Filomat 33:2 (2019), 543–559
https://doi.org/10.2298/FIL1902543K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Problem $\sigma_{od}(n) = \sigma_{od}(n+1)$

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Abstract. Let $\sigma_{od}(n) = \sum_{d|n, 2\nmid d} d$. In this paper, we study the solutions of $\sigma_{od}(n) = \sigma_{od}(n+1)$, their relations to Pell numbers, and some interesting conjectures. Finally, we obtain that the equation $\sigma_{od}(n) = \sigma_{od}(n+1) = \sigma_{od}(n+2) \equiv 1 \pmod 2$ has no solution.

1. Introduction: A Question on Odd Divisor Functions

Let

$$\sigma(n) = \sum_{d|n} d, \text{ and } \sigma_{od}(n) = \sum_{d|n, 2\nmid d} d$$

be the divisor function, and the odd divisor function, respectively, where n is a positive integer. The divisor function and the odd divisor function are important in number theory. They appear naturally as the coefficients of a (quasi-) modular form.

Let $q = e^{2\pi i\tau}$, where τ is a complex variable whose imaginary part is greater than 0. The Dedekind eta function $\eta(\tau)$ is defined as

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Taking the logarithmic derivative of $\eta(\tau)$, i.e., $q\frac{d}{dq}\ln$, or equivalently, $\frac{1}{2\pi i}\frac{d}{d\tau}\ln$, we get

$$E_2(\tau) = \frac{1}{24} + q \sum_{n=1}^{\infty} \frac{-nq^{n-1}}{1-q^n} = \frac{1}{24} - \sum_{n=1}^{\infty} \sigma(n) q^n.$$

2010 *Mathematics Subject Classification*. Primary 11A07
Keywords. Congruence, odd divisor functions
Received: 13 November 2017; Revised: 05 May 2018; Accepted: 01 June 2018
Communicated by Ljubiša D.R. Kočinac
Research was supported by The Research Fund of Balikesir University, Project No: 2016/44
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The eta quotient $\eta(\tau)/\eta(2\tau)$ is equal to

$$\frac{\eta(\tau)}{\eta(2\tau)} = q^{-1/24} \prod_{n=1,2\nmid n}^{\infty} (1-q^n).$$

Similarly, taking the logarithmic derivative, we get

$$E_{2,2}(\tau) = -\frac{1}{24} + q \sum_{n=1,2\nmid n}^{\infty} \frac{-nq^{n-1}}{1-q^n} = -\frac{1}{24} - \sum_{n=1}^{\infty} \sigma_{od}(n) q^n.$$

It is known that $E_2(\tau)$ is a quasi-modular form (see [4]) and $E_{2,2}(\tau)$ is a modular form for the congruence subgroup $\Gamma_0(2)$ (see [5],pp.18-19).

Ramanujan gave a formula for the convolution sum of the divisor function, that is,

$$\sum_{k+l=n} \sigma(k)\sigma(l) = \frac{1}{12}\left\{5\sigma_3(n) + (1-6n)\sigma_1(n)\right\}.$$

Recently, various kinds of convolution sums of the divisor function were studied in [3, 6, 8, 9, 13, 15, 17, 22].

A formula of the convolution sum of the odd divisor function was given in [16, (11)], [22, p. 130], that is,

$$\sum_{k+l=n} \sigma_{od}(k)\sigma_{od}(l) = \frac{1}{24}(11\sigma_3(n) - \sigma_3(2n) - 2\sigma_{od}(n)).$$

Kim and Bayad [18] introduced several definitions and properties of odd divisor functions.

In this paper, we will study a new question on the odd divisor function. There is an unsolved problem on the divisor function, which asks that if $\sigma(n) = \sigma(n+1)$ infinitely often ([11, p. 103], [21, p. 166]). Erdös [7] made the much stronger conjecture that for every integer $k \geq 1$ there is an n such that $\sigma(n) = \sigma(n+1) = \cdots = \sigma(n+k)$ has infinitely many solutions for each k. In this study, we are interested in the question whether $\sigma_{od}(n) = \sigma_{od}(n+1)$ infinitely often or not. By computer, we find all the solutions of $\sigma_{od}(n) = \sigma_{od}(n+1)$ up to $n \leq 2^{40}$. We list the prime factorizations of these n and $n+1$ in a table (see the Appendix, https://drive.google.com/open?id=1zuZ6DbgKUg7ueMMtbC6SVRhP9W8Exxgc).

From the Appendix, we find the following statements in Conjecture 1.1 are true up to $n \leq 2^{40}$. We conjecture that they are true for all $n \geq 2$ (except the trivial case: $\sigma_{od}(1) = \sigma_{od}(2)$).

Conjecture 1.1. *Assume that $\sigma_{od}(n) = \sigma_{od}(n+1)$ and $n \geq 2$ is an integer. Then*

(i) $4 \nmid n$ *and* $4 \nmid (n+1)$*;*

(ii) *The even one of n and $n+1$ has at least four distinct odd prime factors;*

(iii) *The odd one of n and $n+1$ is not a prime;*

(iv) *Neither n nor $n+1$ is a square.*

A natural number n is called perfect if $\sigma(n) = 2n$. Euclid found, and Euler proved that all the even perfect numbers are of the form $2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are both primes (or equivalently, $2^p - 1$ is a Mersenne prime). On the other hand, no odd perfect number is known up to now. Let $\omega(n)$ be the number of distinct prime factors of n. The main result of [20] shows that $\omega(n) \geq 9$ for n being odd perfect.

Analogous to the perfect numbers, a natural number n is called quasi-perfect (resp., almost-perfect) if $\sigma(n) - 2n = 1$ (resp., -1). The only known almost-perfect numbers are powers of 2 (p.74 of [11]). For quasi-perfect numbers, Cattaneo [2] showed that they are odd squares. But still none of them is found. Hagis and Cohen [12] proved $\omega(n) \geq 7$ for n being a quasi-perfect number.

It seems to be mysterious that no odd perfect number, no odd almost-perfect number and no odd quasi-perfect number are found up to now.

A natural number n is called near-perfect if $\sigma(n) - 2n = 2$ in [19]. There is no odd near-perfect number up to 10^{10} by computer searching (See [19, Remark 2.4]). The main result of [19] proved that $\omega(n) \geq 6$, for n being an odd near-perfect number.

In special cases, the new question about $\sigma_{od}(n) = \sigma_{od}(n+1)$ is related to perfect numbers and near-perfect numbers. In details,

(i) If n is an odd prime and $4 \nmid n+1$, then $\sigma_{od}(n) = n+1$ and $\sigma_{od}(n+1) = \sigma((n+1)/2)$. Therefore, in this case, $\sigma_{od}(n) = \sigma_{od}(n+1)$ is equivalent to $(n+1)/2$ is an odd perfect number.

(ii) If $n+1$ is an odd prime and $4 \nmid n$, then $\sigma_{od}(n+1) = n+2$ and $\sigma_{od}(n) = \sigma(n/2)$. Therefore, in this case, $\sigma_{od}(n) = \sigma_{od}(n+1)$ is equivalent to $n/2$ is an odd near-perfect number.

