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Fixed Points of Suzuki-Type Generalized Multivalued $(f,\theta,L)-$ Almost Contractions with Applications

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Abstract.
In this paper, we define Suzuki type generalized multivalued almost contraction mappings and prove some related fixed point results. As an application, some coincidence and common fixed point results are obtained. The results proved herein extend the recent results on fixed points of Kikkawa Suzuki type and almost contraction mappings in the frame work of complete metric spaces. We provide examples to show that obtained results are proper generalization of comparable results in the existing literature. Some applications in homotopy, dynamic programming, integral equations and data dependence problems are also presented.

1. Introduction and preliminaries

Let (X,d) be a metric space. We denote $CL(X)$ ($CB(X)$) as the collection of closed (closed and bounded) subsets of X. For $A,B \in CL(X)$, define:

$$D(A,B) = \{\varepsilon > 0 : A \subseteq B_\varepsilon,\ B \subseteq A_\varepsilon\},$$

where

$$B_\varepsilon = \cup_{y\in B} N_\varepsilon(y),$$

and

$$N_\varepsilon(y) = \{x \in X : d(x,y) < \varepsilon\},$$

for some $y \in B$. The Pompeiu-Hausdorff metric H on $CL(X)$ induced by the metric d on X is given as:

$$H(A,B) = \begin{cases} \inf\limits_{\varepsilon} D(A,B) \text{ if } D(A,B) \neq \phi, \\ \infty \text{ if } D(A,B) = \phi. \end{cases}$$

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Recall that a multivalued mapping $T : X \to CL(X)$ is continuous at $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x) = 0 \text{ implies that } \lim_{n\to\infty} H(Tx_n, Tx) = 0.$$

Let $f : X \longrightarrow X$ and $T : X \longrightarrow CL(X)$. An element x in X is said to be:

a A fixed point of f if $x = fx$, the set of all fixed points of f is denoted by $F(f)$;

b A fixed point of T if $x \in Tx$, the set of all fixed points of T is represented by $F(T)$;

c A coincidence point of f and T if $fx \in Tx$, the set of all coincidence points of f and T is denoted by $C(f, T)$;

d A common fixed point of f and T if $x = fx \in Tx$, the set of all common fixed points of f and T is denoted by $F(f, T)$.

The letter $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of all real numbers, set of all non-negative real numbers and the set of all positive integers, respectively.

For $x, y \in X$, set

$$\begin{aligned}
M(x,y) &= \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\},\\
M^f(x,y) &= \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\right\},\\
N(x,y) &= \min\{d(x,Tx), d(y,Tx)\},\\
N^f(x,y) &= \min\{d(x,fx), d(y,fx)\},\\
M_f(x,y) &= \max\left\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\right\},\\
N_f(x,y) &= \min\{d(fx,Tx), d(fy,Tx)\}.
\end{aligned}$$

The well known Banach contraction principle [3] has been generalized in several directions [16–19, 28, 29].

Nadler [22] proved multivalued version of Banach contraction principle as follows:

Theorem 1.1. ([22]) *Let (X, d) be a complete metric space and $T : X \to CB(X)$. If there exists a constant $r \in [0, 1)$ such that*

$$H(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$. Then $F(T)$ is nonempty.

For more results in this direction, we refer to [12, 20, 27].

Suzuki [29] presented an interesting generalization of Banach contraction principle and employed his result to characterize metric completeness.

Throughout this paper, a mapping $\eta : [0, 1) \to (0, 1]$ is defined as

$$\eta(\theta) = \begin{cases} 1 & \text{if } 0 \le \theta < \frac{1}{2}, \\ 1-\theta & \text{if } \frac{1}{2} \le \theta < 1. \end{cases} \tag{1}$$

One interesting extension of Nadler's theorem [22], Ciric's result [11], and Suzuki-type result [29] is due to Djorić and Lazović [13] in complete metric spaces.

Theorem 1.2. ([13]) *Let (X, d) be a complete metric space and $T : X \to CB(X)$. Suppose that there exists $\theta \in [0, 1)$ such that for $x, y \in X$*

$$\eta(\theta)d(x, Tx) \leq d(x, y)$$

implies that

$$H(Tx, Ty) \leq \theta M(x, y).$$

Then there exists $z \in X$ such that $z \in Tz$.

Berinde [7] introduced weak contraction mappings. Later, Berinde et al. [8] extended this concept for multivalued mappings. Berinde et al. [9] modified the definition of multivalued weak contraction to generalized multivalued $(\theta, L)-$strict almost contraction mappings and obtained a fixed point result for such mappings. Note that V. Berinde in [9] generalized the term "weak contraction" as "almost contraction", so these terms are interchangeable. Kamran [15] introduced the notion of multivalued weak contraction mappings for a hybrid pair of mappings (f, T) as follows:

Definition 1.3. ([15]) Let (X, d) be a metric space and (f, T) a hybrid pair of mappings. A mapping T is generalized multivalued $(f, \theta, L)-$weak contraction if there exist constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(fx, fy) + Ld(fy, Tx)$$

for all $x, y \in X$.

Abbas [1] further generalized the concept of weak contraction mappings.

To obtain common fixed points of hybrid pair (f, T), Abbas et al. [2] introduced the notion of $T-$weakly commuting and $w-$compatible mappings.

Definition 1.4. ([2]) A mapping f is called $T-$weakly commuting at $x \in X$ if $f^2x \in Tfx$.

Definition 1.5. ([2]) A hybrid pair (f, T) is $w-$compatible if $fTx \subseteq Tfx$ whenever $x \in C(T, f)$.

Motivated by the work of Djorić et al. [13] and Abbas [1]) we give following definitions.

Definition 1.6. A mapping $T : X \to CL(X)$ is called Suzuki-type generalized multivalued $(\theta, L)-$almost contraction if there exist constants $\theta \in [0, 1)$ and $L \geq 0$ such that for any $x, y \in X$ with $x \neq y$

$$\eta(\theta)d(x, Tx) \leq d(x, y)$$

implies that

$$H(Tx, Ty) \leq \theta M(x, y) + LN(x, y). \tag{2}$$

Definition 1.7. Let (f, T) be a hybrid pair. A mapping T is called Suzuki-type generalized multivalued $(f, \theta, L)-$almost contraction if there exist constants $\theta \in [0, 1)$ and $L \geq 0$ such that for any $x, y \in X$ with $x \neq y$

$$\eta(\theta)d(fx, Tx) \leq d(fx, fy) \tag{3}$$

implies that

$$H(Tx, Ty) \leq \theta M_f(x, y) + LN_f(x, y). \tag{4}$$

If f is an identity mapping on X in the above definition, then Suzuki-type generalized multivalued $(f, \theta, L)-$almost contraction mapping becomes Suzuki-type generalized multivalued $(\theta, L)-$almost contraction.

2. Fixed Points of Suzuki-Type Generalized Multivalued $(\theta, L)-$Almost Contractions

In this section, we first obtain fixed point result of Suzuki-type generalized multivalued $(\theta, L)-$almost contractions and then coincidence and common fixed point results of Suzuki-type generalized multivalued $(f, \theta, L)-$almost contraction mapping.

The following result complements and extends the comparable results in [1, 9, 13, 15, 22, 29].

