMDF),

$$co_C(A)(x_\lambda) = \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(A)} \bigwedge_{x_\mu \not\leq B \geq G} [C(B)]' \leq \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(A)} \bigwedge_{x_\mu \not\leq B \geq G} [(\mathcal{T} \wedge C)(B)]'.$$

Conversely, for $G \in \mathfrak{F}(A)$, $B \in L^X$, $\{B_i\}_{i\in I} \overset{dir}{\subseteq} L^X$ with $x_\mu \not\leq B \geq A \geq G$ and $\bigvee_{i\in I}^{dir} B_i = B$, there is $i_G \in I$ such that $x_\mu \not\leq B_{i_G} \geq G$. Thus, by (LMCL0),

$$\begin{aligned} co_{\mathcal{C}}(A)(x_\lambda) &= \bigwedge_{\mu<\lambda} \bigwedge_{x_\mu \not\leq B \geq A} [\mathcal{C}(B)]' \\ &= \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(A)} \bigwedge_{x_\mu\not\leq B\geq A\geq G} \bigwedge_{\bigvee_{i\in I}^{dir} B_i = B} \bigvee_{i\in I} [\mathcal{B}(B_i)]' \\ &\geq \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(A)} \bigwedge_{x_\mu\not\leq B\geq A\geq G} \bigwedge_{\bigvee_{i\in I}^{dir} B_i = B} [\mathcal{B}(B_{i_G})]' \\ &\geq \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(A)} \bigwedge_{x_\mu\not\leq D\geq G} [\mathcal{B}(D)]' \\ &\geq \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(A)} \bigwedge_{x_\mu\not\leq D\geq G} [(\mathcal{T} \wedge \mathcal{C})(D)]'. \end{aligned}$$

(2) ⇒ (3). Clear.
(3) ⇒ (1). Let $A \in L^X$. By (LMDF), we have

$$\begin{aligned} \mathcal{C}(A) &= \bigwedge_{x_\lambda \not\leq A} [co_{\mathcal{C}}(A)(x_\lambda)]' \\ &= \bigwedge_{x_\lambda\not\leq A} \bigvee_{\mu<\lambda} \bigwedge_{F\in\mathfrak{F}(A)} [co_{\mathcal{C}}(F)(x_\mu)]' \\ &= \bigwedge_{x_\lambda\not\leq A} \bigvee_{\mu<\lambda} \bigwedge_{F\in\mathfrak{F}(A)} \bigvee_{x_\mu\not\leq B\geq F} (\mathcal{T} \wedge \mathcal{C})(B) \\ &\leq \bigwedge_{x_\lambda\not\leq A} [co_{\mathcal{C}_{\mathcal{T}\wedge\mathcal{C}}}(A)(x_\lambda)]' = \mathcal{C}_{\mathcal{T}\wedge\mathcal{C}}(A). \end{aligned}$$

On the other hand, by (LMC3), we have

$$\mathcal{C}_{\mathcal{T}\wedge\mathcal{C}}(A) = \bigvee_{\bigvee_{i\in I}^{dir} A_i = A} \bigwedge_{i\in I} (\mathcal{T} \wedge \mathcal{C})(B_i) \leq \bigvee_{\bigvee_{i\in I}^{dir} A_i = A} \bigwedge_{i\in I} \mathcal{C}(B_i) \leq \mathcal{C}(A).$$

Thus $\mathcal{C}(A) = \mathcal{C}_{\mathcal{T}\wedge\mathcal{C}}(A)$ showing that $\mathcal{T} \wedge \mathcal{C}$ is an (L, M)-fuzzy convex base of $\mathcal{C}$. □

Corollary 3.4. *An (L, M)-fuzzy cotopology $\mathcal{T}$ is compatible with an (L, M)-fuzzy convex structure $\mathcal{C}$ on X iff for any $A \in L^X$ and any $x_\lambda \in J(L^X)$, one of the following conditions holds:*
(1) $co_{\mathcal{C}}(A)(x_\lambda) = \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(A)} cl_{\mathcal{T}\wedge\mathcal{C}}(G)(x_\mu)$;
(2) $co_{\mathcal{C}}(F)(x_\lambda) = \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} cl_{\mathcal{T}\wedge\mathcal{C}}(G)(x_\mu)$.

Remark 3.5. (1) An $(L, \mathbf{2})$-fuzzy topological-convex space is reduced to be an L-topological-convex space [6]. That is, $\mathcal{C}$ has an L-convex base $\mathcal{B}$ such that $\mathcal{B} \subseteq \mathcal{T} \cap \mathcal{C}$, where a subset $\mathcal{B} \subseteq \mathcal{C}$ is an L-convex base of $\mathcal{C}$, if there is an up-directed subset $\mathcal{B}_1 \subseteq \mathcal{B}$ such that $A = \bigvee \mathcal{B}_1$ for any $A \in \mathcal{C}$. In addition, $\mathcal{T}$ is compatible with $\mathcal{C}$ iff $co_{\mathcal{C}}(F) = \bigvee_{G\in\mathfrak{F}(F)} cl_{\mathcal{T}\wedge\mathcal{C}}(G)$ for any $F \in \mathfrak{F}(L^X)$.

(2) A $(\mathbf{2}, M)$-fuzzy topological-convex space $(X, \mathcal{T}, \mathcal{C})$ is called an M-fuzzifying topological-convex space [23], where the M-fuzzifying cotopology $\mathcal{T}$ is compatible with the M-fuzzifying convex structure $\mathcal{C}$. That is, there is an M-fuzzifying convex base $\mathcal{B} : 2^X \to M$ of $\mathcal{C}$ such that $\mathcal{B} \leq \mathcal{T} \wedge \mathcal{C}$. Further, by Theorem 3.3 and Corollary 3.4, $(X, \mathcal{T}, \mathcal{C})$ is an M-fuzzifying topological-convex space iff for $F \in 2^X_{fin}$ and $x \in X$,

$$co_{\mathcal{C}}(F)(x) = \bigwedge_{x\notin B\geq F} [(\mathcal{T} \wedge \mathcal{C})(B)]' = cl_{\mathcal{T}\wedge\mathcal{C}}(F)(x).$$

(3) A $(\mathbf{2},\mathbf{2})$-fuzzy topological-convex space is a topological-convex space [22].

Theorem 3.6. *Let $(X,\mathcal{T})$ be an (L,M)-fuzzy cotopological space, and let $\mathcal{C}_{\mathcal{T}}: L^X \to M$ be the (L,M)-fuzzy convex structure generated by $\mathcal{T}$, that is,*

$$\forall A \in L^X,\ \ \mathcal{C}_{\mathcal{T}}(A) = \bigvee_{\bigvee^{dir}_{i\in I} B_i = A} \ \bigwedge_{i\in I} \mathcal{T}(B_i).$$

Then $\mathcal{T}$ is compatible with $\mathcal{C}_{\mathcal{T}}$.

Proof. We have $(X,\mathcal{C}_{\mathcal{T}})$ is an (L,M)-fuzzy convex space [30]. To prove that the desired result, let $A \in L^X$ and $x_\lambda \in J(L^X)$. Since $\mathcal{C}_{\mathcal{T}} \geq \mathcal{T}$, we have $co_{\mathcal{C}_{\mathcal{T}}}(A)(x_\lambda) \leq cl_{\mathcal{T}}(A)(x_\lambda)$.

Let $G \in \mathfrak{F}(A)$ and $\mu \in \beta^*(\lambda)$. For all $B \in L^X$ and $\{D_i\}_{i\in I} \overset{dir}{\subseteq} L^X$ with $x_\lambda \not\leq B \geq A$ and $\bigvee_{i\in I} D_i = B$, there is $D_G \in \{D_i\}_{i\in I}$ such that $G \leq D_G$. Thus

$$co_{\mathcal{C}_{\mathcal{T}}}(A)(x_\mu) = \bigwedge_{x_\mu \not\leq B \geq A} \ \bigwedge_{\bigvee^{dir}_{i\in I} D_i = B} \ \bigvee_{i\in I} \ \bigvee_{y_\eta \not\leq D_i} cl_{\mathcal{T}}(D_i)(y_\eta) \geq \bigwedge_{x_\mu \not\leq B \geq A} \ \bigwedge_{\bigvee^{dir}_{i\in I} D_i = B} cl_{\mathcal{T}}(D_G)(x_\mu) \geq cl_{\mathcal{T}}(G)(x_\mu).$$

Hence $co_{\mathcal{C}_{\mathcal{T}}}(A)(x_\mu) \geq \bigvee_{G\in\mathfrak{F}(A)} cl_{\mathcal{T}}(G)(x_\mu)$ and

$$co_{\mathcal{C}_{\mathcal{T}}}(A)(x_\lambda) \geq \bigwedge_{\mu<\lambda} \ \bigvee_{G\in\mathfrak{F}(A)} cl_{\mathcal{T}}(G)(x_\mu) \geq \bigwedge_{\mu<\lambda} \ \bigvee_{G\in\mathfrak{F}(A)} co_{\mathcal{C}_{\mathcal{T}}}(G)(x_\mu) = co_{\mathcal{C}_{\mathcal{T}}}(A)(x_\lambda).$$

From this result and $\mathcal{T} \leq \mathcal{C}_{\mathcal{T}}$, we conclude that

$$co_{\mathcal{C}_{\mathcal{T}}}(A)(x_\lambda) \ \ = \ \ \bigwedge_{\mu<\lambda} \ \bigvee_{G\in\mathfrak{F}(A)} \ \bigwedge_{x_\mu \not\leq B \geq G} [(\mathcal{T} \wedge \mathcal{C}_{\mathcal{T}})(B)]'.$$

Therefore $\mathcal{T}$ is compatible with $\mathcal{C}_{\mathcal{T}}$. □

By Theorem 3.6, many mathematical structures induce (L,M)-fuzzy topological-convex spaces. We list some of them as follows.