Since no odd perfect number and no odd near-perfect number is found up to now, this gives some evidence of Conjecture 1.1 (iii). Assume that $\sigma_{od}(n) = \sigma_{od}(n+1)$ and $n \geq 2$ is an integer. Then we prove that the even one of n and $n+1$ has at least three distinct odd prime factors. Moreover, if the odd one of n and $n+1$ is a prime power p^t, then the even one of n and $n+1$ has at least 4 distinct odd prime factors. This gives a partial result on Conjecture 1.1 (ii).

Bayad and Kim [18] suggest notions of polygon-shape number, n-gon, order, convex, area, and prime. Our result give several information of a study of polygon-shape number, for example, we see examples satisfying the difference of area of $n+1$-gon and n-gon $A(n+1) - A(n) = \frac{1}{2}$ with $\sigma_{od}(n) = \sigma_{od}(n+1)$.

The paper is organized as follows. In Section 2, we derive some basic conditions for the solutions of $\sigma_{od}(n) = \sigma_{od}(n+1)$. In Section 3, we prove the equations $\sigma_{od}(p^t) = \sigma_{od}(p^t+1)$ with $p^t + 1 = 2 \cdot p_1^{t_1} p_2^{t_2} p_3^{t_3}$ and $\sigma_{od}(p^t - 1) = \sigma_{od}(p^t)$ with $p^t - 1 = 2 \cdot p_1^{t_1} p_2^{t_2} p_3^{t_3}$ have no solutions. In Section 4, we obtain the equation $\sigma_{od}(n) = \sigma_{od}(n+1) = \sigma_{od}(n+2) \equiv 1 \pmod 2$ has no solution. The solutions of $\sigma_{od}(n) = \sigma_{od}(n+1) (n \leq 2^{40})$ are given in the Appendix, https://drive.google.com/open?id=1zuZ6DbgKUg7ueMMtbC6SVRhP9W8Exxgc

2. General Results

Lemma 2.1. *The integers $t_1, t_2 \cdots t_s$ are positive. Let $n = q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s}$ be the prime factorization of a positive integer n. Then*

$$\left(1 + \frac{1}{q_1}\right) \cdots \left(1 + \frac{1}{q_s}\right) \leq \frac{\sigma(n)}{n} < \left(1 + \frac{1}{q_1 - 1}\right) \cdots \left(1 + \frac{1}{q_s - 1}\right).$$

Proof. Since

$$\sigma(n) = \sigma(q_1^{t_1}) \cdots \sigma(q_s^{t_s}) = (1 + q_1 + \cdots + q_1^{t_1}) \cdots (1 + q_s + \cdots + q_s^{t_s}),$$

we get

$$\frac{\sigma(n)}{n} = (1 + q_1^{-1} + \cdots + q_1^{-t_1}) \cdots (1 + q_s^{-1} + \cdots + q_s^{-t_s}).$$

Letting $t_1 = t_2 = \cdots = t_s = 1$, we get

$$\frac{\sigma(n)}{n} \geq \left(1 + \frac{1}{q_1}\right)\left(1 + \frac{1}{q_2}\right) \cdots \left(1 + \frac{1}{q_s}\right). \quad (1)$$

Letting $t_1, t_2, \cdots, t_s$ all go to $+\infty$, we get

$$\frac{\sigma(n)}{n} < \left(1 + \frac{1}{q_1 - 1}\right)\left(1 + \frac{1}{q_2 - 1}\right) \cdots \left(1 + \frac{1}{q_s - 1}\right). \quad (2)$$

□

Theorem 2.2. *The integers $t_0, t_1, t_2 \cdots t_s$ are positive. Let $n \geq 2$ be an integer. Assume $\sigma_{od}(n) = \sigma_{od}(n+1)$. Then*

(i) *If n is an odd integer and $n+1 = 2^{t_0} q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s}$ is the prime factorization of $n+1$,*

(ii) *If n is an even integer and $n = 2^{t_0} q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s}$ is the prime factorization of n, then, $s \geq 3$. Moreover, if $s = 3$, then the only possibility of $\{q_1, q_2, q_3\}$ is $\{3,5,7\}$, $\{3,5,11\}$, or $\{3,5,13\}$.*

Proof. (i) Directly from definition of $\sigma_{od}(n)$ and $\sigma_{od}(n+1)$, we see that

$$\sigma(n) = \sigma(q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s}).$$

Dividing it by $n+1$, we get

$$\frac{\sigma(n)}{n+1} = \frac{\sigma(q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s})}{n+1} = \frac{1}{2^{t_0}} \frac{\sigma(q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s})}{q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s}}. \tag{3}$$

Obviously,

$$\frac{\sigma(n)}{n+1} \geq 1 \text{ and } t_0 \geq 1.$$

So, we get

$$\frac{\sigma(q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s})}{q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s}} \geq 2^{t_0} \geq 2.$$

By inequality (2), we get

$$\left(1 + \frac{1}{q_1 - 1}\right)\left(1 + \frac{1}{q_2 - 1}\right) \cdots \left(1 + \frac{1}{q_s - 1}\right) > 2. \tag{4}$$

Since

$$\left(1 + \frac{1}{3-1}\right)\left(1 + \frac{1}{5-1}\right) = \frac{15}{8} < 2,$$

by equation (4), we get $s \geq 3$.

Now assume $s = 3$ and $q_1 < q_2 < q_3$. Since

$$\left(1 + \frac{1}{5-1}\right)\left(1 + \frac{1}{7-1}\right)\left(1 + \frac{1}{11-1}\right) = \frac{77}{48} < 2,$$

by equation (4), we get $q_1 = 3$. Since

$$\left(1 + \frac{1}{3-1}\right)\left(1 + \frac{1}{7-1}\right)\left(1 + \frac{1}{11-1}\right) = \frac{77}{40} < 2,$$

by equation (4), we get $q_2 = 5$. From

$$\left(1 + \frac{1}{3-1}\right)\left(1 + \frac{1}{5-1}\right)\left(1 + \frac{1}{13-1}\right) = \frac{195}{96} > 2,$$
$$\left(1 + \frac{1}{3-1}\right)\left(1 + \frac{1}{5-1}\right)\left(1 + \frac{1}{17-1}\right) = \frac{255}{128} < 2,$$

we conclude that the only possibility of q_3 is 7, 11, or 13. Therefore, we find

$$(q_1, q_2, q_3) = (3,5,7),\ (3,5,11) \text{ or } (3,5,13). \tag{5}$$

(ii) In a similar way, we get

$$\frac{\sigma(n+1)}{n} = \frac{\sigma(q_1^{t_1}q_2^{t_2}\cdots q_s^{t_s})}{n} = \frac{1}{2^{t_0}}\frac{\sigma(q_1^{t_1}q_2^{t_2}\cdots q_s^{t_s})}{q_1^{t_1}q_2^{t_2}\cdots q_s^{t_s}}. \tag{6}$$

Obviously,

$$\frac{\sigma(n+1)}{n} > 1 \text{ and } \mathrm{t}_0 \geq 1.$$

So, we get

$$\frac{\sigma(q_1^{t_1}q_2^{t_2}\cdots q_s^{t_s})}{q_1^{t_1}q_2^{t_2}\cdots q_s^{t_s}} > 2^{t_0} \geq 2.$$

Then, again by inequality (2), we can get equation (4). The rest procedure is as the same as in the proof of Theorem 2.2(i). □

Remark 2.3. If $n = 103 \cdot 263 = 27089$ and $n+1 = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 43$, then $\sigma_{od}(n) = \sigma_{od}(n+1)$.
And if $n = 2 \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 157 = 18693990$ and $n+1 = 37 \cdot 41 \cdot 12323 = 18693991$, then $\sigma_{od}(n) = \sigma_{od}(n+1)$.
Therefore, the bound 4 in Conjecture 1.1 (ii) is right.