Theorem 2.1. *Let (X, d) be a complete metric space and T a Suzuki-type generalized multivalued $(\theta, L)-$almost contraction mapping. Then $F(T)$ is nonempty.*

Proof. Let $\theta_1 \in \mathbb{R}$ with $0 \le \theta < \theta_1 < 1$, $u_1 \in X$ and $h = \frac{1}{\sqrt{\theta}}$. As Tu_1 is nonempty, we can choose $u_2 \in Tu_1$. Since $h > 1$, there exists $u_3 \in Tu_2$ such that $d(u_2, u_3) \le hH(Tu_1, Tu_2)$. If $u_2 = u_1$, then $u_1 \in Tu_1$ and hence the result. Suppose that $u_2 \ne u_1$. Note that $\eta(\theta) \le 1$. Thus

$$\eta(\theta)d(u_1, Tu_1) \le d(u_1, Tu_1) \le d(u_1, u_2),$$

implies that

$$\begin{aligned}
d(u_2, u_3) &\le \frac{1}{\sqrt{\theta}}H(Tu_1, Tu_2) \le \frac{1}{\sqrt{\theta}}\left(\theta M(u_1, u_2) + LN(u_1, u_2)\right) \\
&\le \sqrt{\theta}\max\left\{d(u_1, u_2), d(u_1, Tu_1), d(u_2, Tu_2), \frac{d(u_1, Tu_2) + d(u_2, Tu_1)}{2}\right\} + \\
&\quad \frac{L}{\sqrt{\theta}}\min\{d(u_1, Tu_1), d(u_2, Tu_1)\} \\
&\le \sqrt{\theta}\max\left\{d(u_1, u_2), d(u_2, u_3), \frac{d(u_1, u_2) + d(u_2, u_3)}{2}\right\} \\
&\le \sqrt{\theta}d(u_1, u_2) \le \sqrt{\theta_1}d(u_1, u_2).
\end{aligned}$$

Continuing this way, we obtain a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ and $u_{n+1} \ne u_n$ and it satisfies:

$$d(u_n, u_{n+1}) \le \sqrt{\theta_1}d(u_{n-1}, u_n)$$

and

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \le \sum_{n=1}^{\infty} (\sqrt{\theta_1})^{n-1} d(u_1, u_2) < \infty.$$

Thus $\{u_n\}$ is a Cauchy sequence in X. Assume that there exists $z \in X$ such that $\lim_{n\to\infty} u_n = z$. We claim that

$$d(z, Tx) \le \theta\max\{d(z, x), d(x, Tx)\} \tag{5}$$

for all $z \ne x$. Since $\lim_{n\to\infty} u_n = z$, there exists $n_0 \in \mathbb{N}$ such that

$$d(u_n, z) \le \frac{1}{3}d(z, x)$$

holds for all $n \ge n_0$. Also, $u_n \ne x$ for all $n \ge n_0$. As $u_{n+1} \in Tu_n$, we have

$$\begin{aligned}
\eta(\theta)d(u_n, Tu_n) &\le d(u_n, Tu_n) \le d(u_n, u_{n+1}) \\
&\le d(u_n, z) + d(z, u_{n+1}) \\
&\le \frac{2}{3}d(z, x).
\end{aligned}$$

Hence, for any $n \geq n_0$ we have

$$\begin{aligned}\eta(\theta)d(u_n, Tu_n) &\leq \frac{2}{3}d(z,x) = d(z,x) - \frac{1}{3}d(z,x)\\ &\leq d(z,x) - d(z,u_n)\\ &\leq d(u_n,x).\end{aligned} \tag{6}$$

Thus

$$\eta(\theta)d(u_n, Tu_n) \leq d(u_n, x) \tag{7}$$

implies that

$$\begin{aligned}d(u_{n+1}, Tx) &\leq H(Tu_n, Tx)\\ &\leq \theta \max\left\{d(u_n,x), d(u_n,Tu_n), d(x,Tx), \frac{d(u_n,Tx)+d(x,Tu_n)}{2}\right\} +\\ &\quad L\min\{d(u_n,Tu_n), d(x,Tu_n)\}.\end{aligned} \tag{8}$$

That is,

$$\begin{aligned}d(u_{n+1}, Tx) &\leq \theta \max\left\{d(u_n,x), d(u_n,u_{n+1}), d(x,Tx), \frac{d(u_n,Tx)+d(x,u_{n+1})}{2}\right\} +\\ &\quad L\min\{d(u_n,u_{n+1}), d(x,u_{n+1})\}.\end{aligned}$$

On taking limit as $n \to \infty$, we have

$$\begin{aligned}d(z,Tx) &\leq \theta \max\left\{d(z,x), d(x,Tx), \frac{d(z,Tx)+d(x,z)}{2}\right\}\\ &\leq \theta \max\{d(z,x), d(x,Tx)\}.\end{aligned}$$

Consequently,

$$d(z,Tx) \leq \theta \max\{d(z,x), d(x,Tx)\}, \tag{9}$$

holds for all $x \neq z$. Now we prove that $z \in Tz$. For this, we consider the following cases:

(i) Let $0 \leq \theta < 1/2$.

Assume on contrary that $z \notin Tz$. We choose an element $a \in Tz$ such that

$$2\theta d(a,z) < d(z,Tz).$$

Clearly $a \neq z$. From (5) with $x = a$, we have

$$d(z,Ta) \leq \theta \max\{d(z,a), d(a,Ta)\}. \tag{10}$$

Now $\eta(\theta)d(z,Tz) \leq d(z,Tz) \leq d(z,a)$ implies that

$$\begin{aligned}d(a,Ta) &\leq H(Tz,Ta) \leq \theta \max\left\{d(z,a), d(z,Tz), d(a,Ta), \frac{d(z,Ta)+d(a,Tz)}{2}\right\} +\\ &\quad L\min\{d(z,Tz), d(a,Tz)\}\\ &\leq \theta \max\left\{d(z,a), d(a,Ta), \frac{d(z,Ta)+d(a,a)}{2}\right\}\\ &\leq \theta \max\left\{d(z,a), d(a,Ta), \frac{d(z,a)+d(a,Ta)}{2}\right\}.\end{aligned}$$

That is,

$$d(a,Ta) \leq \theta \max\{d(z,a), d(a,Ta)\}. \tag{11}$$

Hence

$$d(a, Ta) \leq \theta d(z, a) < d(z, a).$$

From (10), we have

$$d(z, Ta) \leq \theta d(z, a).$$

Hence

$$\begin{aligned} d(z, Tz) &\leq d(z, Ta) + H(Ta, Tz) \\ &\leq d(z, Ta) + \theta \max\{d(z, a), d(a, Ta)\} \\ &\leq 2\theta d(z, a) < d(z, Tz), \end{aligned}$$

gives a contradiction. Thus $z \in Tz$.

(ii) Let $\frac{1}{2} \leq \theta < 1$. We now show that

$$\begin{aligned} H(Tx, Tz) \leq\ & \theta \max\left\{d(x, z), d(x, Tx), d(z, Tz), \frac{d(x, Tz) + d(z, Tx)}{2}\right\} + \\ & L \min\{d(x, Tx), d(z, Tx)\}, \end{aligned} \tag{12}$$

holds for all $x \in X$ with $x \neq z$. For each positive integer $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z, y_n) \leq d(z, Tx) + \frac{1}{n} d(x, z).$$

In this case we have

$$\begin{aligned} d(x, Tx) &\leq d(x, y_n) \\ &\leq d(x, z) + d(z, y_n) \\ &\leq d(x, z) + d(z, Tx) + \frac{1}{n} d(x, z). \end{aligned}$$

Hence from (5) we get

$$d(x, Tx) \leq d(x, z) + \theta \max\{d(z, x), d(x, Tx)\} + \frac{1}{n} d(x, z). \tag{13}$$