Example 3.7. Let (X,d) be an (L,M)-fuzzy metric space [17], where the (L,M)-fuzzy pseudo-quasi-metric $d: J(L^X) \times J(L^X) \to [0,+\infty)(M)$ satisfies: for $x_\lambda, y_\mu, z_\gamma \in J(L^X)$,

(LMd1) $d(x_\lambda, x_\lambda)(0+) = \bigvee_{t>0} d(x_\lambda, x_\lambda)(t) = \bot$;

(LMd2) $d(x_\lambda, y_\mu) = d(y_\mu, x_\lambda)$;

(LMd3) $d(x_\lambda, z_\gamma)(r+s) \leq d(x_\lambda, y_\mu)(r) \vee d(y_\mu, z_\gamma)(s)$ for all $r, s > 0$.

Let $cl_d: L^X \to M^{J(L^X)}$ be defined by:

$$cl_d(A)(x_\lambda) = \bigwedge_{r>0} \ \bigvee_{y_\mu \leq A} [d(x_\lambda, y_\mu)(r)]',$$

for any $A \in L^X$ and any $x_\lambda \in J(L^X)$. Then cl_d is an (L,M)-fuzzy closure operator inducing an (L,M)-fuzzy topology $\mathcal{T}_d$ [17]. Thus $(X,\mathcal{T}_d,\mathcal{C}_d)$ is an (L,M)-fuzzy topological-convex space, where $\mathcal{C}_d$ is the (L,M)-fuzzy convex structure generated by $\mathcal{T}_d$.

Example 3.8. Let $H(L^X)$ be the family of all mappings $d: L^X \to L^X$ satisfying:

(1) $A \leq d(A)$ for $A \in L^X$;

(2) $d(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} d(A_i)$ for $\{A_i\}_{i\in I} \subseteq L^X$.

The biggest element $d_\top$ of $H(L^X)$ which is defined by: $d_\top(\underline{\bot}) = \underline{\bot}$ and $d_\top(A) = \underline{\top}$ for all $A \in L^X \backslash \{\underline{\bot}\}$. For all $d, e \in H(L^X)$, we have $d \wedge e, d \circ e \in H(L^X)$, where $d \wedge e, d \circ e : L^X \to L^X$ are defined by:

$$(d \wedge e)(A) = \bigvee_{x_\lambda \in \beta^*(A)} d(x_\lambda) \wedge e(x_\lambda)$$

and $(d \circ e)(A) = d(e(A))$ for any $A \in L^X$.

The mapping $\mathcal{FU} : H(L^X) \to M$ is called an (L, M)-fuzzy quasi-uniformity and $(X, \mathcal{FU})$ is called an (L, M)-fuzzy quasi-uniform space and [34], if

(FQU1) $\mathcal{FU}(d_\top) = \top$;

(FQU2) $\mathcal{FU}(d \wedge e) = \mathcal{FU}(d) \wedge \mathcal{FU}(d)$ for $d, e \in H(L^X)$;

(FQU3) $\mathcal{FU}(d) = \bigvee_{e \circ e \leq d} \mathcal{FU}(e)$ for $d \in H(L^X)$.

Let $\mathcal{T}_{\mathcal{FU}} : L^X \to M$ be defined by:

$$\forall A \in L^X,\ \mathcal{T}_{\mathcal{FU}}(A) = \bigwedge_{x_\lambda \not\leq^* A'} \bigvee_{x_\lambda \not\leq d(A')} \mathcal{FU}(d),$$

Then $\mathcal{T}_{\mathcal{FU}}$ is an (L, M)-fuzzy topology [34]. So $(X, \mathcal{T}_{\mathcal{FU}}, \mathcal{C}_{\mathcal{FU}})$ is an (L, M)-fuzzy topological convex space, where $\mathcal{C}_{\mathcal{FU}}$ is the (L, M)-fuzzy convex structure generated by $\mathcal{T}_{\mathcal{FU}}$.

Example 3.9. A mapping $\mathcal{F} : L^X \to M$ is called an (L, M)-fuzzy filer on X and $(X, \mathcal{F})$ is called an (L, M)-fuzzy filter space, if

(LMF1) $\mathcal{F}(\underline{\perp}) = \perp$ and $\mathcal{F}(\underline{\top}) = \top$;

(LMF2) $\mathcal{F}(A \wedge B) = \mathcal{F}(A) \wedge \mathcal{F}(B)$ for $A, B \in L^X$.

The family of all (L, M)-fuzzy filters on X is denoted by $\mathcal{F}_{LM}(X)$. For $\mathcal{F}, \mathcal{G} \in \mathcal{F}_{LM}$, the order $\mathcal{F} \leq \mathcal{G}$ implies $\mathcal{F}(A) \leq \mathcal{G}(A)$ for all $A \in L^X$.

For each $x_\lambda \in J(L^X)$, define $\hat{q}(x_\lambda) : L^X \to M$ by:

$$\forall A \in L^X,\ \hat{q}(x_\lambda)(A) = \begin{cases} \top, & x_\lambda \hat{q} A, \\ \perp, & \text{otherwise}, \end{cases}$$

where $x_\lambda \hat{q} A$ means $x_\lambda \not\leq A'$. Then $\hat{q}(x_\lambda)$ is an (L, M)-fuzzy filter.

A mapping $c : \mathcal{F}_{LM}(X) \to M^{J(L^X)}$ is called an (L, M)-fuzzy convergence structure and the pair (X, c) is called an (L, M)-fuzzy convergence space, if

(LFC1) $c(\hat{q}(x_\lambda))(x_\lambda) = \top$ for all $x_\lambda \in J(L^X)$;

(LFC2) $\mathcal{F} \leq \mathcal{G}$ implies that $c(\mathcal{F}) \leq c(\mathcal{G})$ for all $\mathcal{F}, \mathcal{G} \in \mathcal{F}_{LM}(X)$.

Let $\mathcal{T}_c : L^X \to M$ be defined by: for any $A \in L^X$,

$$\mathcal{T}_c(A) = \bigwedge_{x_\lambda \hat{q} A} \bigwedge_{\mathcal{F} \in \mathcal{F}_{LM}(X)} [c(\mathcal{F})(x_\lambda) \to \mathcal{F}(A)].$$

Then $\mathcal{T}_c$ is an (L, M)-fuzzy topology [8]. Thus $(X, \mathcal{T}_c, \mathcal{C}_c)$ is an (L, M)-fuzzy topological-convex space, where $\mathcal{C}_c$ is the (L, M)-fuzzy convex structure generated by $\mathcal{T}_c$.

Remark 3.10. (1) In (2) of Remark 3.5, we know that if X is a set equipped with an M-fuzzifying cotopology $\mathcal{T}$ and an M-fuzzifying convex structure $\mathcal{C}$, then the triple $(X, \mathcal{T}, \mathcal{C})$ is an M-fuzzifying topological-convex space iff

$$co_{\mathcal{C}}(F)(x) = \bigwedge_{x \notin B \supseteq F} [(\mathcal{T} \wedge \mathcal{C})(B)]' = cl_{\mathcal{T} \wedge \mathcal{C}}(F)(x)$$

for any $F \in 2^X_{fin}$ and any $x \in X$.