A positive integer n is called *abundant*, *perfect*, or *deficient* according as $\sigma(n) > 2n, = 2n, < 2n$.

Proposition 2.4. *Let $n \geq 2$ be an integer. Assume $\sigma_{od}(n) = \sigma_{od}(n+1)$.*

(i) *If n is an odd integer and $n+1 = 2^{t_0}q_1^{t_1}q_2^{t_2}q_3^{t_3}$ is the prime factorization of $n+1$, then n is always deficient,*

(ii) *If n is an even integer and $n = 2^{t_0}q_1^{t_1}q_2^{t_2}q_3^{t_3}$ is the prime factorization of n, then, $n+1$ is always deficient.*

Proof. (i) By Theorem 2.2, we know that the only possibility of $\{q_1, q_2, q_3\}$ is $\{3, 5, 7\}$ $\{3, 5, 11\}$,or $\{3, 5, 13\}$. Firstly, we consider the case of $n+1 = 2^{t_0}3^{t_1}5^{t_2}7^{t_3}$. By equation (3) and inequality (2), we get

$$\begin{aligned}\frac{\sigma(n)}{n+1} &= \frac{1}{2^{t_0}}\frac{\sigma(3^{t_1}5^{t_2}7^{t_s})}{3^{t_1}5^{t_2}7^{t_s}} \\ &< \frac{1}{2^{t_0}}\left(1+\frac{1}{3-1}\right)\left(1+\frac{1}{5-1}\right)\left(1+\frac{1}{7-1}\right).\end{aligned} \tag{7}$$

Since $t_0 \geq 1$, we get

$$\frac{\sigma(n)}{n+1} < \frac{35}{32}. \tag{8}$$

V. Annapurna [1] proved that

$$\sigma(n) < \frac{6}{\pi^2}n\sqrt{n} \tag{9}$$

for every natural number $n \neq 1, 2, 3, 4, 6, 8$. Since $n \geq 2 \cdot 3^{t_1}5^{t_2}7^{t_3} - 1 \geq 209$, from (9), we have

$$\begin{aligned}\frac{\sigma(n)}{n} - \frac{\sigma(n)}{n+1} &= \sigma(n)\frac{1}{n(n+1)} \\ &< \frac{6}{\pi^2}\frac{n\sqrt{n}}{n(n+1)} \\ &= \frac{6\sqrt{n}}{\pi^2(n+1)} \leq \frac{3}{\pi^2}\end{aligned} \tag{10}$$

by $2\sqrt{n} \le n+1$. By (8) and (10), $\frac{\sigma(n)}{n} = \frac{35}{32} + \frac{3}{\pi^2} < 2$.

Finally, we consider the case of $n+1 = 2^{t_0}3^{t_1}5^{t_2}11^{t_3}$ and $n+1 = 2^{t_0}3^{t_1}5^{t_2}13^{t_3}$. Similarly, as in the case of $n+1 = 2^{t_0}3^{t_1}5^{t_2}11^{t_3}$, we have

$$\frac{\sigma(n)}{n+1} < \frac{1}{2}\left(1+\frac{1}{3-1}\right)\left(1+\frac{1}{5-1}\right)\left(1+\frac{1}{11-1}\right) = \frac{33}{32};$$

while in the case of $n+1 = 2^{t_0}3^{t_1}5^{t_2}13^{t_3}$, we have

$$\frac{\sigma(n)}{n+1} < \frac{1}{2}\left(1+\frac{1}{3-1}\right)\left(1+\frac{1}{5-1}\right)\left(1+\frac{1}{13-1}\right) = \frac{65}{64}.$$

Therefore, using the same method, we can derive that $\sigma(n) < 2n$ in both cases. Summing up, Proposition 2.4 (i) is proved.

(ii) By equation (6), we know that

$$\frac{\sigma(n+1)}{n+1} < \frac{\sigma(n+1)}{n} = \frac{1}{2^{t_0}}\frac{\sigma(q_1^{t_1}q_2^{t_2}q_3^{t_3})}{q_1^{t_1}q_2^{t_2}q_3^{t_3}}.$$

Therefore, by Lemma 2.1, we get

$$\frac{\sigma(n+1)}{n+1} < \frac{1}{2}\left(1+\frac{1}{q_1-1}\right)\left(1+\frac{1}{q_2-1}\right)\left(1+\frac{1}{q_3-1}\right).$$

By Theorem 2.2, we know that the only possibility of $\{q_1, q_2, q_3\}$ is $\{3,5,7\}$, $\{3,5,11\}$, $\{3,5,13\}$. Hence,

$$\frac{\sigma(n+1)}{n+1} < \frac{1}{2}\left(1+\frac{1}{3-1}\right)\left(1+\frac{1}{5-1}\right)\left(1+\frac{1}{7-1}\right) = \frac{35}{32}.$$

So, $n+1$ is deficient. □

3. Results on Conjecture 1.1 (*ii*)

In this section, we want to give some partial results on Conjecture 1.1 (*ii*). To prove Conjecture 1.1 (*ii*), we only need to exclude the case that the even one of n and $n+1$ has prime factorization $2^{t_0}q_1^{t_1}q_2^{t_2}q_3^{t_3}$ with

$$(q_1, q_2, q_3) = (3,5,7),\ (3,5,11) \text{ or } (3,5,13).$$

by Theorem 2.2.

Lemma 3.1. *Let $n \ge 2$ be an integer. Assume $\sigma_{od}(n) = \sigma_{od}(n+1)$.*

(i) *If n is an odd integer and $n+1 = 2^{t_0}q_1^{t_1}q_2^{t_2}q_3^{t_3}$ is the prime factorization of $n+1$, then t_0 must be equal to 1,*

(ii) *If n is an even integer and $n = 2^{t_0}q_1^{t_1}q_2^{t_2}q_3^{t_3}$ is the prime factorization of n, then t_0 must be equal to 1.*

Proof. (i) Assume $t_0 \ge 2$. From equation (3), we get

$$\frac{\sigma(n)}{n+1} = \frac{1}{2^{t_0}}\frac{\sigma(q_1^{t_1}q_2^{t_2}q_3^{t_3})}{q_1^{t_1}q_2^{t_2}q_3^{t_3}}.$$

Since $t_0 \ge 2$, we have

$$\frac{\sigma(q_1^{t_1}q_2^{t_2}q_3^{t_3})}{q_1^{t_1}q_2^{t_2}q_3^{t_3}} \ge 4\frac{\sigma(n)}{n+1} \ge 4.$$

By inequality (2), we get

$$\left(1+\frac{1}{q_1-1}\right)\left(1+\frac{1}{q_2-1}\right)\left(1+\frac{1}{q_3-1}\right) > 4. \tag{11}$$

From equation (5), we get

$$\left(1+\frac{1}{q_1-1}\right)\left(1+\frac{1}{q_2-1}\right)\left(1+\frac{1}{q_3-1}\right) \le \frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6} = \frac{35}{16},$$

which contradicts to equation (11). Therefore, $t_0 = 1$.