If

$$\max\{d(z, x), d(x, Tx)\} = d(x, z),$$

then by (13), we have

$$\begin{aligned} d(x, Tx) &\leq d(x, z) + \theta d(z, x) + \frac{1}{n} d(x, z) \\ &= [(1 + \theta) + \frac{1}{n}] d(x, z) \end{aligned}$$

which implies that

$$\left[\frac{1}{1 + \theta}\right] d(x, Tx) \leq \left[1 + \frac{1}{(1 + \theta)n}\right] d(x, z).$$

As $\eta(\theta) = 1 - \theta$, it follows that

$$\begin{aligned} \eta(\theta) d(x, Tx) &= (1 - \theta) d(x, Tx) \\ &\leq \left(\frac{1}{1 + \theta}\right) d(x, Tx) \leq \left[1 + \frac{1}{(1 + \theta)n}\right] d(x, z). \end{aligned}$$

On taking limit as $n \to \infty$, we obtain that

$$\eta(\theta)d(x, Tx) \leq d(x, z).$$

If

$$d(x, z) < d(x, Tx),$$

then by (2) we have

$$d(x, Tx) \leq d(x, z) + \theta d(x, Tx) + \frac{1}{n} d(x, z),$$

and hence

$$(1 - \theta)d(x, Tx) \leq (1 + \frac{1}{n})d(x, z).$$

On taking limit as $n \to \infty$, we have

$$(1 - \theta)d(x, Tx) \leq d(x, z).$$

That is,

$$\eta(\theta)d(x, Tx) \leq d(x, z),$$

and hence the claim follows.
Since $u_{n+1} \neq u_n$ for each $n \in \mathbb{N}$, we have $u_{n+1} \neq z$ or $u_n \neq z$, and the set $I = \{n : u_n \neq z\}$ is infinite. From (12) with $x = u_n$, $n \in I$, we have

$$\begin{aligned} d(u_{n+1}, Tz) &\leq H(Tu_n, Tz) \\ &\leq \theta \max\left\{d(u_n, z), d(u_n, Tu_n), d(z, Tz), \frac{d(u_n, Tz) + d(z, Tu_n)}{2}\right\} + \\ &\quad L \min\{d(u_n, Tu_n), d(z, Tu_n)\} \\ &\leq \theta \max\left\{d(u_n, z),\ d(u_n, u_{n+1}), d(z, Tz), \frac{d(u_n, Tz) + d(z, u_{n+1})}{2}\right\} + \\ &\quad L \min\{d(u_n, u_{n+1}), d(z, u_{n+1})\}. \end{aligned}$$

On taking limit as $n \to \infty$, we obtain that

$$d(z, Tz) \leq \theta d(z, Tz)$$

which implies that $d(z, Tz) = 0$ and hence $z \in Tz$. □

Example 2.2. Let $X = \{\alpha, \beta, \gamma, \delta, \zeta\}$ and $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ be the metric defined by

$$\begin{aligned} &d(\alpha, \beta) = d(\alpha, \gamma) = 5, \\ &d(\beta, \zeta) = d(\gamma, \delta) = d(\gamma, \zeta) = d(\beta, \gamma) = 10, \\ &d(\alpha, \delta) = d(\alpha, \zeta) = 12, \\ &d(\beta, \delta) = 8, \\ &d(\delta, \zeta) = 2, \\ &d(x, x) = 0 \text{ and } d(x, y) = d(y, x) \text{ for all } x, y \in X. \end{aligned}$$

Define the mapping $T : X \longrightarrow CL(X)$ by

$$Tx = \begin{cases} \{\alpha\} \text{ if } x \in \{\alpha, \beta, \gamma\} \\ \{\alpha, \beta\} \quad \text{if } x = \delta \\ \{\gamma\} \quad \text{if } x = \zeta. \end{cases}$$

Note that T is Suzuki-type generalized multivalued $(\theta, L)-$almost contraction with $\theta = \frac{3}{4}$ and $L = 2$. In particular,

$$\begin{aligned}
\eta(\theta)d(\delta, T\delta) &= 2 \leq d(\delta, \zeta) \text{ implies that} \\
H(T\delta, T\zeta) &= H(\{\alpha, \beta\}, \gamma) = 10 \leq \frac{35}{2} = \theta d(\delta, \zeta) + L\min\{d(\delta, \beta), d(\zeta, \beta)\}, \text{ and} \\
H(T\zeta, T\delta) &= H(\gamma, \{\alpha, \beta\}) = 10 \leq \frac{43}{2} = \theta d(\zeta, \delta) + L\min\{d(\zeta, \gamma), d(\delta, \gamma)\}.
\end{aligned}$$

Moreover, $x = \alpha$ is a fixed point of T in X. On the other hand, if we take $x = \delta$ and $y = \zeta$, then

$$\begin{aligned}
M(\delta, \zeta) &= \max\left\{d(\delta, \zeta), d(\delta, T\delta), d(\zeta, T\zeta), \frac{d(\delta, T\zeta) + d(\zeta, T\delta)}{2}\right\}, \\
&= \max\left\{d(\delta, \zeta), d(\delta, \{\alpha, \beta\}), d(\zeta, \gamma), \frac{d(\delta, \gamma) + d(\zeta, \{\alpha, \beta\})}{2}\right\}, \\
&= \max\{2, 8, 10, 10\} = 10.
\end{aligned}$$

Note that

$$H(T\delta, T\zeta) = 10 > \frac{15}{2} = \theta M(\delta, \zeta).$$

Hence Theorem 1.2 and Theorem 3 in [11] are not applicable in this case.

Corollary 2.3. *Let (X, d) be a complete metric space and $T : X \to CL(X)$. If there exist constants $\theta \in [0, 1)$ and $L \geq 0$ such that for any $x, y \in X$ with $x \neq y$*

$$\eta(\theta)d(x, Tx) \leq d(x, y)$$

implies that

$$H(Tx, Ty) \leq \theta \max\{d(x, y), d(x, Tx), d(y, Ty)\} + LN(x, y).$$

Then $F(T)$ is nonempty.

Corollary 2.4. *Let (X, d) be a complete metric space and $T : X \to CL(X)$. If there exist positive constants α, β, γ with $\theta = \alpha + \beta + \gamma < 1$ and $L \geq 0$ such that for any $x, y \in X$ with $x \neq y$,*

$$\eta(\theta)d(x, Tx) \leq d(x, y)$$

implies that

$$H(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + LN(x, y).$$

Then there exists $z \in X$ such that $z \in Tz$.

Corollary 2.5. *Let (X, d) be a complete metric space and $f : X \to X$. If there exists $0 \leq \theta < 1$ and $L \geq 0$ such that for any $x, y \in X$ with $x \neq y$*

$$\eta(\theta)d(x, fx) \leq d(x, y) \text{ implies that } d(fx, fy) \leq \theta M^f(x, y) + LN^f(x, y).$$

Then f has a unique fixed point.