However, for an (L, M)-fuzzy cotopology $\mathcal{T}$ and an (L, M)-fuzzy convex structure $\mathcal{C}$, the condition that

$$co_{\mathcal{C}}(F)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq F} [(\mathcal{T} \wedge \mathcal{C})(B)]'$$

is just a sufficiency for the compatibility of $\mathcal{T}$ and $\mathcal{C}$. For example, let $X = \{x\}$ and $L = M = [0, 1]$. Define a mapping $\mathcal{T} : L^X \to M$ by:

$$\mathcal{T}(x_r) = \begin{cases} 1, & r \in \{0\} \cup [\frac{1}{2}, 1], \\ 0, & r \in (0, \frac{1}{2}). \end{cases}$$

Then $\mathcal{T}$ is an (L,M)-fuzzy topology and $(X,\mathcal{T},\mathcal{C}_\mathcal{T})$ is an (L,M)-fuzzy topological-convex space by Theorem 3.6. But it fails to satisfy the described condition.

In fact, $x_{\frac{1}{2}} \in \mathfrak{J}(L^X)$ and $x_{\frac{2}{3}} \in J(L^X)$. In addition, we have

$$co_{\mathcal{C}_\mathcal{T}}(x_{\frac{1}{2}})(x_{\frac{2}{3}}) = \bigwedge_{x_{\frac{2}{3}} \not\leq B \geq x_{\frac{1}{2}}} \bigwedge_{\bigvee^{dir}_{i\in I} D_i = B} \bigvee_{i\in I}[\mathcal{T}(D_i)]' \leq \bigvee_{0<r<\frac{1}{2}}[\mathcal{T}(x_r)]' = \frac{1}{2}$$

and $cl_\mathcal{T}(x_{\frac{1}{2}})(x_{\frac{2}{3}}) = \bigwedge_{x_{\frac{2}{3}} \not\leq B \geq x_{\frac{1}{2}}}[(\mathcal{T}\wedge\mathcal{C})(B)]' = 1$.

(2) By Theorem 3.6, an (L,M)-fuzzy cotopology induces an (L,M)-fuzzy topological-convex space. In fact, in an (L,M)-fuzzy topological-convex space $(X,\mathcal{T},\mathcal{C})$, $\mathcal{C}$ needs not to be generated by $\mathcal{T}$, that is, $\mathcal{C} \neq \mathcal{C}_\mathcal{T}$. For example, let $X = \{x, y\}$ and $L = M = [0,1]$. Define two mappings $\mathcal{T}, \mathcal{C} : L^X \to M$ by

$$\mathcal{C}(A) = \begin{cases} 1, & A \in \{\underline{0}, \underline{1}, x_{\frac{1}{2}}, y_{\frac{1}{2}}\}, \\ 0, & \text{otherwise;} \end{cases} \quad \text{and} \quad \mathcal{T}(A) = \begin{cases} 1, & A \in \{\underline{0}, \underline{1}, x_{\frac{1}{2}}, y_{\frac{1}{2}}, \underline{\tfrac{1}{2}}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X,\mathcal{T},\mathcal{C})$ is an (L,M)-fuzzy topological-convex space with $\mathcal{C} \neq \mathcal{C}_\mathcal{T}$.

(3) For an (L,M)-fuzzy cotopology $\mathcal{T}$ on X, $\mathcal{C}_\mathcal{T}$ is an (L,M)-fuzzy Alexander topology. To prove this, it is sufficient to prove that

$$\forall \{A_i\}_{i\in I} \subseteq L^X,\ \mathcal{C}_\mathcal{T}(\bigvee_{i\in I} A_i) \geq \bigwedge_{i\in I} \mathcal{C}_\mathcal{T}(A_i).$$

Let $a \prec \bigwedge_{i\in I}\mathcal{C}_\mathcal{T}(A_i)$. Then for each $i \in I$, there is $\varphi_i = \{D_{ij}\}_{j\in J_i} \overset{dir}{\subseteq} L^X$ such that $\bigvee_{j\in J_i} D_{ij} = A_i$ and $a \leq \bigwedge_{j\in J_i}\mathcal{T}(D_{ij})$. Let $\psi = \bigcup_{i\in I}\psi_i$ and $\varphi = \{\sqcup\phi : \psi \subseteq 2^{\psi}_{fin}\}$, where $\sqcup\phi$ stands for $\bigvee\phi$ when ϕ is finite. Then φ is up-directed satisfying $\bigvee_{i\in I} A_i = \bigvee\varphi$ and $\mathcal{T}(\sqcup\phi) \geq a$ for each $\phi \in 2^{\psi}_{fin}$. Thus $\mathcal{C}_\mathcal{T}(\bigvee_{i\in I} A_i) \geq \bigwedge_{\sqcup\phi\in\varphi}\mathcal{T}(\sqcup\phi) \geq a$. So $\bigwedge_{i\in I}\mathcal{C}_\mathcal{T}(A_i) \leq \mathcal{C}_\mathcal{T}(\bigvee_{i\in I} A_i)$.

Hence $\mathcal{C}_\mathcal{T}$ is an (L,M)-fuzzy Alexander topology.

(4) In an (L,M)-fuzzy topological-convex space $(X,\mathcal{T},\mathcal{C})$, $\mathcal{C}$ needs not to be an (L,M)-fuzzy Alexander topology. The example in (2) is of this type.

To obtain more characterizations of (L,M)-fuzzy topological-convex spaces, we construct a new (L,M)-fuzzy topology by an (L,M)-fuzzy cotopology and an (L,M)-fuzzy convex structure as follows.

Lemma 3.11. *Let X be a set equipped with an (L,M)-fuzzy cotopology $\mathcal{T}$ and an (L,M)-fuzzy convex structure $\mathcal{C}$. Define $\mathcal{T}_w : L^X \to M$ by:*

$$\forall A \in L^X,\ \mathcal{T}_w(A) = \bigvee_{\bigwedge_{i\in I} B_i = A} \bigwedge_{i\in I}(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(B_i),$$

where $\sqcup$ stands for finite joins and $(\mathcal{T}\wedge\mathcal{C})^{\sqcup} : L^X \to M$ is defined by:

$$\forall B \in L^X,\ (\mathcal{T}\wedge\mathcal{C})^{\sqcup}(B) = \bigvee_{\sqcup_{i\in I} D_i = B} \bigwedge_{i\in I}(\mathcal{T}\wedge\mathcal{C})(D_i).$$

Then $\mathcal{T}_w$ is an (L,M)-fuzzy cotopology satisfying $\mathcal{T}_w \leq \mathcal{T}$. In addition, $\mathcal{T}_w$ is the least (L,M)-fuzzy cotopology containing $\mathcal{T}\wedge\mathcal{C}$.

Proof. We prove that $\mathcal{T}_w$ satisfies (LMT1)–(LMT3).

(LMT1). It is clear since $\mathcal{T}_w \geq (\mathcal{T}\wedge\mathcal{C})^{\sqcup} \geq \mathcal{T}\wedge\mathcal{C}$.

(LMT2). Let $\{A_i\}_{i\in I} \subseteq L^X$ and $a \prec \bigwedge_{i\in I}\mathcal{T}_w(A_i)$. Then $a \prec \mathcal{T}_w(A_i)$ for each $i \in I$. Thus, for each $i \in I$, there is a family $\{D_{ij}\}_{j\in J_i} \subseteq L^X$ such that $\bigwedge_{j\in J_i} D_{ij} = A_i$ and $\bigwedge_{j\in J_i}(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(D_{ij}) \geq a$. Note that $\bigwedge_{i\in I} A_i = \bigwedge_{i\in I}\bigwedge_{j\in J_i} D_{ij}$. We have

$$\mathcal{T}_w(\bigwedge_{i\in I} A_i) \geq \bigwedge_{i\in I}\bigwedge_{j\in J_i}(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(D_{ij}) \geq a.$$

Hence $\bigwedge_{i\in I}\mathcal{T}_w(A_i)\le\mathcal{T}_w(\bigwedge_{i\in I}A_i)$.

(LMT3). Let $A,B\in L^X$. We firstly prove that

$$(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(A\vee B)\ge(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(A)\wedge(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(B).$$

Let $a\prec(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(A)\wedge(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(B)$. Then there are finite subsets $\{D_i\}_{i\in I},\{D_j\}_{j\in J}\subseteq L^X$ such that $\sqcup_{i\in I}D_i=A$, $\sqcup_{j\in J}D_i=B$ and $(\mathcal{T}\wedge\mathcal{C})(D_i)\wedge(\mathcal{T}\wedge\mathcal{C})(D_j)\ge a$ for any $i\in I$ and any $j\in J$. Since $\sqcup\{D_k:k\in I\cup J\}=A\vee B$, we have

$$(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(A\vee B)=\bigvee_{\sqcup_{k\in K}B_k=A\vee B}\ \bigwedge_{k\in K}(\mathcal{T}\wedge\mathcal{C})(B_k)\ge\bigwedge_{k\in I\cup J}(\mathcal{T}\wedge\mathcal{C})(D_k)\ge a.$$

So $(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(A\vee B)\ge(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(A)\wedge(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(B)$.