(ii) Similarly we can obtain the desired result. □

By Lemma 3.1, we only need to exclude the case: $n+1 = 2\cdot 3^{t_1}5^{t_2}q^{t_3}$ with $q = 7, 11$ or 13, and the case: $n = 2\cdot 3^{t_1}5^{t_2}q^{t_3}$ with $q = 7, 11$ or 13, in order to prove the Conjecture 1.1 (*ii*). But this seems to be hard. The reason, we thought, might be that we do not know the prime factorization of the odd one of n and $n+1$. In the following, we assume the odd one of n and $n+1$ is a prime power p^t. Under this condition, we prove Conjecture 1.1.

The following lemma will be used in the proof of Lemma 3.3.

Lemma 3.2. *Let p and q be primes and t, t_1, t_2, t_3 be positive integers.*

(i) *If $p^t + 1 = 2\cdot 3^{t_1}5^{t_2}q^{t_3}$, then t must be odd and $p \equiv 29 \pmod{60}$. Moreover, if $t_1 \ge 2$ and $p \equiv 2, 5 \pmod 9$, then $3|t$,*

(ii) *If $p^t - 1 = 2\cdot 3^{t_1}5^{t_2}q^{t_3}$, then t must be odd and $p \equiv 31 \pmod{60}$. Moreover, if $t_1 \ge 2$ and $p \equiv 4, 7 \pmod 9$, then $3|t$.*

Proof. (i) From $p^t + 1 = 2\cdot 3^{t_1}5^{t_2}q^{t_3}$, we get

$$p^t + 1 \equiv 2 \pmod 4, \tag{12}$$

$$p^t + 1 \equiv 0 \pmod 3, \tag{13}$$

$$p^t + 1 \equiv 0 \pmod 5. \tag{14}$$

Clearly, p is a prime not equaling to either of 2, 3 and 5. Since $p^2 \equiv 1 \pmod 3$, if t is even, then $p^t + 1 \equiv 2 \pmod 3$, which contradicts to (13). Therefore, t must be odd. If $p \equiv -1 \pmod 4$, then $p^t + 1 \equiv 0 \pmod 4$ as t is odd, which contradicts to (12). So $p \equiv 1 \pmod 4$. If $p \equiv 1 \pmod 3$, then $p^t + 1 \equiv 2 \pmod 3$, which contradicts to (13). So $p \equiv -1 \pmod 3$. If $p \equiv 1 \pmod 5$, then $p^t + 1 \equiv 2 \pmod 5$, which contradicts to (14). If $p \equiv 2, 3 \pmod 5$, from (14), we get $t \equiv 2 \pmod 4$. Since t is odd, it is still a contradiction. So $p \equiv -1 \pmod 5$. Summing up, $p \equiv 29 \pmod{60}$ by the Chinese Reminder theorem. Moreover, if $t_1 \ge 2$, then

$$p^t + 1 \equiv 0 \pmod 9. \tag{15}$$

Since $p \equiv -1 \pmod 3$, p must $\equiv 2, 5, 8 \pmod 9$. If $p \equiv 2, 5 \pmod 9$, then $t \equiv 3 \pmod 6$. Therefore, $3|t$.

(ii) In a similar way, we get (ii). □

Lemma 3.3. *Let t, t_1, t_2, t_3 be positive integers. Then:*

(i) *If $p^t + 1 = 2\cdot 3^{t_1}5^{t_2}7^{t_3}$, there does not exist an odd prime p satisfying $\sigma_{od}(p^t) = \sigma_{od}(p^t + 1)$.*

(ii) *If $p^t - 1 = 2\cdot 3^{t_1}5^{t_2}7^{t_3}$, there does not exist an odd prime p satisfying $\sigma_{od}(p^t) = \sigma_{od}(p^t - 1)$.*

(iii) *If $p^t + 1 = 2\cdot 3^{t_1}5^{t_2}11^{t_3}$, there does not exist an odd prime p satisfying $\sigma_{od}(p^t) = \sigma_{od}(p^t + 1)$.*

(iv) *If $p^t - 1 = 2\cdot 3^{t_1}5^{t_2}11^{t_3}$, there does not exist an odd prime p satisfying $\sigma_{od}(p^t) = \sigma_{od}(p^t - 1)$.*

(v) *If* $p^t + 1 = 2 \cdot 3^{t_1} 5^{t_2} 13^{t_3}$, *there does not exist an odd prime p satisfying* $\sigma_{od}(p^t) = \sigma_{od}(p^t + 1)$.

(vi) *If* $p^t - 1 = 2 \cdot 3^{t_1} 5^{t_2} 13^{t_3}$, *there does not exist an odd prime p satisfying* $\sigma_{od}(p^t) = \sigma_{od}(p^t - 1)$.

Proof. (i) Assume $p^t + 1 = 2 \cdot 3^{t_1} 5^{t_2} 7^{t_3}$ and $\sigma_{od}(p^t) = \sigma_{od}(p^t + 1)$. We will seek a contradiction. We have

$$\sigma(p^t) = \sigma(3^{t_1} 5^{t_2} 7^{t_3}). \tag{16}$$

Dividing (16) by $2 \cdot 3^{t_1} 5^{t_2} 7^{t_3}$, we get

$$\frac{\sigma(p^t)}{p^t + 1} = \frac{35}{32}(1 - \frac{1}{3^{t_1+1}})(1 - \frac{1}{5^{t_2+1}})(1 - \frac{1}{7^{t_3+1}}). \tag{17}$$

By the inequality (2), we have

$$1 \le \frac{\sigma(p^t)}{p^t + 1} < \frac{\sigma(p^t)}{p^t} < 1 + \frac{1}{p-1}. \tag{18}$$

Denote

$$A_1(t_1, t_2, t_3) = \frac{35}{32}(1 - \frac{1}{3^{t_1+1}})(1 - \frac{1}{5^{t_2+1}})(1 - \frac{1}{7^{t_3+1}}). \tag{19}$$

Combining (17), (18) and (19), we get

$$1 \le A_1(t_1, t_2, t_3) < 1 + \frac{1}{p-1}. \tag{20}$$

In the following, we will seek a lower bound of $A_1(t_1, t_2, t_3)$ (shortly, $lb(A_1(t_1, t_2, t_3))$), which is strictly greater than 1, case by case. Then by inequality (20), we will get an upper bound of p. Note that $A_1(t_1, t_2, t_3)$ is monotonic increasing with each variable t_i, where $1 \le i \le 3$. Note that

$$A_1(1, t_2, t_3) < \frac{35}{32} \cdot \frac{8}{9} = \frac{35}{36} < 1,$$

which contradicts to (20). So $t_1 \ge 2$. We will divide the discussion into two cases: $t_1 = 2$ and $t_1 \ge 3$.