Proof. The existence of the fixed point of f follows from Theorem 2.1. For uniqueness, assume that there exist $z_1, z_2 \in X$ with $z_1 \neq z_2$ such that $z_1 = fz_1$ and $z_2 = fz_2$. Then

$$\eta(\theta)d(z_1, fz_1) \leq d(z_1, fz_1) = d(z_1, z_1) = 0 \leq d(z_1, z_2)$$

implies that

$$\begin{aligned} d(z_1, z_2) &= d(fz_1, fz_2) \\ &\leq \theta \max\left\{d(z_1, z_2), d(z_1, fz_1), d(z_2, fz_2), \frac{d(z_2, fz_1) + d(z_1, fz_2)}{2}\right\} + \\ & L\min\{d(z_1, fz_1), d(z_2, fz_1)\} \\ &\leq \theta \max\{d(z_1, z_2), d(z_1, z_1), d(z_2, z_2)\} + L\min\{d(z_1, z_1), d(z_2, z_1)\} \\ &\leq \theta d(z_1, z_2) \end{aligned}$$

which is contradiction to our supposition that $z_1 \neq z_2$. Hence the result. □

Example 2.6. Let X and the metric $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ be as given in Example 2.2. Define the mapping $f : X \longrightarrow X$ by $f(\alpha) = f(\beta) = f(\gamma) = \alpha$, $f(\delta) = \beta$ and $f(\zeta) = \gamma$. Note that that for any $x, y \in X$ with $x \neq y$

$$\begin{aligned} \eta(\theta)d(x, fx) &\leq d(x, y) \text{ implies that} \\ d(fx, fy) &\leq \theta M^f(x, y) + LN^f(x, y) \end{aligned}$$

where $\theta = \frac{3}{4}$and $L = 2$. Thus all the conditions of Corollary 2.5 are satisfied. In particular,

$$\begin{aligned} \eta(\theta)d(\delta, f\delta) &= 2 \leq d(\delta, \zeta) \text{ implies that} \\ H(f\delta, f\zeta) &= d(\beta, \gamma) = 10 \leq \frac{35}{2} = \theta d(\delta, \zeta) + L\min\{d(\delta, \beta), d(\zeta, \beta)\}, \text{ and} \\ H(f\zeta, f\delta) &= d(\gamma, \beta) = 10 \leq \frac{43}{2} = \theta d(\zeta, \delta) + L\min\{d(\zeta, \gamma), d(\delta, \gamma)\}. \end{aligned}$$

Moreover $x = \alpha$ is a unique fixed point of f. On the other hand, if we take $x = \delta$ and $y = \zeta$, then

$$\begin{aligned} M^f(\delta, \zeta) &= \max\left\{d(\delta, \zeta), d(\delta, f\delta), d(\zeta, f\zeta), \frac{d(\delta, f\zeta) + d(\zeta, f\delta)}{2}\right\}, \\ &= \max\left\{d(\delta, \zeta), d(\delta, \beta), d(\zeta, \gamma), \frac{d(\delta, \gamma) + d(\zeta, \beta)}{2}\right\}, \\ &= \max\{2, 8, 10, 10\} = 10, \end{aligned}$$

and

$$d(f\delta, f\zeta) = d(\beta, \gamma) = 10 > \frac{15}{2} = \theta M^f(\delta, \zeta).$$

Thus, Theorem 3 in [11] is not applicable in this case.

We now state the following Lemma in [14] which is crucial to prove a coincidence point result for a hybrid pair (f, T).

Lemma 2.7. ([14]) *Let X be a nonempty set and $g : X \to X$. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one.*

Theorem 2.8. *Let (X, d) be a metric space, (f, T) a hybrid pair with $T(X) \subseteq f(X)$. If T is a Suzuki-type generalized multivalued $(f, \theta, L)-$almost contraction and $f(X)$ is a complete subspace of X. Then T and f have a coincidence point. Also $F(f, T) \neq \phi$ if any of the following conditions holds:*

a *T and f are $w-$compatible, $\lim_{n\to\infty} f^n x = u$ for some $x \in C(T, f)$, $u \in X$ and f is continuous at u.*

b *f is $T-$weakly commuting for some $x \in C(T, f)$ and $f^2 x = fx$.*

c *f is continuous at x for some $x \in C(T, f)$ and for some $u \in X$, $\lim_{n\to\infty} f^n u = x$.*

Proof. By Lemma 2.7, there is a subset of E of X such that $f : X \to X$ is one-to-one and $f(E) = f(X)$. As $f(X)$ is complete, $f(E)$ is complete. Define the mapping $\mathcal{A} : f(E) \to CB(X)$ by

$$\mathcal{A}(fx) = T(x), \text{ for all } fx \in f(E). \tag{14}$$

Since f is one-to-one on E, so $\mathcal{A}$ is well defined. Now

$$\eta(\theta)d(fx, \mathcal{A}(fx)) = \eta(\theta)d(fx, Tx) \leq d(fx, fy)$$

implies that

$$\begin{aligned}
H(\mathcal{A}fx, \mathcal{A}fy) &= H(Tx, Ty) \leq \theta M_f(x, y) + LN_f(x, y) \\
&= \theta \max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\right\} + \\
&\quad L \min\{d(fx, Tx), d(fy, Tx)\} \\
&= \theta \max\left\{d(fx, fy), d(fx, \mathcal{A}fx), d(fy, \mathcal{A}fy), \frac{d(fx, \mathcal{A}fy) + d(fy, \mathcal{A}fx)}{2}\right\} + \\
&\quad L \min\{d(fx, \mathcal{A}fx), d(fy, \mathcal{A}fx)\}.
\end{aligned}$$

Let $fx = x^*$ and $fy = y^*$, then we obtain that

$$\eta(\theta)d(x^*, \mathcal{A}x^*) \leq d(x^*, y^*)$$

and hence

$$\begin{aligned}
H(\mathcal{A}x^*, \mathcal{A}y^*) &\leq \theta \max\left\{d(x^*, y^*), d(x^*, \mathcal{A}x^*), d(y^*, \mathcal{A}y^*), \frac{d(x^*, \mathcal{A}y^*) + d(y^*, \mathcal{A}x^*)}{2}\right\} + \\
&\quad L \min\{d(x^*, \mathcal{A}x^*), d(y^*, \mathcal{A}x^*)\} \\
&= \theta M(x^*, y^*) + LN(x^*, y^*).
\end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied and there exists $u \in f(E)$ such that $u \in \mathcal{A}u$. Now, we prove that T and f have a coincidence point. As $T(X) \subseteq f(X)$, there exist $u_1 \in X$ such that $fu_1 = u$. Thus

$$fu_1 \in \mathcal{A}fu_1 = Tu_1.$$

Hence $C(T, f)$ is nonempty. Now suppose that condition (a) holds. That is, for some $x \in C(T, f)$, we have $\lim_{n\to\infty} f^n x = u$ where $u \in X$ and f is continuous at u. So u is a fixed point of f. As T and f are w−compatible, $f^n x \in C(T, f)$ for all $n \geq 1$. That is, for all $n \geq 1$, $f^n x \in T(f^{n-1}x)$. By (3), we obtain that

$$\begin{aligned}
\eta(\theta)d(f^n x, Tf^{n-1}x) &\leq d(f^n x, Tf^{n-1}x) = 0 \\
&\leq d(ff^{n-1}x, fu)
\end{aligned}$$

which implies that

$$\begin{aligned}
d(fu, Tu) &\leq d(fu, f^n x) + d(f^n x, Tu) \leq d(fu, f^n x) + H(T(f^{n-1}x), Tu) \\
&\leq d(fu, f^n x) + \theta M(ff^{n-1}x, fu) + LN(ff^{n-1}x, fu) \\
&\leq d(fu, f^n x) + \theta \max\left\{d(ff^{n-1}x, fu), d(ff^{n-1}x, Tf^{n-1}x), d(fu, Tu), \frac{d(ff^{n-1}x, Tu) + d(fu, Tf^{n-1}x)}{2}\right\} + \\
&\quad L \min\{d(ff^{n-1}x, Tf^{n-1}x), d(fu, Tf^{n-1}x)\} \\
&\leq d(fu, f^n x) + \theta \max\left\{d(f^n x, fu), d(f^n x, f^n x), d(fu, Tu), \frac{d(f^n x, Tu) + d(fu, f^n x)}{2}\right\} + \\
&\quad L \min\{d(f^n x, f^n x), d(fu, f^n x)\}.
\end{aligned}$$