To prove that $\mathcal{T}_w(A\vee B)\ge\mathcal{T}_w(A)\wedge\mathcal{T}_w(B)$, let $a\prec\mathcal{T}_w(A)\wedge\mathcal{T}_w(B)$. Then there are $\{D_i\}_{i\in I},\{D_j\}_{j\in J}\subseteq L^X$ such that $\bigwedge_{i\in I}D_i=A$, $\bigwedge_{j\in J}D_j=B$, $\bigwedge_{i\in I}(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(D_j)\ge a$ and $\bigwedge_{j\in J}(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(D_j)\ge a$. Let $D_{ij}=D_i\vee D_j$ for all $i\in I$ and $j\in J$. We have $A\vee B=\bigwedge_{i\in I,j\in J}D_i\vee D_j=\bigwedge_{i\in I,j\in J}D_{ij}$. Thus

$$\mathcal{T}_w(A\vee B)=\bigvee_{\bigwedge_{k\in K}H_k=A\vee B}\ \bigwedge_{k\in K}(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(H_k)\ge\bigwedge_{i\in I,j\in J}(\mathcal{T}\wedge\mathcal{C})^{\sqcup}(D_{ij})\ge a.$$

Hence $\mathcal{T}_w(A\vee B)\ge\mathcal{T}_w(A)\wedge\mathcal{T}_w(B)$.

Therefore $\mathcal{T}_w$ is an (L,M)-fuzzy cotopology.

Let $A\in L^X$. By (LMT3) and (LMT2), we have

$$\mathcal{T}_w(A)=\bigvee_{\bigwedge_{i\in I}B_i=A}\ \bigwedge_{i\in I}\ \bigvee_{\bigsqcup_{j\in J_i}B_{ij}=B_i}\ \bigwedge_{j\in J_i}(\mathcal{T}\wedge\mathcal{C})(B_{ij})\le\bigvee_{\bigwedge_{i\in I}B_i=A}\ \bigwedge_{i\in I}\mathcal{T}(B_i)=\mathcal{T}(A).$$

Thus $\mathcal{T}_w\le\mathcal{T}$. Finally, if $\mathcal{D}$ is an (L,M)-fuzzy cotopology on X containing $\mathcal{T}\wedge\mathcal{C}$, then $\mathcal{D}(\sqcup_{i\in I}B_i)\ge\bigwedge_{i\in I}\mathcal{D}(B_i)\ge\bigwedge_{i\in I}(\mathcal{T}\wedge\mathcal{C})(B_i)$ for any finite subset $\{B_i\}_{i\in I}\subseteq L^X$. Hence $\mathcal{D}\ge(\mathcal{T}\wedge\mathcal{C})^{\sqcup}$ showing that $\mathcal{D}\ge\mathcal{T}_w$. Therefore $\mathcal{T}_w$ is the least (L,M)-fuzzy cotopology containing $\mathcal{T}\wedge\mathcal{C}$. □

Theorem 3.12. *Let X be equipped with an (L,M)-fuzzy cotopology $\mathcal{T}$ and an (L,M)-fuzzy convex structure $\mathcal{C}$. For $F\in\mathfrak{F}(L^X)$ and $a\in M$, we denote $D_F=\bigvee\psi(F,a)$, where*

$$\mathfrak{F}_F(L^X)=\{H\in\mathfrak{F}(L^X):F\in\mathfrak{F}(H)\}$$

and

$$\psi(F,a)=\bigcap_{H\in\mathfrak{F}_F(L^X)}co_{\mathcal{C}}(H)_{[a]}.$$

Then the following conditions are equivalent:

(1) $\mathcal{T}$ is compatible with $\mathcal{C}$;

(2) $\mathcal{T}_w$ is compatible with $\mathcal{C}$;

(3) $cl_{\mathcal{T}_w}(D_F)_{[a]}\subseteq\psi(F,a)$;

(4) $cl_{\mathcal{T}}(D_F)_{[a]}\subseteq\psi(F,a)$.

(5) $co_{\mathcal{C}}(F)(x_\lambda)=\bigwedge_{\mu\prec\lambda}\bigvee_{G\in\mathfrak{F}(F)}\bigwedge_{x_\mu\not\leq D\ge G}\bigvee_{y_\eta\not\leq D}[cl_{\mathcal{T}}\vee co_{\mathcal{C}}](D)(y_\eta)$;

(6) $co_{\mathcal{C}}(F)(x_\lambda)=\bigwedge_{\mu\prec\lambda}\bigvee_{G\in\mathfrak{F}(F)}\bigwedge_{x_\mu\not\leq D\ge G}\bigvee_{y_\eta\not\leq D}[cl_{\mathcal{T}_w}\vee co_{\mathcal{C}}](D)(y_\eta)$;

(7) $cl_{\mathcal{T}\wedge\mathcal{C}}(D_F)_{[a]}\subseteq\psi(F,a)$;

(8) $co_{\mathcal{C}}(F)(x_\lambda)=\bigwedge_{\mu\prec\lambda}\bigvee_{G\in\mathfrak{F}(F)}[\bigvee_{A\in L^X}(cl_{\mathcal{T}}(A)(x_\lambda)\wedge\bigwedge_{y_\mu\prec A}co_{\mathcal{C}}(G)(y_\mu))]$.

Proof. (1) ⇒ (2). By Lemma 3.11 and Theorem 3.3, we have

$$co_{\mathcal{C}}(F)(x_\lambda) \leq \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} \bigwedge_{x_\mu \not\leq B\geq G} [(\mathcal{T}_w \wedge \mathcal{C})(B)]'.$$

Conversely, let $a \in \beta^*(\top)$ with $a \not\leq co_{\mathcal{C}}(F)(x_\lambda)$. There is $b \in \beta^*(a)$ such that $b \not\leq co_{\mathcal{C}}(F)(x_\lambda)$. Thus

$$b \not\leq co_{\mathcal{C}}(F)(x_\lambda) = \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} \bigwedge_{x_\mu \not\leq B\geq G} \bigvee_{y_\eta \not\leq B} [cl_{\mathcal{T}} \vee co_{\mathcal{C}}](B)(y_\eta).$$

Thus there is $\mu < \lambda$ such that for all $G \in \mathfrak{F}(F)$, there is $x_\mu \not\leq B_G \geq G$ such that

$$[(\mathcal{T} \wedge \mathcal{C})(B_G)]' = \bigvee_{y_\eta \not\leq B_G} [cl_{\mathcal{T}} \vee co_{\mathcal{C}}](B_G)(y_\eta) \not\geq b.$$

Hence $y_\eta \notin cl_{\mathcal{T}}(B_G)_{[b]} \cup co_{\mathcal{C}}(B_G)_{[b]}$ for all $y_\eta \not\leq B_G$. Take $E = \bigvee co_{\mathcal{C}}(B_G)_{[b]}$. Then $E = \bigvee cl_{\mathcal{T}}(B_G)_{[b]} = B_G \not\geq x_\mu$. In addition, by $\mathcal{T}_w \leq \mathcal{T}$, we have

$$\bigwedge_{x_\mu \not\leq B\geq G} [(\mathcal{T}_w \wedge \mathcal{C})(B)]' \leq \bigwedge_{x_\mu \not\leq B\geq E} [(\mathcal{T}_w \wedge \mathcal{C})(B)]' \leq [(\mathcal{T} \wedge \mathcal{C})(B_G)]'.$$

Thus $b \not\leq \bigwedge_{x_\mu \not\leq B\geq G}[\mathcal{T}_w(B) \wedge \mathcal{C}(B)]'$. From this result and $b \in \beta^*(a)$, we conclude that

$$a \not\leq \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} \bigwedge_{x_\mu \not\leq B\geq G} [(\mathcal{T}_w \wedge \mathcal{C})(B)]'.$$

By arbitrariness of $a \in M$, we have

$$\bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} \bigwedge_{x_\mu \not\leq B\geq G} [(\mathcal{T}_w \wedge \mathcal{C})(B)]' \leq co_{\mathcal{C}}(F)(x_\lambda).$$

Therefore $\mathcal{T}_w$ is compatible with $\mathcal{C}$.