Case: $t_1 = 2$:

If $t_2 \ge 2$, then $A_1(2, t_2, t_3)$ is equal to or greater than

$$\frac{35}{32}(1 - \frac{1}{3^3})(1 - \frac{1}{5^3})(1 - \frac{1}{7^2}) = \frac{1612}{1575} = 1 + \frac{1}{42.5675...}.$$

By inequality (20), $p < 44$. If $t_3 \ge 4$, then $A_1(2, t_2, t_3)$ is equal to or greater than

$$\frac{35}{32}(1 - \frac{1}{3^3})(1 - \frac{1}{5^2})(1 - \frac{1}{7^5}) = \frac{36413}{36015} = 1 + \frac{1}{90.4899...}.$$

By inequality (20), $p < 92$. Otherwise, we have $t_2 = 1$ and $t_3 \le 3$. Since $2 \cdot 3^2 5^1 7^3 < 2^{40}$, by the table in the Appendix, we have $\sigma_{od}(p^t) \ne \sigma_{od}(p^t + 1)$ with $p^t + 1 = 2 \cdot 3^{t_1} 5^{t_2} 7^{t_3}$. Therefore, these finitely many cases can be ruled out.

Case: $t_1 \ge 3$:

In this case, $A_1(t_1, t_2, t_3)$ is equal to or greater than

$$\frac{35}{32}(1 - \frac{1}{3^4})(1 - \frac{1}{5^2})(1 - \frac{1}{7^2}) = \frac{64}{63} = 1 + \frac{1}{63}.$$

By inequality (20), $p < 64$. Finally, by the above argument, we only need to consider the case $p < 92$. We use congruent method to exclude the case $p < 92$. Since $p^t + 1 = 2 \cdot 3^{t_1} 5^{t_2} 7^{t_3}$, by Lemma 3.2, $p = 29$ or 89, and t is odd. From $p^t + 1 = 2 \cdot 3^{t_1} 5^{t_2} 7^{t_3}$, we also get

$$p^t + 1 \equiv 0 \pmod 7. \tag{21}$$

If $p = 29$, then $p^t + 1 \equiv 2 \pmod 7$, which contradicts to (21). If $p = 89$, from (21), we get $t \equiv 3 \pmod 6$. Hence $3|t$. Since t is odd, we get

$$89^3 + 1 | 89^t + 1 = 2 \cdot 3^{t_1} 5^{t_2} 7^{t_3}.$$

But $89^3 + 1 = 2 \times 3^3 \times 5 \times 7 \times 373$. This is a contradiction.

(ii) Similarly, we get

$$A_1(t_1, t_2, t_3) = 1 + \frac{1}{p-1} + \frac{1}{p^t - 1}. \tag{22}$$

By the Appendix, we find $\sigma_{od}(p^t) \neq \sigma_{od}(p^t - 1)$ with $p^t - 1 = 2 \cdot 3^{t_1} 5^{t_2} 7^{t_3}$ and $p^t - 1 \leq 2^{40}$. So we can assume $p^t - 1 > 2^{40}$. Thus, we have

$$1 + \frac{1}{p-1} + \frac{1}{p^t - 1} < 1 + \frac{1}{p-1} + \frac{1}{2^{40}}. \tag{23}$$

Combining (22) and (23), we get

$$1 < A_1(t_1, t_2, t_3) < 1 + \frac{1}{p-1} + \frac{1}{2^{40}}. \tag{24}$$

In the following, we will seek a lower bound of $A_1(t_1, t_2, t_3)$ (shortly, $lb(A_1(t_1, t_2, t_3))$), which is strictly greater than 1, case by case. Then by inequality (24), we will get an upper bound of p. Compared inequality (24) with inequality (20), they have a difference 2^{-40}. Since 2^{-40} is very close to 0, we will get an upper bound of p (Shortly, ub(p)), which is very close to that in Lemma 3.2(i). To get a better understanding of $ub(p)$ and $lb(A_1(t_1, t_2, t_3))$, we give the following table.

t_1	t_2	t_3	$lb(A_1(t_1, t_2, t_3))$	$ub(p)$
2	≥ 2	≥ 1	1.0234...	43.5675...
2	≥ 1	≥ 4	1.0110...	91.4899...
≥ 3	≥ 1	≥ 1	1.0158...	64.0000...

TABLE 1. $lb(A_1(t_1, t_2, t_3))$ and $ub(p)$

From Table 1, we conclude the prime $p < 92$. Now we use the congruent method. Since $p^t - 1 = 2 \cdot 3^{t_1} 5^{t_2} 7^{t_3}$, by Lemma 3.2(ii), $p \equiv 31 \pmod{60}$ and t is odd. Therefore, $p = 31$. From $p^t - 1 = 2 \cdot 3^{t_1} 5^{t_2} 7^{t_3}$, we get

$$p^t - 1 \equiv 0 \pmod 7. \tag{25}$$

Since $p \equiv 3 \pmod 7$, from (25), we get $6|t$. Since t is odd, it is a contradiction.

(iii) Assume $p^t + 1 = 2 \cdot 3^{t_1} 5^{t_2} 11^{t_3}$ and $\sigma_{od}(p^t) = \sigma_{od}(p^t + 1)$. We will seek a contradiction. Using the same method as in (i), we have

$$\frac{\sigma(p^t)}{p^t + 1} = \frac{33}{32}(1 - \frac{1}{3^{t_1+1}})(1 - \frac{1}{5^{t_2+1}})(1 - \frac{1}{11^{t_3+1}}). \tag{26}$$

Denote

$$A_2(t_1, t_2, t_3) = \frac{33}{32}(1 - \frac{1}{3^{t_1+1}})(1 - \frac{1}{5^{t_2+1}})(1 - \frac{1}{11^{t_3+1}}). \tag{27}$$

Similarly, we get

$$1 \le A_2(t_1, t_2, t_3) < 1 + \frac{1}{p-1}. \tag{28}$$

In the following, we will still seek a lower bound of $A_2(t_1, t_2, t_3)$, hence, an upper bound of p, by inequality (28). Note that

$$A_2(2, t_2, t_3) < \frac{33}{32} \cdot (1 - \frac{1}{3^3}) = \frac{143}{144} < 1,$$

and

$$A_2(t_1, 1, t_3) < \frac{33}{32} \cdot (1 - \frac{1}{5^2}) = \frac{99}{100} < 1,$$

which contradict to (28). Therefore, $t_1 \ge 3$ and $t_2 \ge 2$. We divide the discussion into two cases $t_1 = 3$ and $t_1 \ge 4$.

Case 1: $t_1 = 3$.

If $t_2 \ge 6$, then $A_2(3, t_2, t_3)$ is greater than or equal to

$$\frac{33}{32}(1 - \frac{1}{3^4})(1 - \frac{1}{5^7})(1 - \frac{1}{11^2}) = 1 + \frac{1}{99.1268...}.$$

From (28), we get $p < 101$. If $t_3 \ge 3$, then $A_2(3, t_2, t_3)$ is greater than or equal to

$$\frac{33}{32}(1 - \frac{1}{3^4})(1 - \frac{1}{5^3})(1 - \frac{1}{11^4}) = 1 + \frac{1}{97.0745...}.$$

From (28), we get $p < 99$. The rest cases are $t_2 \le 5$ and $t_3 \le 2$. Since $2 \times 3^3 5^5 11^2 < 2^{40}$, by the table in the Appendix, we have $\sigma_{od}(p^t) \ne \sigma_{od}(p^t + 1)$ with $p^t + 1 = 2 \cdot 3^{t_1} 5^{t_2} 11^{t_3}$. Therefore, these finitely many cases can be ruled out.