On taking limit $n \to \infty$ we have

$$d(fu, Tu) \leq \theta d(fu, Tu).$$

Since $\theta < 1$, $d(fu, Tu) = 0$ and $fu \in Tu$. Hence $u = fu \in Tu$. Now suppose that (b) hold. That is, for some $x \in C(T, f)$ and f is $T-$weakly commuting and $f^2x = fx$, then

$$fx = f^2x \in T(fx).$$

Hence, $fx \in F(f, T)$. Now suppose that condition (c) holds true, that is for some $x \in C(T, f)$ and for some $u \in X$, $\lim_{n\to\infty} f^n u = x$. Since f is continuous at x, we get

$$x = fx \in T(x).$$

□

Example 2.9. Let $X = [1, 4]$ be equipped with a usual metric. Define $T : X \to CL(X)$ and $f : X \to X$ by $T(x) = [1, 2]$ and $f(x) = 4 - \frac{3}{4}x$ for all $x \in X$. Clearly all the conditions in Theorem 2.8 are satisfied. Note that

$$C(f, T) = \left[\frac{8}{3}, 4\right].$$

Note that $F(f, T)$ is empty in this case.

Example 2.10. Let $X = [0, 1]$ with usual metric $d(x, y) = \left|x - y\right|$. Define $T : X \to CL(X)$ and $f : X \to X$ by

$$Tx = \left[0, \frac{\sin x}{2}\right] \text{ and } fx = \frac{2}{3}x$$

for all $x \in X$. If $\sin x = \sin y$, then $H(Tx, Ty) = 0$. If $\sin x \neq \sin y$, then

$$\begin{aligned} H(Tx, Ty) &\leq \frac{3}{4}d(fx, fy) \\ &\leq \theta M_f(x, y) + LN_f(x, y) \end{aligned}$$

for all x, y in X with $\theta = \frac{3}{4}$. Thus all the conditions of Theorem 2.8 are satisfied. Moreover, $0 \in C(T, f)$.

Corollary 2.11. *Let (X, d) be a complete metric space. If (f, T) is a hybrid pair of mappings such that for any $x, y \in X$,*

$$\eta(\theta)d(fx, Tx) \leq d(fx, fy)$$

implies that

$$H(Tx, Ty) \leq \theta \max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\} + LN_f(x, y).$$

Then $C(f, T) \neq \phi$. Moreover $F(f, T) \neq \phi$ if any one of given conditions holds:

a *T and f are $w-$compatible, $\lim_{n\to\infty} f^n x = u$ for some $x \in C(T, f)$, $u \in X$, and f is continuous at u;*

b *f is $T-$weakly commuting for some $x \in C(T, f)$ and fx is fixed point of f, that is $f^2x = fx$;*

c *f is continuous at x for some $x \in C(T, f)$ and for some $u \in X$, $\lim_{n\to\infty} f^n u = x$.*

Corollary 2.12. *Let (X, d) be a complete metric space and and (f, T) a hybrid pair of mappings. If there exist positive constants α, β, γ with $\theta = \alpha + \beta + \gamma < 1$ and $L \geq 0$ such that for any $x, y \in X$ with $x \neq y$,*

$$\eta(\theta)d(fx, Tx) \leqslant d(fx, fy)$$

implies

$$H(Tx, Ty) \leq \alpha d(fx, fy) + \beta d(fx, Tx) + \gamma d(fy, Ty) + LN_f(x, y).$$

Then $C(f, T) \neq \phi$. Moreover $F(f, T) \neq \phi$ if any of the following conditions holds:

a *T and f are $w-$compatible, $\lim_{n\to\infty} f^n x = u$ for some $x \in C(T, f)$, $u \in X$, and f is continuous at u.*

b *f is $T-$weakly commuting for some $x \in C(T, f)$ and fx is fixed point of f, that is $f^2x = fx$.*

c *f is continuous at x for some $x \in C(T, f)$ and for some $u \in X$, $\lim_{n\to\infty} f^n u = x$.*

For a self mapping, Theorem 2.8 becomes:

Corollary 2.13. *Let (X, d) be a metric space and f, $T : X \to X$ with $T(X) \subseteq f(X)$. Suppose that $f(X)$ is a complete subspace of X and for any $x, y \in X$, we have*

$$\begin{aligned} \eta(\theta)d(x, Tx) &\leq d(fx, fy) \Longrightarrow \\ d(Tx, Ty) &\leq \theta \max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\right\} \\ &+L \min\{d(fx, Tx), d(fy, Tx)\} \end{aligned}$$

Then $C(f, T)$ is nonempty. Further $F(f, T)$ is nonempty and singleton provided that f and T are commuting at $x \in C(f, T)$.

Proof. Using theorem 2.8 it follows that $C(f, T) \neq \phi$. Let $x \in C(f, T)$, that is $fx = Tx$. As f and T are commuting at x, so $f^2x = fTx = Tfx$. We now show that $fx = f^2x$. If not, then we have

$$\eta(\theta)(fx, Tfx) \leq d(fx, Tfx) \leq d(fx, f^2x)$$

which implies that

$$\begin{aligned} d(fx, f^2x) &\leq d(Tx, Tfx) \leq \theta M_f(x, fx) + LN_f(x, fx) \\ &\leq \theta \max\left\{d(fx, ffx), d(fx, Tx), d(ffx, Tfx), \frac{d(fx, Tfx) + d(ffx, Tx)}{2}\right\} + \\ &\quad L \min\{d(fx, Tx), d(ffx, Tx)\} \\ &\leq \theta \max\left\{d(fx, ffx), d(fx, fx), d(ffx, ffx), \frac{d(fx, ffx) + d(ffx, fx)}{2}\right\} + \\ &\quad L \min\{d(fx, fx), d(ffx, fx)\} \\ &\leq \theta d(fx, ffx). \end{aligned}$$

Hence $(1 - \theta)d(fx, ffx) = 0$ gives a contradiction. Consequently, $F(f, T) \neq \phi$. For uniqueness of the common fixed point of f and T, Suppose that there exists z_1, z_2 in $F(f, T)$ such that $z_1 \neq z_2$. Clearly, $\eta(\theta)d(z_1, Tz_1) \leq$

$d(fz_1, fz_2)$. Hence

$$\begin{aligned} d(fz_1, fz_2) &= d(Tz_1, Tz_2) \leq \theta M_f(z_1, z_2) + LN_f(z_1, z_2) \\ &\leq \theta \max\left\{d(fz_1, fz_2), d(fz_1, Tz_1), d(fz_2, Tz_2), \frac{d(fz_1, Tz_2) + d(fz_2, Tz_1)}{2}\right\} + \\ &\quad L\min\{d(fz_1, Tz_1), d(fz_2, Tz_1)\} \\ &\leq \theta \max\left\{d(fz_1, fz_2), d(fz_1, fz_1), d(fz_2, fz_2), \frac{d(fz_1, fz_2) + d(fz_2, fz_1)}{2}\right\} + \\ &\quad L\min\{d(fz_1, fz_1), d(fz_2, fz_1)\} \\ &\leq \theta d(fz_1, fz_2), \end{aligned}$$

a contradiction and the result follows. □

3. Application in Dynamic Programming

Suppose that E and F are Banach spaces and $W \subseteq E$ and $D \subseteq F$ are state and decision spaces, respectively. A state space is the set of all feasible state and a decision space is the resultant network formed by the nodes of feasible states and all the feasible decisions. The main objective is to find the optimal decision in the given state space using dynamic programming related with the problem of solving nonlinear-functional equations

$$\begin{cases} p(x) = \sup\limits_{y\in D}\{g(x, y) + \Phi(x, y, p(\tau(x, y)))\}, \text{ for } x \in W, \\ q(x) = \sup\limits_{y\in D}\{h(x, y) + \Psi(x, y, q(\tau(x, y)))\}, \text{ for } x \in W, \\ \qquad \text{where} \\ \tau : W \times D \to W,\ g, h : W \times D \to \mathbb{R}, \text{ and } \Phi, \Psi : W \times D \times \mathbb{R} \to \mathbb{R}. \end{cases} \tag{15}$$

For detailed discussion on this topic, we refer to [4–6, 10, 23, 26].