(2) ⇒ (3). If $x_\lambda \notin \psi(F, a)$, then there is $H \in \mathfrak{F}_F(L^X)$ such that $x_\lambda \notin co_{\mathcal{C}}(H)_{[a]}$. Since $F \in \mathfrak{F}(H)$, there is $R \in \mathfrak{F}(H)$ such that $F \in \mathfrak{F}(R)$. Thus

$$\begin{aligned}
a &\not\leq co_{\mathcal{C}}(H)(x_\lambda) \\
&= \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(H)} cl_{\mathcal{T}_w\wedge\mathcal{C}}(G)(x_\mu) \\
&\geq \bigvee_{G\in\mathfrak{F}(H)} cl_{\mathcal{T}_w\wedge\mathcal{C}}(G)(x_\lambda) \\
&= \bigvee_{G\in\mathfrak{F}(H)} \bigwedge_{x_\lambda \not\leq B\geq G} \bigvee_{y_\eta \not\leq B} [cl_{\mathcal{T}_w} \vee co_{\mathcal{C}}](B)(y_\eta) \\
&\geq \bigwedge_{x_\lambda \not\leq B\geq R} \bigvee_{y_\eta \not\leq B} [cl_{\mathcal{T}_w} \vee co_{\mathcal{C}}](B)(y_\eta).
\end{aligned}$$

Hence there is $B \in L^X$ such that $x_\lambda \not\leq B \geq R$ and

$$a \not\leq \bigvee_{y_\eta \not\leq B} [cl_{\mathcal{T}_w} \vee co_{\mathcal{C}}](B)(y_\eta) = [(\mathcal{T}_w \wedge \mathcal{C})(B)]'.$$

So $y_\eta \notin cl_{\mathcal{T}_w}(B)_{[a]} \cup co_{\mathcal{C}}(B)_{[a]}$ for any $y_\eta \not\leq B$. So

$$\bigvee co_{\mathcal{C}}(B)_{[a]} = \bigvee cl_{\mathcal{T}_w}(B)_{[a]} = B \quad \text{and} \quad D_F \leq \bigvee co_{\mathcal{C}}(R)_{[a]} \leq \bigvee co_{\mathcal{C}}(B)_{[a]} = B.$$

This implies that $x_\lambda \not\leq B \geq D_F$ and

$$cl_{\mathcal{T}_w}(D_F)(x_\lambda) = \bigwedge_{x_\lambda \not\leq W \geq D_F} \bigwedge_{\bigwedge_{i\in I} H_i = W} \bigvee_{i\in I} [(\mathcal{T} \wedge C)^{\sqcup}(H_i)]' \leq [(\mathcal{T}_w \wedge C)(B)]'.$$

So $x_\lambda \notin cl_{\mathcal{T}_w}(D_F)_{[a]}$ and $cl_{\mathcal{T}_w}(D_F)_{[a]} \subseteq \psi(F, a)$.

(3) ⇒ (4). We have $cl_{\mathcal{T}_w} \geq cl_{\mathcal{T}}$ by $\mathcal{T}_w \leq \mathcal{T}$. So $cl_{\mathcal{T}}(D_F)_{[a]} \subseteq cl_{\mathcal{T}_w}(D_F)_{[a]} \subseteq \psi(F, a)$.

(4) ⇒ (5). By (LMDF) and (LMCO4), we have

$$co_C(F)(x_\lambda) = \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} \bigwedge_{x_\mu \not\leq D \geq G} \bigvee_{y_\eta \not\leq D} co_C(D)(y_\eta) \leq \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} \bigwedge_{x_\mu \not\leq D \geq G} \bigvee_{y_\eta \not\leq D} [cl_{\mathcal{T}} \vee co_C](D)(y_\eta).$$

Conversely, suppose that

$$co_C(F)(x_\lambda) \not\geq \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} \bigwedge_{x_\mu \not\leq D \geq G} \bigvee_{y_\eta \not\leq D} [cl_{\mathcal{T}} \vee co_C](D)(y_\eta).$$

So there is $a \in \beta^*(\top)$ such that $a \not\leq co_C(F)(x_\lambda)$ and

$$a \prec \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} \bigwedge_{x_\mu \not\leq D \geq G} \bigvee_{y_\eta \not\leq D} [cl_{\mathcal{T}} \vee co_C](D)(y_\eta).$$

Since $x_\lambda \notin co_C(F)_{[a]}$, there is $\mu_0 \prec \lambda$ with $x_{\mu_0} \notin co_C(F)_{[a]}$ and $x_{\mu_0} \not\leq \bigvee co_C(F)_{[a]}$ by (LMCL0). Since

$$a \prec \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} \bigwedge_{x_\mu \not\leq D \geq G} \bigvee_{y_\eta \not\leq D} [cl_{\mathcal{T}} \vee co_C](D)(y_\eta),$$

there is $G_0 \in \mathfrak{F}(F)$ such that

$$a \prec \bigwedge_{x_{\mu_0} \not\leq D \geq G_0} \bigvee_{y_\eta \not\leq D} [cl_{\mathcal{T}} \vee co_C](D)(y_\eta).$$

Since $x_{\mu_0} \not\leq \bigvee co_C(F)_{[a]} \geq D_{G_0} \geq G_0$, we have $a \prec \bigvee_{y_\eta \not\leq D_{G_0}} [cl_{\mathcal{T}} \vee co_C](D_{G_0})(y_\eta)$. So there is $y_\eta \not\leq D_{G_0}$ with $a \leq [cl_{\mathcal{T}} \vee co_C](D_{G_0})(y_\eta)$. Hence $a \leq cl_{\mathcal{T}}(D_{G_0})(y_\eta)$ or $a \leq co_C(D_{G_0})(y_\eta)$. If $a \leq cl_{\mathcal{T}}(D_{G_0})(y_\eta)$, then $y_\eta \in cl_{\mathcal{T}}(D_{G_0})_{[a]} \subseteq \psi(G_{\mu_0}, a)$. If $a \leq co_C(D_{G_0})(y_\eta)$, then

$$y_\eta \in co_C(D_{G_0})_{[a]} \subseteq \bigcap_{H\in\mathfrak{F}_{G_0}(L^X)} co_C(\bigvee co_C(H)_{[a]})_{[a]} \subseteq \bigcap_{H\in\mathfrak{F}_{G_0}(L^X)} co_C(H)_{[a]} = \psi(G_0, a).$$

They imply $y_\eta \leq D_{G_0}$ which is a contradiction. Thus

$$co_C(F)(x_\lambda) \geq \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} \bigwedge_{x_\mu \not\leq D \geq G} \bigvee_{y_\eta \not\leq D} [cl_{\mathcal{T}} \vee co_C](D)(y_\eta).$$

(5) ⇒ (1). We have

$$co_C(F)(x_\lambda) = \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} \bigwedge_{x_\mu \not\leq D \geq G} \bigvee_{y_\eta \not\leq D} [cl_{\mathcal{T}} \vee co_C](D)(y_\eta)] = \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} \bigwedge_{x_\lambda \not\leq D \geq F} [\mathcal{T}(D) \wedge C(D)]'.$$

Therefore $(X, \mathcal{T}, C)$ is an (L, M)-fuzzy topological-convex space.

(3) ⇔ (6). Similar to (4) ⇔ (5).

(1) ⇒ (7). Let $x_\lambda \notin \psi(F, a)$. Similar to (2) ⇒ (3), we can find some $B \in L^X$ such that $x_\lambda \not\leq B \geq D_F$, $a \not\leq [(\mathcal{T} \wedge C)(B)]'$, $\bigvee co_C(B)_{[a]} = \bigvee cl_{\mathcal{T}}(B)_{[a]} = B$ and

$$D_F \leq \bigvee co_C(R)_{[a]} \leq \bigvee co_C(B)_{[a]} = B.$$

Thus $cl_{\mathcal{T}\wedge C}(D)(x_\lambda) \leq [(\mathcal{T} \wedge C)(B)]' \not\geq a$. Hence $x_\lambda \notin cl_{\mathcal{T}\wedge C}(D)_{[a]}$. Therefore $cl_{\mathcal{T}\wedge C}(D_F)_{[a]} \subseteq \psi(F, a)$.

(7) ⇒ (3). Let $F \in \mathfrak{F}(L^X)$. Since $\mathcal{T} \wedge C \leq \mathcal{T}_w$, we have $cl_{\mathcal{T}_w} \leq cl_{\mathcal{T}\wedge C}$. Thus

$$cl_{\mathcal{T}_w}(D_F)_{[a]} \subseteq cl_{\mathcal{T}\wedge C}(D_F)_{[a]} \subseteq \psi(F, a).$$

Therefore (3) holds.