Case 2: $t_2 \ge 4$.

In this case, $A_2(t_1, t_2, t_3)$ is greater than or equal to

$$\frac{33}{32}(1 - \frac{1}{3^5})(1 - \frac{1}{5^3})(1 - \frac{1}{11^2}) = 1 + \frac{1}{96.4285...}.$$

From (28), we get $p < 98$. We can conclude the suitable prime number p must < 101. Now we use the congruent method. Since $p^t + 1 = 2 \cdot 3^{t_1} 5^{t_2} 11^{t_3}$, by Lemma 3.2, $p \equiv 29$ or 89, and t is odd. Also, we get

$$p^t + 1 \equiv 0 \pmod{11}. \tag{29}$$

If $p = 29$, then $29 \equiv 2 \pmod 9$. As $t_1 \ge 2$, by Lemma 3.2, we get $3|t$. Since t is odd, we get

$$29^3 + 1 | 29^t + 1 = 2 \cdot 3^{t_1} 5^{t_2} 7^{t_3}.$$

But $29^3 + 1 = 2 \times 3^2 \times 5 \times 271$. This is a contradiction.

If $p = 89$, then $p^t + 1 \equiv 2 \pmod{11}$, which contradicts to (29).

(iv) Assume $p^t - 1 = 2 \cdot 3^{t_1} 5^{t_2} 11^{t_3}$ and $\sigma_{od}(p^t) = \sigma_{od}(p^t - 1)$. We will seek a contradiction. Similarly as in (iii), we denote

$$A_2(t_1, t_2, t_3) = \frac{33}{32}(1 - \frac{1}{3^{t_1+1}})(1 - \frac{1}{5^{t_2+1}})(1 - \frac{1}{11^{t_3+1}}). \tag{30}$$

Using the same method as in (ii), we get

$$A_2(t_1, t_2, t_3) = 1 + \frac{1}{p-1} + \frac{1}{p^t - 1} \tag{31}$$

and

$$1 < A_2(t_1, t_2, t_3) < 1 + \frac{1}{p-1} + \frac{1}{2^{40}}. \tag{32}$$

From $1 < A_2(t_1, t_2, t_3)$, similarly as in (iii), we get

$$t_1 \geq 3 \text{ and } t_2 \geq 2.$$

Similarly as in (iii), we will get a lower bound of $A_2(t_1, t_2, t_3)$, hence, an upper bound of p by inequality (32), case by case. For a better understanding, we list them in the following table.

t_1	t_2	t_3	$lb(A_2(t_1, t_2, t_3))$	$ub(p)$
3	≥ 6	≥ 1	1.0100...	100.1268...
3	≥ 2	≥ 3	1.0103...	98.0745...
≥ 4	≥ 2	≥ 1	1.0103...	97.4285...

TABLE 2. $lb(A_2(t_1, t_2, t_3))$ and $ub(p)$

From Table 2, we conclude $p < 101$. Now we use the congruent method. Since $p^t - 1 = 2 \cdot 3^{t_1} 5^{t_2} 11^{t_3}$, by (ii), $p \equiv 31 \pmod{60}$ and t is odd. Therefore, $p = 31$. Since $p \equiv 4 \pmod 9$ and $t_1 \geq 2$, we get $3|t$, by (ii). Therefore,

$$31^3 - 1 | 31^t - 1 = 2 \cdot 3^{t_1} 5^{t_2} 11^{t_3}.$$

But $11^3 - 1 = 2 \times 5 \times 7 \times 19$. A contradiction.

(v) Assume $p^t + 1 = 2 \cdot 3^{t_1} 5^{t_2} 13^{t_3}$ and $\sigma_{od}(p^t) = \sigma_{od}(p^t + 1)$. We will seek a contradiction. Using the same method as in (i), we have

$$\frac{\sigma(p^t)}{p^t + 1} = \frac{65}{64}(1 - \frac{1}{3^{t_1+1}})(1 - \frac{1}{5^{t_2+1}})(1 - \frac{1}{13^{t_3+1}}). \tag{33}$$

Denote

$$A_3(t_1, t_2, t_3) = \frac{65}{64}(1 - \frac{1}{3^{t_1+1}})(1 - \frac{1}{5^{t_2+1}})(1 - \frac{1}{13^{t_3+1}}) \tag{34}$$

Similarly, we get

$$1 \leq A_3(t_1, t_2, t_3) < 1 + \frac{1}{p-1}. \tag{35}$$

In the following, we will still seek a lower bound of $A_3(t_1, t_2, t_3)$, hence, an upper bound of p by inequality (35). Note that

$$A_3(2, t_2, t_3) < \frac{65}{64} \cdot (1 - \frac{1}{3^3}) = \frac{845}{864} < 1,$$

and

$$A_2(t_1, 1, t_3) < \frac{65}{64} \cdot (1 - \frac{1}{5^2}) = \frac{39}{40} < 1,$$

which contradict to (35). Therefore, $t_1 \geq 3$ and $t_2 \geq 2$. We divide the discussion into six cases $t_1 = 3, 4, 5, 6, 7$ and $t_1 \geq 8$.

Case 1: $t_1 = 3$.

Note that

$$A_3(3, 2, t_3) < \frac{65}{64} \cdot (1 - \frac{1}{3^4})(1 - \frac{1}{5^3}) = \frac{403}{405} < 1,$$

and

$$A_3(3,t_2,1) < \frac{65}{64}\cdot(1-\frac{1}{3^4})(1-\frac{1}{13^2}) = \frac{350}{351} < 1,$$

which contradict to (35). Therefore, $t_2 \geq 3$ and $t_3 \geq 2$ in this case. If $t_2 \geq 5$, then $A_2(3,t_2,t_3)$ is greater than or equal to

$$\frac{65}{64}(1-\frac{1}{3^4})(1-\frac{1}{5^6})(1-\frac{1}{13^3}) = 1+\frac{1}{389.7601...}$$

From (35), we get $p < 391$. If $t_3 \geq 5$, then $A_2(3,t_2,t_3)$ is greater than or equal to

$$\frac{65}{64}(1-\frac{1}{3^4})(1-\frac{1}{5^4})(1-\frac{1}{13^6}) = 1+\frac{1}{675.0945...}.$$

From (35), we get $p < 677$. The rest cases are $t_2 \leq 4$ and $t_3 \leq 4$. Since $2\times 3^3 5^4 13^4 < 2^{40}$, these finitely many cases can be ruled out by the table in the Appendix.

Case 2: $t_1 = 4$.