In this section, we study the existence and uniqueness of the bounded solution of the above equations . Let $B(W)$ be the set of all bounded real-valued functions on W. For an arbitrary $h \in B(W)$, define a norm as on W as $\|h\| = \sup\limits_{x\in W} |h(x)|$. The space of all bounded real functional $(B(W), \|\cdot\|)$ endowed with the metric d induced by the supremum norm is defined by

$$d(h, k) = \sup_{x\in W} |h(x) - k(x)| \tag{16}$$

for all $h, k \in B(W)$. Note that $B(W)$ is a complete space. Suppose that the following conditions hold:

(C1): Φ, Ψ, g and h are bounded.

(C2): There exists constants $\theta \in [0, 1)$ and $L \geq 0$ such that for every $(x, y) \in W \times D$, $h, k \in B(W)$ and $a \in W$,

$$\eta(\theta)d(Sh(a), Th(a)) \leq d(Sh(a), Sk(a))$$

implies that

$$\left|\Phi(x, y, h(a)) - \Phi(x, y, k(a))\right| \leq \theta M_S(h(a), k(a)) + LN_S(h(a), k(a))$$

where $M_S(h(a), k(a))$, $N_S(h(a), k(a))$ and $\eta(\theta)$ are as given in section (1). Now for $x \in W$, $h \in B(W)$, mappings T and S are defined as

$$\begin{aligned} Th(x) &= \sup_{y\in D}\{g(x, y) + \Phi(x, y, h(\tau(x, y)))\}, \\ Sh(x) &= \sup_{y\in D}\{h(x, y) + \Psi(x, y, h(\tau(x, y)))\}. \end{aligned}$$

C3: For any $h \in B(W)$, there exists $k \in B(W)$ such that for $x \in W$, we have

$$Th(x) = Sk(x).$$

C4: There exists $h \in B(W)$ such that

$$Th(x) = Sh(x) \text{ implies that } STh(x) = TSh(x)$$

Theorem 3.1. *Suppose that (C1)-(C4) are satisfied, then the system of equation (15) has a unique, bounded and common solution in $B(W)$.*

Proof. Note that T is selfmap on $B(W)$ Let $h_1, h_2 \in B(W)$. Then for every real number α and $x \in W$, there exist $y_1, y_2 \in D$ such that

$$T(h_1(a)) < g(x, y_1) + \Phi(x, y_1, h_1(\tau_1)) + \alpha \tag{17}$$
$$T(h_2(a)) < g(x, y_2) + \Phi(x, y_2, h_2(\tau_2)) + \alpha, \tag{18}$$

where $\tau_1 = \tau(x, y_1)$ and $\tau_2 = \tau(x, y_2)$.
Thus

$$T(h_1(a)) \geq g(x, y_2) + \Phi(x, y_2, h_1(\tau_1)), \tag{19}$$
$$T(h_2(a)) \geq g(x, y_1) + \Phi(x, y_1, h_2(\tau_2)). \tag{20}$$

From (17) and (20), we obtain that

$$\begin{aligned} T(h_1(a)) - T(h_2(a)) &< \Phi(x, y_1, h_1(\tau_1)) - \Phi(x, y_1, h_2(\tau_2)) + \alpha \\ &\leq \left|\Phi(x, y_1, h_1(\tau_1)) - \Phi(x, y_1, h_2(\tau_2))\right| + \alpha \\ &\leq \theta M_S(h_1(a), h_2(a)) + LN_S(h_1(a), h_2(a)) + \alpha. \end{aligned} \tag{21}$$

Similarly, (18) and (19) imply that

$$\begin{aligned} T(h_2(a)) - T(h_1(a)) &< \Phi(x, y_2, h_1(\tau_1)) - \Phi(x, y_1, h_2(\tau_1)) + \alpha \\ &\leq \left|\Phi(x, y_2, h_1(\tau_1)) - \Phi(x, y_1, h_2(\tau_1))\right| + \alpha \\ &\leq \theta M_S(h_1(a), h_2(a)) + LN_S(h_1(a), h_2(a)) + \alpha. \end{aligned} \tag{22}$$

Hence from (21) and (22), we have

$$|T(h_1(a)) - T(h_2(a))| \leq \theta M_S(h_1(a), h_2(a)) + LN_S(h_1(a), h_2(a)) + \alpha. \tag{23}$$

Since (23) holds true for any $x \in W$ and for an arbitrary $x > 0$, therefore

$$\eta(\theta) d(S(h_1), T(h_1)) \leq d(S(h_1), S(h_2))$$

implies that

$$d(T(h_1), T(h_2)) \leq \theta M_S(h_1(a), h_2(a)) + LN_S(h_1(a), h_2(a)).$$

Thus all the conditions of Corollary 2.13 hold for T and S, and hence the system of equation (15) has a unique, common and bounded solution. □

4. Application in Integral Equations

As an application of Corollary 2.13, the solution of the system of Volterra type integral equations will be discussed in this section.

Such system can be represented as:

$$\begin{aligned} u(t) &= \int_0^t \Phi(t,s,u(s))ds + g(t) \\ w(t) &= \int_0^t \Psi(t,s,w(s))ds + f(t) \end{aligned}$$

for $t \in [0,a]$, where $a > 0$. Let $C([0;a];\mathbb{R})$ be the space of all continuous functions defined on $[0,a]$. For $u \in C([0;a];\mathbb{R})$, define supremum norm as, $\|u\|_\tau = \sup_{t\in[0,a]}\{u(t)e^{-\tau t}\}$ where $\tau > 0$. Let $C([0;a];\mathbb{R})$ be endowed with the metric given by

$$d_\tau(u,v) = \sup_{t\in([0,a])} \|u(t) - v(t)\|_\tau$$

for all $u, v \in C([0;a];\mathbb{R})$. Note that $C([0;a];\mathbb{R};\|\cdot\|_\tau)$ is a Banach space.
For further details in this direction, we refer to [4].