(4) ⇒ (8). By (LMDF), (LMCL2) and (LMCL0),

$$co_{\mathcal{C}}(F)(x_\lambda) \leq \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} [\bigvee_{A\in L^X} cl_{\mathcal{T}}(A)(x_\mu) \wedge \bigwedge_{y_\eta<A} co_{\mathcal{C}}(G)(y_\eta)].$$

Conversely, suppose that

$$co_{\mathcal{C}}(F)(x_\lambda) \not\geq \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} [\bigvee_{A\in L^X} cl_{\mathcal{T}}(A)(x_\mu) \wedge \bigwedge_{y_\eta<A} co_{\mathcal{C}}(G)(y_\eta)].$$

So there is $a \in \beta^*(\top)$ such that $a \not\leq co_{\mathcal{C}}(F)(x_\lambda)$ and

$$a \prec \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} [\bigvee_{A\in L^X} cl_{\mathcal{T}}(A)(x_\mu) \wedge \bigwedge_{y_\eta<A} co_{\mathcal{C}}(G)(y_\eta)].$$

By $a \not\leq co_{\mathcal{C}}(F)(x_\lambda)$, there is $\gamma \prec \lambda$ such that $a \not\leq co_{\mathcal{C}}(F)(x_\gamma)$. Further, there is $G_\gamma \in \mathfrak{F}(F)$ such that

$$a \prec \bigvee_{A\in L^X} cl(A)(x_\gamma) \wedge \bigwedge_{y_\eta<A} co(G_\gamma)(y_\eta).$$

Thus there is $A \in L^X$ such that $a \leq cl_{\mathcal{T}}(A)(x_\gamma) \wedge \bigwedge_{y_\eta<A} co_{\mathcal{C}}(G_\gamma)(y_\eta)$. So

$$A = \bigvee_{y_\eta<A} y_\eta \leq \bigvee co_{\mathcal{C}}(G_\gamma)_{[a]} \leq \bigvee \psi(G_\gamma, a) = D_{G_\gamma}.$$

Hence we obtain from (4) that

$$cl_{\mathcal{T}}(A)_{[a]} \subseteq cl_{\mathcal{T}}(D_{G_\gamma})_{[a]} \subseteq \psi(G_\gamma, a) \subseteq co_{\mathcal{C}}(F)_{[a]}.$$

So $x_\gamma \in cl_{\mathcal{T}}(A)_{[a]} \subseteq co_{\mathcal{C}}(F)_{[a]}$ which contradicts $a \not\leq co(F)(x_\gamma)$. So we conclude that

$$co(F)(x_\lambda) \geq \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(F)} [\bigvee_{A\in L^X} cl_{\mathcal{T}}(A)(x_\mu) \wedge \bigwedge_{y_\eta<A} co_{\mathcal{C}}(G)(y_\eta)].$$

(8) ⇒ (4). If $x_\lambda \notin \psi(F, a)$, then there is $H \in \mathfrak{F}_F(L^X)$ such that $x_\lambda \notin co_{\mathcal{C}}(H)_{[a]}$. Since $F \in \mathfrak{F}(H)$, there is $R \in \mathfrak{F}(H)$ such that $F \in \mathfrak{F}(R)$. Thus

$$\begin{aligned} a &\not\leq co_{\mathcal{C}}(H)(x_\lambda) \\ &= \bigwedge_{\mu<\lambda} \bigvee_{G\in\mathfrak{F}(H)} [\bigvee_{A\in L^X} cl_{\mathcal{T}}(A)(x_\mu) \wedge \bigwedge_{y_\eta<A} co_{\mathcal{C}}(G)(y_\eta)] \\ &\geq cl_{\mathcal{T}}(D_F)(x_\lambda) \wedge \bigwedge_{y_\eta<D_F} co_{\mathcal{C}}(R)(y_\eta). \end{aligned}$$

This shows that $a \not\leq cl(D_F)(x_\lambda) \wedge \bigwedge_{y_\eta<D_F} co(R)(y_\eta)$. Further, $\bigwedge_{y_\eta<D_F} co_{\mathcal{C}}(R)(y_\eta) \geq a$ by $\psi(F, a) \subseteq co(R)_{[a]}$. Thus $a \not\leq cl_{\mathcal{T}}(D_F)(x_\lambda)$. Hence $x_\lambda \notin cl_{\mathcal{T}}(D_F)_{[a]}$. Therefore $cl_{\mathcal{T}}(D_F)_{[a]} \subseteq \psi(F, a)$. □

4. (L,M)-Fuzzy Topological-Convex Enclosed Relation Spaces

In this section, we define (L,M)-fuzzy topological-convex enclosed relation spaces, by which we characterize (L,M)-fuzzy topological-convex spaces. Before this, we introduce the notion of (L,M)-fuzzy topological enclosed relation spaces as follows.

Definition 4.1. A binary relation $\mathcal{R}: L^X \times L^X \to M$ is called an (L,M)-fuzzy convex enclosed relation and the pair $(X,\mathcal{R})$ is called an (L,M)-fuzzy convex enclosed relation space, if $\mathcal{R}$ satisfies

(LMCER1) $\mathcal{R}(\underline{\bot},\underline{\bot}) = \top$;

(LMCER2) $\mathcal{R}(A,B) > \bot$ implies $A \leq B$;

(LMCER3) $\mathcal{R}(A, \bigwedge_{i\in I} B_i) = \bigwedge_{i\in I} \mathcal{R}(A,B_i)$;

(LMCER4) $\mathcal{R}(A,B) \leq \bigvee_{C\in L^X} \mathcal{R}(A,C) \wedge \mathcal{R}(C,B)$;

(LMCER5) $\mathcal{R}(\bigvee_{i\in I}^{dir} A_i, B) = \bigwedge_{i\in I} \mathcal{R}(A_i,B)$.

Let $(X,\mathcal{R}_X)$ and $(Y,\mathcal{R}_Y)$ be (L,M)-fuzzy convex enclosed relation spaces. A mapping $f: X \to Y$ is called an (L,M)-fuzzy convex enclosed relation preserving (or, (L,M)-fuzzy CERP) mapping, if

$$\mathcal{R}_Y(f^{\to}(U), V) \leq \mathcal{R}_X(U, f_L^{\leftarrow}(V))$$

for any $U \in L^X$ and any $V \in L^Y$.

Example 4.2. (1) Let $X = \{x\}$ and $L = M = [0,1]$. Define $\mathcal{R}_1: L^X \times L^X \to M$ by

$$\mathcal{R}_1(x_s,x_t) = \begin{cases} 1, & s = t = 0, \\ \frac{1}{2}, & s \leq \frac{1}{2} \leq t, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X,\mathcal{R}_1)$ is an (L,M)-fuzzy convex enclosed relation space.

(2) Let $X = \{x,y\}$ and $L = M = [0,1]$. Define $\mathcal{R}_2: L^X \times L^X \to M$ by

$$\mathcal{R}_2(A,B) = \begin{cases} 1, & A = B = \underline{0}, \\ \frac{1}{2}, & z \in X, A \leq z_{\frac{1}{2}} \leq B, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X,\mathcal{R}_2)$ is an (L,M)-fuzzy convex enclosed relation space.

Similar to the relations between (L,M)-fuzzy topological enclosed relations and (L,M)-fuzzy cotopologies discussed in [20], the following result shows that there is a one-to-one correspondence between (L,M)-fuzzy convex enclosed relations and (L,M)-fuzzy convex structures.

Theorem 4.3. *(1) Let $(X,\mathcal{C})$ be an (L,M)-fuzzy convex space. Define $\mathcal{R}_{\mathcal{C}}: L^X \times L^X \to M$ by:*

$$\forall A,B \in L^X,\ \mathcal{R}_{\mathcal{C}}(A,B) = \bigwedge_{x_\lambda \not\leq B} \bigvee_{x_\lambda \not\leq C \geq A} \mathcal{C}(C).$$

Then $\mathcal{R}_{\mathcal{C}}$ is an (L,M)-fuzzy convex enclosed relation.

(2) Let $(X,\mathcal{R})$ be an (L,M)-fuzzy convex enclosed relation space. Define $\mathcal{C}_{\mathcal{R}}: L^X \to M$ by

$$\forall A \in L^X,\ \mathcal{C}_{\mathcal{R}}(A) = \bigwedge_{x_\lambda \not\leq A} \bigvee_{x_\lambda \not\leq B} \mathcal{R}(A,B).$$

Then $\mathcal{C}_{\mathcal{R}}$ is an (L,M)-fuzzy convex structure with $\mathcal{R}_{\mathcal{C}_{\mathcal{R}}} = \mathcal{R}$. Further, if $(X,\mathcal{C})$ is an (L,M)-fuzzy convex space, then $\mathcal{C}_{\mathcal{R}_{\mathcal{C}}} = \mathcal{C}$.

(4) If $f: (X,\mathcal{C}_X) \to (Y,\mathcal{C}_Y)$ is an (L,M)-fuzzy CP mapping, then $f: (X,\mathcal{R}_{\mathcal{C}_X}) \to (Y,\mathcal{R}_{\mathcal{C}_Y})$ is an (L,M)-fuzzy CERP mapping.