If $t_2 \geq 3$, then $A_3(4,t_2,t_3)$ is greater than or equal to

$$\frac{65}{64}(1-\frac{1}{3^5})(1-\frac{1}{5^4})(1-\frac{1}{13^2}) = 1+\frac{1}{259.6153...}.$$

From (35), we get $p < 261$. If $t_3 \geq 2$, then $A_3(4,t_2,t_3)$ is greater than or equal to

$$\frac{65}{64}(1-\frac{1}{3^5})(1-\frac{1}{5^3})(1-\frac{1}{13^3}) = 1+\frac{1}{345.1588...}.$$

From (35), we get $p < 347$. The rest cases are $t_2 = 2$ and $t_3 = 1$. Since $2\times 3^4 5^2 13 < 2^{40}$, this case can be ruled out by the table in the Appendix.

Case 3~5: $t_1 = 5,6,7$. These cases can be discussed similarly as the case: $t_1 = 4$. Precisely, in each case, if $t_2 \geq 3$ or $t_3 \geq 2$, we will get a larger lower bound of $A_3(t_1,t_2,t_3)$, hence, a smaller upper bound of p, in this case than in the case: $t_1 = 4$. Otherwise, $t_2 = 2$ and $t_3 = 1$. Since $2\times 3^7 5^2 13 < 2^{40}$, it can be excluded by the table in the Appendix.

Case 6: $t_1 \geq 8$. In this case, $A_3(t_1,t_2,t_3)$ is equal to or greater than

$$\frac{65}{64}(1-\frac{1}{3^9})(1-\frac{1}{5^3})(1-\frac{1}{13^2}) = 1+\frac{1}{672.2336...}.$$

By inequality (35), $p < 674$. Summing up, the suitable prime p must < 677. Now, we use the congruent method. Since $p^t + 1 = 2\cdot 3^{t_1}5^{t_2}13^{t_3}$, by (i), t is odd and

$$p \in \{29,\ 89, 149,\ 269,\ 389,\ 449,\ 509,\ 569\}.$$

From $p^t + 1 = 2\cdot 3^{t_1}5^{t_2}13^{t_3}$, we get

$$p^t + 1 \equiv 0 \pmod{13}. \tag{36}$$

If $p \equiv 2,6,7,11 \pmod{13}$, then $t \equiv 6 \pmod{12}$. Since t is odd, this is impossible. If $p \equiv 5,8 \pmod{13}$, then $t \equiv 2 \pmod 4$. Since t is odd, this is impossible. If $p \equiv 1 \pmod{13}$, then $p^t \equiv 1 \pmod{13}$, which contradicts to (36). If $p \equiv 3,9 \pmod{13}$, then $p^t \equiv 1,3,9 \pmod{13}$, which contradicts to (36). Therefore, $p \equiv 4,10,12 \pmod{13}$. This implies $p = 389$ or $p = 569$. If $p = 569$, since t is odd,

$$p+1 | p^t + 1 = 2\cdot 3^{t_1}5^{t_2}13^{t_3}.$$

But $p + 1 = 570 = 2\times 3\times 5\times 19$. A contradiction! If $p = 389$, then $p \equiv 2 \pmod 9$. Since $t_1 \geq 2$, by (i), we get $3|t$. Since t is odd,

$$389^3 + 1 | 389^t + 1 = 2\cdot 3^{t_1}5^{t_2}13^{t_3}.$$

But $389^3 + 1 = 2 \times 3 \times 5 \times 13 \times 50311$. A contradiction!

(vi) Assume $p^t - 1 = 2 \cdot 3^{t_1} 5^{t_2} 13^{t_3}$ and $\sigma_{od}(p^t) = \sigma_{od}(p^t - 1)$. We will seek a contradiction. Similarly as in (v), we denote

$$A_4(t_1, t_2, t_3) = \frac{33}{32}(1 - \frac{1}{3^{t_1+1}})(1 - \frac{1}{5^{t_2+1}})(1 - \frac{1}{13^{t_3+1}}). \quad (37)$$

Using the same method as in (ii), we get

$$A_4(t_1, t_2, t_3) = 1 + \frac{1}{p-1} + \frac{1}{p^t - 1} \quad (38)$$

and

$$1 < A_4(t_1, t_2, t_3) < 1 + \frac{1}{p-1} + \frac{1}{2^{40}}. \quad (39)$$

From $1 < A_3(t_1, t_2, t_3)$, like in (v), we get

$$t_1 \geq 3, \ \ t_2 \geq 2 \text{ and if } t_1 = 3, \text{ then } t_2 \geq 3 \text{ and } t_3 \geq 2.$$

Similarly as in (v), we will get a lower bound of $A_4(t_1, t_2, t_3)$, hence, an upper bound of p by inequality (39), case by case. For a better understanding, we list them in the following table.

t_1	t_2	t_3	$lb(A_4(t_1, t_2, t_3))$	$ub(p)$
3	≥ 5	≥ 2	1.0025...	390.7601...
3	≥ 3	≥ 5	1.0014...	676.0945...
4	≥ 3	≥ 1	1.0038...	260.6153...
4	≥ 2	≥ 2	1.0028...	346.1588...
5	≥ 3	≥ 1	1.0066...	152.1194...
5	≥ 2	≥ 2	1.0056...	177.6778...
...	...	...	...	...
≥ 8	≥ 2	≥ 1	1.0014...	673.2336...

TABLE 3. $lb(A_4(t_1, t_2, t_3))$ and $ub(p)$

Hence, we consider the prime numbers $p < 677$. Since $p^t - 1 = 2 \cdot 3^{t_1} 5^{t_2} 13^{t_3}$, by (ii), t is odd and

$$p \in \{31, 151, 211, 271, 331, 571, 631\}. \quad (40)$$

From $p^t - 1 = 2 \cdot 3^{t_1} 5^{t_2} 13^{t_3}$, we get

$$p^t - 1 \equiv 0 \pmod{13}. \quad (41)$$

Since t is odd, from (41), the order of p modulo 13 must be 1 or 3. Therefore, $p \equiv 1, 3, 9 \pmod{13}$, which contradicts to (40). □

By Theorem 2.2, Lemma 3.1 and Lemma 3.3, we get the following corollary.

Corollary 3.4. *Let $p^t + 1 = 2^{t_0} q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s}$ (resp., $p^t - 1 = 2^{t_0} q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s}$) with p and q_i $(1 \leq i \leq s)$ being odd distinct prime numbers. If $\sigma_{od}(p^t) = \sigma_{od}(p^t + 1)$ (resp., $\sigma_{od}(p^t - 1) = \sigma_{od}(p^t)$) then $s \geq 4$.*

Remark 3.5. Suppose that $n + 1 = 2^{t_0} q_1^{t_1} \cdots q_s^{t_s}$ and $t_0 \geq 2$. Assume $\sigma_{od}(n) = \sigma_{od}(n + 1)$, then

$$\sigma(n) = \sigma(q_1^{t_1} \cdots q_s^{t_s}) = (\sum_{k_1=0}^{t_1} q_1^{k_1}) \cdots (\sum_{k_s=0}^{t_s} q_1^{k_s}).$$

It is obvious that $\sigma(n) - (n+1) \geq 0$, then we have

$$n + 1 + (\sigma(n) - (n+1)) = \sigma(q_1^{t_1} \cdots q_s^{t_s}) = (\sum_{k_1=0}^{t_1} q_1^{k_1}) \cdots (\sum_{k_s=0}^{t_s} q_1^{k_s})$$

and

$$n + 1 \leq (\sum_{k_1=0}^{t_1} q_1^{k_1}) \cdots (\sum_{k_s=0}^{t_s} q_1^{k_s}). \tag{42}$$