Theorem 4.1. *Assume the following conditions are satisfied:*

(i) $\Phi \times \Psi : [0,a] \times [0,a] \times \mathbb{R} \to \mathbb{R}$ *and* $g, f : [0,a] \to \mathbb{R}$ *are continuous;*

Define

$$Tu(t) = \int_0^t \Phi(t,s,u(s))ds + g(t) \tag{24}$$

$$Su(t) = \int_0^t \Psi(t,s,u(s))ds + f(t). \tag{25}$$

Suppose there exists a $\tau \geq 1$ *such that*

$$|\Phi(t,s,u) - \Phi(t,s,v)| \leqslant \tau[\theta M_S(u,v) + LN_S(u,v)]$$

for all $t, s \in [0,a]$ *and* $u, v \in C([0;a];\mathbb{R})$

(ii) *There exists* $u, v \in C([0;a];\mathbb{R})$ *such that* $Tu(t) = Su(t)$ *implies* $TSu(t) = STu(t)$. *Then the system of integral equations given in (24) and (25) has a unique common solution.*

Proof. By assumption (ii), we have

$$\begin{aligned} |Tu(t) - Tv(t)| &= \int_0^t |K_1(t,s,u(s)) - K_1(t,s,v(s))|\, ds \\ &\leq \int_0^t \tau(\theta M_S(u,v) + LN_S(u,v))e^{\tau s}e^{-\tau s}ds \leq \int_0^t \tau[(\theta M_S(u,v) + LN_S(u,v))e^{-\tau s}]e^{\tau s}ds \\ &\leq \int_0^t \tau \|\theta M_S(u,v) + LN_S(u,v)\|_\tau e^{\tau s}ds \leq \tau \|\theta M_S(u,v) + LN_S(u,v)\|_\tau \int_0^t e^{\tau s}ds \\ &\leq \tau \|\theta M_S(u,v) + LN_S(u,v)\|_\tau \frac{1}{\tau}e^{\tau t} \leq \|\theta M_S(u,v) + LN_S(u,v)\|_\tau e^{\tau t}, \end{aligned}$$

which implies that $|Tu(t) - Tv(t)|\, e^{-\tau t} \leqslant \|\theta M_S(u,v) + LN_S(u,v)\|_\tau$. That is

$$\|Tu(t) - Tv(t)\|_\tau \leqslant \|\theta M_S(u,v) + LN_S(u,v)\|_\tau .$$

So all the conditions of Corollary (2.13) are satisfied. Hence the given system of integral equations has a unique common solution. □

5. Application in Data Dependence

Following are some definitions needed in the sequel (see also, [21, 24, 25]).

Definition 5.1. A multivalued mapping $T : X \to P(X)$ is called multivalued weakly Picard (briefly MWP) operator if and only if for every $x \in X$ and for every $y \in T(x)$, there exists a sequence $\{x_n\}$ such that

(d-1) $x_0 = x$ and $x_1 = y$,

(d-2) $x_{n+1} \in Tx_n \ \forall n \in \mathbb{N}$,

(d-3) the sequence $\{x_n\}$ converges to the fixed point of T.

A sequence defined above is known as a sequence of successive approximations of T starting from (x, y). Let $G(T) = \{(x, y) : y \in Tx\}$ be the graph of MWP−operator T. Define T^∞ from $G(T)$ into $P(Fix(T))$ as follows:

$$T^\infty(x,y) = \{z \in F(T) : \text{ there exists a sequence of successive approximations of } T \text{ starting from } (x,y) \text{ that converges to } z\}.$$

Definition 5.2. ([25]) Let $c > 0$. A MWP−operator T is known as c−multivalued weakly Picard (briefly c−MWP) operator if there exists a selection t^∞ of T^∞ such that

$$d(x, t^\infty(x,y)) \leq c d(x,y) \tag{26}$$

for all $(x, y) \in G(T)$.

In the following, we present a data dependence result for Suzuki-type generalized multivalued (θ, L)−almost contraction mappings.

Theorem 5.3. *Let (X, d) be a complete metric space and $T_i : X \to CL(X)$ Suzuki-type generalized multivalued (θ_i, L_i)−almost contractions for each $i \in \{1, 2\}$. If there exists $\lambda > 0$ such that $H(T_1x, T_2x) \leq \lambda$, for all $x \in X$. Then:*

(a) $F(T_i) \in CB(X)$, $i \in \{1, 2\}$;

(b) *Each T_i is an MWP operator and satisfies*

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\lambda}{1 - \max\{r_1, r_2\}}. \tag{27}$$

Proof. From theorem 2.1, $F(T_i)$ is nonempty for each $i \in \{1, 2\}$. Choose a convergent sequence $x_n \in F(T_1)$ be such that $x_n \to x$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} d(x_n, x) = 0. \tag{28}$$

Note that

$$\eta(\theta)(d(x_n, T_1x_n)) = 0 \leq d(x_n, x).$$

Thus

$$\begin{aligned} d(x, T_1x) &\leq d(x, x_n) + d(x_n, T_1x) \leq d(x, x_n) + H(T_1x_n, T_1x) \\ &\leq d(x, x_n) + \theta_1 \max\left\{ d(x, x_n), d(x, T_1x), d(T_1x_n, x_n), \frac{d(x, T_1x_n) + d(x_n, T_1x)}{2} \right\} \\ &\quad + L_1 \min\{d(x_n, T_1x_n), d(x, T_1x_n)\} \\ &\leq d(x, x_n) + \theta_1 d(x, x_n). \end{aligned}$$

On taking limit as $n \to \infty$, we obtain that $d(x, T_1x) = 0$, that is, $x \in T_1x$. Hence $F(T_1)$ is closed. Similarly, we can show that $F(T_2)$ is closed. Following arguments similar to those in proof of Theorem 2.1, each T_i is an MWP operator. Now we prove that $H(F(T_1), F(T_2)) \leq \dfrac{\lambda}{1 - \max\{r_1, r_2\}}$. Let $a > 1$. Then for an arbitrary $x_0 \in F(T_1)$, there exists $x_1 \in T_2x_0$ such that

$$d(x_0, x_1) \leq aH(T_1x_0, T_2x_0).$$

As $x_1 \in T_2x_0$, there exists $x_2 \in T_2x_1$ such that

$$\eta(\theta_2)(d(x_0, T_2x_0)) \leq \eta(\theta_2)d(x_0, x_1) \leq d(x_0, x_1),$$

which implies that

$$\begin{aligned} d(x_1, x_2) &\leq aH(T_2x_0, T_2x_1) \\ &\leq a\theta_2 \max\left\{ d(x_0, x_1), d(x_0, T_2x_0), d(x_1, T_2x_1), \frac{d(x_0, T_2x_1) + d(x_1, T_2x_0)}{2} \right\} \\ &\quad + aL \min\{d(x_0, T_2x_0), d(x_1, T_2x_0)\} \\ &\leq a\theta_2 d(x_0, x_1). \end{aligned}$$

Continuing this way, we can obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in T_2x_n$ and

$$\begin{aligned} d(x_n, x_{n+1}) &\leq a\theta_2 d(x_n, x_{n+1}) \\ &\leq \ldots \leq (a\theta_2)^n d(x_0, x_1). \end{aligned}$$

Thus

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \\ &\leq (ar_2)^n d(x_0, x_1) + \ldots + (ar_2)^{n+p-1} d(x_0, x_1) \\ &\leq \frac{(a\theta_2)^n}{1 - a\theta_2} d(x_0, x_1). \end{aligned} \tag{29}$$

Choose $1 < a < \min\{\frac{1}{\theta_1}, \frac{1}{\theta_2}\}$. Hence $\{x_n\}$ is a Cauchy sequence in X. Consequently, there exists z in X such that $x_n \to z$ as $n \to \infty$. Following arguments similar to those given in proof of Theorem 2.1, it follows that $z \in T_2z$. By (29), we obtain that

$$d(x_n, z) \leq \frac{(a\theta_2)^n}{1 - a\theta_2} d(x_0, x_1)$$

Thus, in particular

$$d(x_0, z) \leq \frac{1}{1 - a\theta_2} d(x_0, x_1) \leq \frac{a\lambda}{1 - a\theta_2}. \tag{30}$$

Similarly, we conclude that for each $z_0 \in F(T_2)$, there is an $x \in F(T_1)$ such that

$$d(z_0, x) \leq \frac{1}{1 - a\theta_1} d(z_0, z_1) \leq \frac{a\lambda}{1 - a\theta_1}. \tag{31}$$

By (30) and (31), we have

$$H(F(T_1), F(T_2)) \leq \frac{\lambda}{1 - \max\{\theta_1, \theta_2\}}.$$

□

6. Application in Homotopy

We first present a local fixed point theorem for Suzuki-type generalized multivalued $(\theta, L)-$almost contractions.