(5) If $f: (X,\mathcal{R}_X) \to (Y,\mathcal{R}_Y)$ is an (L,M)-fuzzy CERP mapping, then $f: (X,\mathcal{C}_{\mathcal{R}_X}) \to (Y,\mathcal{C}_{\mathcal{R}_Y})$ is an (L,M)-fuzzy CP mapping.

Definition 4.4. Let X be a set equipped with an (L,M)-fuzzy topological enclosed relation $\mathcal{E}$ and an (L,M)-fuzzy convex enclosed relation $\mathcal{R}$. The triple $(X,\mathcal{E},\mathcal{R})$ is called an (L,M)-fuzzy topological-convex enclosed relation space, if $\mathcal{E}$ is compatible with $\mathcal{R}$, that is, for any $F \in \mathfrak{F}(L^X)$ and any $B \in L^X$,

$$\mathcal{R}(F,B) \leq \bigwedge_{G\in\mathfrak{F}(F)} \bigvee_{D\in L^X} [\mathcal{R},\mathcal{E}](G,D,B),$$

where $[\mathcal{R},\mathcal{E}](G,D,B) = \mathcal{R}(G,D) \wedge \mathcal{R}(D,D) \wedge \mathcal{E}(D,D) \wedge \mathcal{E}(D,B)$.

Let $(X,\mathcal{E}_X,\mathcal{R}_X)$ and $(Y,\mathcal{E}_Y,\mathcal{R}_Y)$ be (L,M)-fuzzy topological-convex enclosed relation spaces, a mapping $f: X \to Y$ is called an (L,M)-fuzzy topological-convex enclosed relation preserving (or, (L,M)-fuzzy TCERP) mapping, if $f:(X,\mathcal{E}_X) \to (Y,\mathcal{E}_Y)$ is an (L,M)-fuzzy TERP mapping and $f:(X,\mathcal{R}_X) \to (Y,\mathcal{R}_Y)$ is an (L,M)-fuzzy CERP mapping.

The category of (L,M)-fuzzy topological-convex enclosed relation spaces and (L,M)-fuzzy topological-convex enclosed relation preserving mappings is denoted by (L,M)-**TCERS**.

Example 4.5. (1) Let $X = \{x\}$ and $L = M = [0,1]$. Let $\mathcal{R}_1 : L^X \times L^X \to M$ be defined as in (1) of Example 4.2. If $\mathcal{E}_1 : L^X \times L^X \to M$ is defined by $\mathcal{E}_1 = \mathcal{R}_1$, then $(X,\mathcal{E}_1,\mathcal{R}_1)$ is an (L,M)-fuzzy topological-convex enclosed relation space.

(2) Let $X = \{x,y\}$ and $L = M = [0,1]$. Let $\mathcal{R}_2 : L^X \times L^X \to M$ be defined as in (2) of Example 4.2. Define $\mathcal{E}_2 : L^X \times L^X \to M$ by

$$\mathcal{E}_2(A,B) = \begin{cases} 1, & A = B = \underline{0}, \\ \frac{1}{2}, & z \in X, A \leq z_{\frac{1}{2}} \leq B, \\ \frac{1}{2}, & A \leq \underline{\tfrac{1}{2}} \leq B, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X,\mathcal{E}_2,\mathcal{R}_2)$ is an (L,M)-fuzzy topological-convex enclosed relation space.

Next, we discuss relations between (L,M)-**TCS** and (L,M)-**TCERS**.

Theorem 4.6. *For an (L,M)-fuzzy topological-convex enclosed relation space $(X,\mathcal{E},\mathcal{R})$, the triple $(X,\mathcal{T}_{\mathcal{E}},\mathcal{C}_{\mathcal{R}})$ is an (L,M)-fuzzy topological-convex space.*

Proof. Let $F \in \mathfrak{F}(L^X)$ and $x_\lambda \in J(L^X)$. By (LMDF),

$$co_{\mathcal{C}_{\mathcal{R}}}(F)(x_\lambda) \leq \bigwedge_{\mu\prec\lambda} \bigvee_{H\in\mathfrak{F}(F)} cl_{\mathcal{T}_{\mathcal{E}}\wedge\mathcal{C}_{\mathcal{R}}}(H)(x_\mu).$$

On the other hand, to prove that

$$co_{\mathcal{C}_{\mathcal{R}}}(F)(x_\lambda) \geq \bigwedge_{\mu\prec\lambda} \bigvee_{H\in\mathfrak{F}(F)} cl_{\mathcal{T}_{\mathcal{E}}\wedge\mathcal{C}_{\mathcal{R}}}(H)(x_\mu),$$

it is sufficient to prove that for any $\mu \prec \lambda$,

$$\bigwedge_{G\in\mathfrak{F}(F)} [co_{\mathcal{C}_{\mathcal{R}}}(G)(x_\mu)]' \leq \bigwedge_{H\in\mathfrak{F}(F)} [cl_{\mathcal{T}_{\mathcal{E}}\wedge\mathcal{C}_{\mathcal{R}}}(H)(x_\mu)]'.$$

Let $\mu \prec \lambda$ and $a \prec \bigwedge_{G\in\mathfrak{F}(F)}[co_{\mathcal{C}_{\mathcal{R}}}(G)(x_\mu)]'$. Then

$$a \prec [co_{\mathcal{C}_{\mathcal{R}}}(G)(x_\mu)]' = \bigvee_{x_\mu \not\leq B} \mathcal{R}(G,B)$$

for all $G \in \mathfrak{F}(F)$. Thus there is $B_G \in L^X$ such that $x_\mu \not\leq B_G$ and $a \prec \mathcal{R}(G,B_G)$. Hence

$$a \prec \mathcal{R}(G,B_G) \leq \bigwedge_{H\in\mathfrak{F}(G)} \bigvee_{D\in L^X} [\mathcal{R},\mathcal{E}](H,D,B_G).$$

So, for each $H \in \mathfrak{F}(G)$, there is $D_H \in L^X$ such that $a \le [\mathcal{R},\mathcal{E}](H, D_H, B_G)$. This shows $H \le D_H \le B_G$ and $x_\mu \not\le D_H$. Hence

$$\begin{aligned} a &\le \mathcal{E}(D_H, D_H) \wedge \mathcal{R}(D_H, D_H) \\ &\le [\bigwedge_{y_\gamma \not\le D_G} \bigvee_{y_\gamma \not\le R} \mathcal{E}(D_H, R)] \wedge [\bigwedge_{z_\eta \not\le D_H} \bigvee_{z_\eta \not\le T} \mathcal{E}(D_H, T)] \\ &= \mathcal{T}_{\mathcal{E}}(D_H) \wedge \mathcal{C}_{\mathcal{R}}(D_H) \\ &\le \bigvee_{x_\mu \not\le B \ge H} [\mathcal{T}_{\mathcal{E}} \wedge \mathcal{C}_{\mathcal{R}}](B) = [cl_{\mathcal{T}_{\mathcal{E}} \wedge \mathcal{C}_{\mathcal{R}}}(H)(x_\mu)]'. \end{aligned}$$

By arbitrariness of $G \in \mathfrak{F}(F)$ and $H \in \mathfrak{F}(G)$, we have

$$\begin{aligned} a &\le \bigwedge_{G \in \mathfrak{F}(F)} \bigwedge_{H \in \mathfrak{F}(G)} [cl_{\mathcal{T}_{\mathcal{E}} \wedge \mathcal{C}_{\mathcal{R}}}(H)(x_\mu)]' \\ &= \bigwedge_{H \in \bigcup_{G \in \mathfrak{F}(F)} \mathfrak{F}(G)} [cl_{\mathcal{T}_{\mathcal{E}} \wedge \mathcal{C}_{\mathcal{R}}}(H)(x_\mu)]' \\ &= \bigwedge_{H \in \mathfrak{F}(F)} [cl_{\mathcal{T}_{\mathcal{E}} \wedge \mathcal{C}_{\mathcal{R}}}(H)(x_\mu)]'. \end{aligned}$$

By arbitrariness of a and $G \in \mathfrak{F}(F)$, we conclude that

$$co_{\mathcal{C}_{\mathcal{R}}}(F)(x_\lambda) \ge \bigwedge_{\mu < \lambda} \bigvee_{H \in \mathfrak{F}(F)} cl_{\mathcal{T}_{\mathcal{E}} \wedge \mathcal{C}_{\mathcal{R}}}(H)(x_\mu).$$

Therefore $\mathcal{T}_{\mathcal{E}}$ is compatible with $\mathcal{C}_{\mathcal{R}}$. □

Theorem 4.7. *For an (L, M)-fuzzy topological-convex space $(X, \mathcal{T}, \mathcal{C})$, the triple $(X, \mathcal{E}_{\mathcal{T}}, \mathcal{R}_{\mathcal{C}})$ is an (L, M)-fuzzy topological-convex enclosed relation space.*

Proof. Let $F \in \mathfrak{F}(L^X)$ and $x_\lambda \in J(L^X)$. We have

$$co_{\mathcal{C}}(F)(x_\lambda) = \bigwedge_{\mu < \lambda} \bigvee_{G \in \mathfrak{F}(F)} cl_{\mathcal{T} \wedge \mathcal{C}}(G)(x_\mu).$$

Thus it is sufficient to verify that

$$\mathcal{R}_{\mathcal{C}}(F, B) \le \bigwedge_{G \in \mathfrak{F}(F)} \bigvee_{D \in L^X} [\mathcal{R}_{\mathcal{C}}, \mathcal{E}_{\mathcal{T}}](G, D, B).$$

Let $r \in P(M)$ with $\mathcal{R}_{\mathcal{C}}(F, B) \not\le r$. Then there is $s \in \alpha^*(r)$ such that $\mathcal{R}_{\mathcal{C}}(F, B) \not\le s$. Further, there is $t \in \alpha^*(s)$ such that $\mathcal{R}_{\mathcal{C}}(F, B) \not\le t$.

For convenience, we denote $D_G = \bigvee cl_{\mathcal{T} \wedge \mathcal{C}}(G)_{[t']}$ for each $G \in \mathfrak{F}(F)$. We have

$$\mathcal{R}_{\mathcal{C}}(F, B) = \bigwedge_{x_\lambda \not\le B} \bigvee_{\mu < \lambda} \bigwedge_{R \in \mathfrak{F}(F)} [co_{\mathcal{C}}(R)(x_\mu)]' \le \bigwedge_{x_\lambda \not\le B} \bigwedge_{R \in \mathfrak{F}(F)} [co_{\mathcal{C}}(R)(x_\lambda)]'.$$

So $t' \not\le co_{\mathcal{C}}(G)(x_\lambda)$ for all $G \in \mathfrak{F}(F)$ and $x_\lambda \not\le B$.

Fix any $G \in \mathfrak{F}(F)$. We say that $D_G \le B$. Otherwise, there is $x_\lambda \prec D_G$ such that $x_\lambda \not\le B$. Thus $t' \le cl_{\mathcal{T} \wedge \mathcal{C}}(G)(x_\lambda)$. Since $G \in \mathfrak{F}(F)$, there is $R \in \mathfrak{F}(F)$ such that $G \in \mathfrak{F}(R)$. Hence

$$co_{\mathcal{C}}(R)(x_\lambda) = \bigwedge_{\mu < \lambda} \bigvee_{H \in \mathfrak{F}(R)} cl_{\mathcal{T} \wedge \mathcal{C}}(H)(x_\mu) \ge cl_{\mathcal{T} \wedge \mathcal{C}}(G)(x_\lambda) \ge t'.$$

Hence $x_\lambda \not\le B$ and $t' \le co_{\mathcal{C}}(R)(x_\lambda)$. It is a contradiction. So $G \le D_G \le B$.

Further, we say that $\mathcal{R}_C(D_G, D_G) \nleq s$. Otherwise, we have

$$\bigwedge_{x_\lambda \nleq D_G} [co_C(D_G)(x_\lambda)]' = \mathcal{R}_C(D_G, D_G) \leq s$$

which implies that

$$t \in \alpha^*(s) \quad \subseteq \quad \bigcup_{x_\lambda \nleq D_G} \alpha^*([co_C(D_G)(x_\lambda)]').$$

Thus $t \in \alpha^*([co_C(D_G)(x_\lambda)]')$ for some $x_\lambda \nleq D_G$. Hence $x_\lambda \in cl_{\mathcal{T}\wedge C}(D_G)_{[t']} \subseteq cl_{\mathcal{T}\wedge C}(G)_{[t']}$ by (LMCL4*). However, this contradicts $x_\lambda \nleq D_G$. Therefore we conclude that $\mathcal{R}_C(G, D_G) \geq \mathcal{R}_C(D_G, D_G) \nleq s$.

Similarly, we have $\mathcal{E}_{\mathcal{T}}(D_G, B) \geq \mathcal{E}_{\mathcal{T}}(D_G, D_G) \nleq s$. So $[\mathcal{R}_C, \mathcal{E}_{\mathcal{T}}](G, D_G, B) \nleq s$ and

$$\bigwedge_{G\in\mathfrak{F}(F)} \bigvee_{D\in L^X} [\mathcal{R}_C, \mathcal{E}_{\mathcal{T}}](G, D, B) \nleq r.$$

By arbitrariness of r, we have

$$\mathcal{R}_C(F, B) \leq \bigwedge_{G\in\mathfrak{F}(F)} \bigvee_{D\in L^X} [\mathcal{R}_C, \mathcal{E}_{\mathcal{T}}](G, D, B).$$

So $(X, \mathcal{E}_{\mathcal{T}}, \mathcal{R}_C)$ is an (L, M)-fuzzy topological-convex enclosed relation space. □

From Theorems 4.3, 4.6 and 4.7, we have the following results.

Theorem 4.8. *(1) Let $(X, \mathcal{T}_X, C_X)$ and $(Y, \mathcal{T}_Y, C_Y)$ be (L, M)-fuzzy topological-convex spaces. If $f : X \to Y$ is an (L, M)-fuzzy TCP mapping, then $f : (X, \mathcal{E}_{\mathcal{T}_X}, \mathcal{R}_{C_X}) \to (Y, \mathcal{E}_{\mathcal{T}_Y}, \mathcal{R}_{C_Y})$ is an (L, M)-fuzzy TCER preserving mapping.*

(2) Let $(X, \mathcal{E}_X, \mathcal{R}_X)$ and $(Y, \mathcal{E}_Y, \mathcal{R}_Y)$ be (L, M)-fuzzy topological-convex enclosed relation spaces. If $f : X \to Y$ is an (L, M)-fuzzy TCER mapping, then $f : (X, \mathcal{T}_{\mathcal{E}_X}, C_{\mathcal{R}_X}) \to (Y, \mathcal{T}_{\mathcal{E}_Y}, C_{\mathcal{R}_Y})$ is an (L, M)-fuzzy TCP mapping.

Corollary 4.9. (L, M)-**TCS** *is isomorphic to* (L, M)-**TCERS**.

5. Conclusions

The aim of this paper is to introduce and characterized (L, M)-fuzzy topological-convex spaces.

From Remarks 3.1, 3.5 and 3.10, we know that (L, M)-fuzzy topological-convex space is a unified form of topological-convex space and L-topological-convex space and M-fuzzifying topological-convex space. Further, each type of them can be defined by its corresponding convex bases. However, (L, M)-fuzzy topological-convex space or L-topological-convex space has a more complex structure than topological-convex space and M-fuzzifying topological-convex space. In fact, a topological-convex space (resp. an M-fuzzifying topological-convex convex space) can be alternatively defined by relations between the closure operator and the hull operator (resp. the M-fuzzifying closure operator and the M-fuzzifying hull operator). That is, a cotopology (resp. an M-fuzzifying cotopology) $\mathcal{T}$ is compatible with a convex structure (resp. an M-fuzzifying convex structure) C iff $co_C(F) = cl_{\mathcal{T}\cap C}(F)$ (resp. $co_C(F) = cl_{\mathcal{T}\wedge C}(F)$) for any $F \in 2^X_{fin}$. But, the compatibility of (L, M)-fuzzy topological-convex space or L-topological-convex space $(X, \mathcal{T}, C)$ should be defined by the condition that $co_C(F) = \bigvee_{G\in\mathfrak{F}(L^X)} cl_{\mathcal{T}\wedge C}(F)$ (other than $co_C(F) = cl_{\mathcal{T}\wedge C}(F)$) for any $F \in \mathfrak{F}(L^X)$.

As we can see, (L, M)-fuzzy convergence spaces are closely related to (L, M)-fuzzy topological spaces and (L, M)-fuzzy convex spaces [3, 4, 9, 10, 14]. Similar to the compatibility between an (L, M)-fuzzy cotopology and an (L, M)-fuzzy convex structure, it could be possible to discuss the compatibility between an (L, M)-fuzzy convergence structure and an (L, M)-fuzzy convex structure. Further, it could be possible to characterize (L, M)-fuzzy topological-convex spaces by such compatibility.

In Convex Theory, topological-convex spaces is a basic notion in combining Topology Theory and Convex Theory. With such spaces, many combined properties can be investigated including continuities of hull operators, compactness and uniformity of convex spaces. Thus this paper could be helpful in discussing (L, M)-fuzzy topological-convex spaces in the future.

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