Theorem 6.1. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Suppose that $T : B(x_0, r) \to CL(X)$ be Suzuki-type generalized multivalued $(\theta, L)-$almost contraction and $d(x_0, Tx_0) < (1-\theta)r$. Then $F(T) \neq \emptyset$.*

Proof. Choose $0 < s < r$ such that $\widetilde{B}(x_0, s) \subset B(x_0, r)$ and $d(x_0, Tx_0) < (1-\theta)s$. Thus $(1-\theta)s - d(x_0, Tx_0) > 0$. For $\varepsilon = (1-\theta)s - d(x_0, Tx_0) > 0$, there exists $x_1 \in Tx_0$ such that $d(x_0, x_1) < d(x_0, Tx_0) + \varepsilon$. Hence

$$d(x_0, x_1) < (1-\theta)s.$$

Now for $h = \frac{1}{\sqrt{\theta}} > 1$ and $x_1 \in Tx_0$, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq hH(Tx_0, Tx_1)$$

Since $\eta(\theta)d(x_0, Tx_0) \leq \eta(\theta)d(x_0, x_1) \leq d(x_0, x_1)$, therefore we obtain

$$\begin{aligned}
d(x_1, x_2) &\leq hH(Tx_0, Tx_1) = \frac{1}{\sqrt{\theta}}H(Tx_0, Tx_1) \leq \sqrt{\theta}M(x_0, x_1) + \frac{L}{\sqrt{\theta}}N(x_0, x_1) \\
&\leq \sqrt{\theta}\max\left\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{d(x_0, Tx_1) + d(Tx_0, x_1)}{2}\right\} \\
&\quad + \frac{L}{\sqrt{\theta}}\min\{d(x_0, Tx_0), d(x_1, Tx_0)\} \\
&\leq \sqrt{\theta}\max\left\{d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_1) + d(x_1, x_2)}{2}\right\} \\
&\quad + \frac{L}{\sqrt{\theta}}\min\{d(x_0, x_1), d(x_1, x_1)\} \\
&\leq \sqrt{\theta}\max\left\{d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_1) + d(x_1, x_2)}{2}\right\} \\
&\leq \sqrt{\theta}d(x_0, x_1) < \sqrt{\theta}(1-\theta)s.
\end{aligned}$$

Note that $x_2 \in B(x_0, s)$. Indeed, $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) < (1-\theta)s + \sqrt{\theta}(1-\theta)s = (1-\theta)(1+\sqrt{\theta})s < s$. Inductively, we obtain a sequence $\{x_n\}$ which satisfies

i) $x_n \in B(x_0, s)$; for each $n \in \mathbb{N}$,
ii) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$,
iii) $d(x_n, x_{n+1}) \leq (\sqrt{\theta})^n(1-\theta)s$ for each $n \in \mathbb{N}$.

From (iii) the sequence $\{x_n\}$ is Cauchy which converges to some $x \in B(x_0, r)$. Following arguments similar to those in the proof of Theorem 2.1, we have $x \in F(T)$. □

Now we present a homotopy result for "Suzuki-type generalized multivalued $(\theta, L)-$almost contractions".

Theorem 6.2. *Let V an open subset of a complete metric space (X, d). If $G : \overline{V} \times [0, 1] \to P(X)$ satisfies the following conditions*

p-1 $x \notin G(x,t)$, *for each* $x \in \partial V$ *and each* $t \in [0,1]$;

p-2 $G(.,t) : \overline{V} \to P(X)$ *is a "Suzuki-type generalized multivalued* $(\theta, L)-$*almost contraction for each* $t \in [0,1]$;

p-3 *there exists a increasing and continuous function* $\psi : [0,1] \to \mathbb{R}$ *such that*

$$H(G(x,t), G(x,s)) \leq \left|\psi(t) - \psi(s)\right| \text{ for all } t,s \in [0,1] \text{ and each } x \in \overline{V};$$

p-4 $G : \overline{V} \times [0,1] \to P(X)$ *is closed.*

Then $G(.,0)$ *has a fixed point if and only if* $G(.,1)$ *has a fixed point.*

Proof. If z is a fixed point of $G(.,0)$, then (p-1) implies that $z \in V$. Define

$$\Omega = \{(t,x) \in [0,1] \times V \mid x \in G(x,t)\}.$$

Now $(0,z) \in \Omega$ implies $\Omega \neq \emptyset$. We define a partial order on Ω as follows:

$$(t,x) \leq (s,y) \text{ if and only if } t \leq s \text{ and } d(x,y) \leq \frac{2}{1-\theta}\left[\psi(s) - \psi(t)\right].$$

Let N be a totally ordered subset of Ω and $t^* := \sup\{t \mid (t,x) \in N\}$. Suppose that $\{(t_n, x_n)\}$ is a sequence in N such that $(t_n, x_n) \leq (t_{n+1}, x_{n+1})$ and $t_n \to t^*$ as $n \to \infty$. Then

$$d(x_m, x_n) \leq \frac{2}{1-\theta}[\psi(t_m) - \psi(t_n)], \text{ for each } m,n \in \mathbb{N},\ m > n.$$

On taking limit as $m, n \to \infty$, we have $d(x_m, x_n) \to 0$. Thus $\{x_n\}$ is a Cauchy sequence and converges to some x^* in X. As G is closed and $x_n \in G(x_n, t_n)$, $n \in \mathbb{N}$, so $x^* \in G(x^*, t^*)$. From condition (p-1), we have $x^* \in V$. Hence $(t^*, x^*) \in \Omega$. Since N is totally ordered, so $(t,x) \leq (t^*, x^*)$, for each $(t,x) \in N$. That is, (t^*, x^*) is an upper bound of N. By Zorn's Lemma, Ω has a maximal element $(t_0, x_0) \in \Omega$. We now show that $t_0 = 1$. Assume on contrary that $t_0 < 1$. Choose $r = \frac{2}{1-\theta}[\psi(t) - \psi(t_0)] > 0$ with $t \in (t_0, 1]$ such that $B(x_0, r) \subset V$. Note that

$$\begin{aligned} d(x_0, G(x_0,t)) &\leq d(x_0, G(x_0,t_0)) + H(G(x_0,t_0), G(x_0,t)) \\ &\leq [\psi(t) - \psi(t_0)] = \frac{(1-\theta)r}{2} < (1-\theta)r. \end{aligned}$$

Thus $G(.,t) : B(x_0,r) \to CL(X)$ satisfies all the conditions of Theorem 6.1 for all $t \in [0,1]$. Hence, there exists $x \in B(x_0,r)$ such that $x \in G(x,t)$ which implies that $(t,x) \in \Omega$ for all $t \in [0,1]$. Now

$$d(x_0, x) \leq r = \frac{2}{1-\theta}[\psi(t) - \psi(t_0)],$$

gives $(t_0, x_0) < (t, x)$, a contradiction to the maximality of (t_0, x_0). Conversely suppose that $G(.,1)$ has a fixed point, then following the similar arguments to those given above, we show that $G(.,0)$ has a fixed point. □

7. Conclusion

In this article, we generalized already existing definitions in [9] and [15] by proposing the concept of Suzuki-type generalized multivalued $(f, \theta, L)-$ almost contractions. We then proved a fixed point result which is a proper generalization of comparable results in 1.2 in [13]. We studied some applications of our result in (a) dynamic programming, (b) solution of integral equations, (c) in data dependence problem and in Homotopy.

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