Divide (42) by $q_1^{t_1} \cdots q_s^{t_s}$, then

$$2^{t_0} \leq \frac{(\sum_{k_1=0}^{t_1} q_1^{k_1}) \cdots (\sum_{k_s=0}^{t_s} q_1^{k_s})}{q_1^{t_1} \cdots q_s^{t_s}} = \left(\frac{1 - q_1^{-(t_1+1)}}{1 - q_1^{-1}}\right) \cdots \left(\frac{1 - q_s^{-(t_s+1)}}{1 - q_s^{-1}}\right)$$

and

$$\left(\frac{1}{1 - q_1^{-1}}\right) \cdots \left(\frac{1}{1 - q_s^{-1}}\right) > 4.$$

Let $q[1] = 2, q[2] = 3, \cdots, q[i]$ be the i-th prime number. Using Mathematica 9.0, we get

$$\left(\frac{1}{1 - q[2]^{-1}}\right) \cdots \left(\frac{1}{1 - q[21]^{-1}}\right) = \frac{2033432863950094091347}{512616335105064960000} < 4$$

and

$$\left(\frac{1}{1 - q[2]^{-1}}\right) \cdots \left(\frac{1}{1 - q[22]^{-1}}\right) = \frac{160641196252057433216413}{39984074138195066880000} > 4.$$

So, if $\sigma_{od}(n) = \sigma_{od}(n+1)$ and $4|n+1$, then $n+1$ has at least 21 distinct odd prime divisors. Similarly, if $\sigma_{od}(n) = \sigma_{od}(n+1)$ and $4|n$, then n has at least 21 distinct odd prime factors. Therefore, if $n < \prod_{i=2}^{22} p[i] - 1 = 6435289534681345815798169108259$ with $n \equiv 0$ or $-1 \pmod 4$, then $\sigma_{od}(n) \neq \sigma_{od}(n+1)$. Assume $4|n+1$ and $3 \nmid n+1$. Similarly, using Mathematica 9.0, we get

$$\left(\frac{1}{1 - q[3]^{-1}}\right) \cdots \left(\frac{1}{1 - q[140]^{-1}}\right) < 4$$

and

$$\left(\frac{1}{1 - q[3]^{-1}}\right) \cdots \left(\frac{1}{1 - q[141]^{-1}}\right) > 4.$$

So, if $\sigma_{od}(n) = \sigma_{od}(n+1)$ with $4|n+1$ and $3 \nmid n+1$, then $n+1$ has at least 139 odd prime factors. Similarly, if $\sigma_{od}(n) = \sigma_{od}(n+1)$ with $4|n$ and $3 \nmid n$, then n has at least 139 odd prime factors.

4. Results on Conjecture 1.1 (iv)

The following lemma, though simple, is the key point to our proof.

Lemma 4.1. *Let N denote the largest odd integer dividing n. Then $\sigma_{od}(n)$ is odd if and only if N is a perfect square.*

Proof. Glaisher [6, p. 294] considered $\sigma_{od}(n) = \sigma(N)$. In [22, p. 28] $\sigma(n)$ is odd if and only if N is a perfect square. This completes the proof of Lemma 4.1. □

Corollary 4.2. *If $\sigma_{od}(n) = \sigma_{od}(n+1)$ is even, then neither n nor $n+1$ is a square.*

Proof. As in Lemma 4.1, let N denote the largest odd integer dividing n and N' denote the largest odd integer dividing $n+1$. By Lemma 4.1, neither N nor N' is a square. Therefore, neither n nor $n+1$ is a square. □

Remark 4.3. Assume $\sigma_{od}(n) = \sigma_{od}(n+1)$ such that n or $n+1$ is a square. It is easily checked that

$n \pmod 4$	0	1	2	3
$n+1 \pmod 4$	1	2	3	0

TABLE 4. n and $n+1 \pmod 4$

and the possible case of square integers for n and $n+1$ is $n \equiv 0 \pmod 4$ and $n+1 \equiv 1 \pmod 4$. By Corollary 4.2, to prove Conjecture 1.1 (v), we only need to consider the case:

$$\sigma_{od}(n) = \sigma_{od}(n+1) \equiv 1 \pmod 2,$$

that is, $n \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 4$ (see Table 4).

Firstly, we consider the case $n \equiv 1 \pmod 4$, by Lemma 4.1, there exist odd positive integers M and L satisfying $n = M^2$ and $n+1 = 2^l L^2$. Since $n = M^2 \equiv 1 \pmod 4$, l must be 1. Therefore, such pair n and $n+1$, can be parameterized by positive solutions of the negative Pell equation, i.e.,

$$M^2 - 2L^2 = -1, \; n = M^2, \; n+1 = 2L^2. \tag{43}$$

$x = 1$, $y = 1$ is an obvious solution of the equation $x^2 - 2y^2 = -1$, and is fundamental as any smaller solution would have x and $y < 1$. The other positive solutions can be obtained by iteration:

$$x_{m+1} = 3x_m + 4y_m \text{ and } y_{m+1} = 2x_m + 3y_m,$$

that is:

(1, 1), (7, 5), (41, 29), (239, 169), (1393, 985), (8119, 5741), (47321, 33461), (1607521, 1136689), ...,

(46305156912921105124676500756345112056691727724000577129664401793869058047789742202

70447822703484163801, 327426904570336523407706809694401711184861124790238682838820336

04409842361054556976605396860319012519349), ⋯ .

Assume $x_m^2 - 2y_m^2 = -1$ with $m \geq 2$. Using Mathematica 9.0, we checked $\sigma_{od}(x_m^2) \neq \sigma_{od}(2y_m^2)$ satisfying $x_m^2 + 1 = 2y_m^2$ with $m = 2, ..., 135$.

Thus, if $n \leq 463051569129211051246765007563451120566917277240005771296644017938690580477897422027044782270$ that is, ($n < 10^{205}$) and $\sigma_{od}(n) = \sigma_{od}(n+1)$, then neither n nor $n+1$ is a square except $n = 1$.

Secondly, we consider the case $n \equiv 0 \pmod 4$. There exist odd positive integers K, U and l satisfying

$$n+1 = K^2 \text{ and } n = K^2 - 1 = 2^l U^2 \tag{44}$$

by Lemma 4.1. If $l = 2l'$ then we cannot find positive integers K and U satisfying $K^2 - (2^{l'}U)^2 = 1$. By (44), put $l = 2l' + 1$, we consider $\sigma_{od}(n) = \sigma_{od}(n+1)$ satisfying $n+1 = K^2$ and $n = K^2 - 1 = 2(2^{l'}U)^2$. Put $2^{l'}U = U$. Then we get the classical Pell equation $K^2 - 2U^2 = 1$.The solutions of $x^2 - 2w^2 = 1$ are

$$(x_1, w_1) = (3, 2), (x_2, w_2) = (17, 12), (x_3, w_3) = (99, 70), (x_3, w_3) = (99, 70),$$

$(x_4, w_4) = (577, 408), (x_5, w_5) = (3363, 2378), .., (x_{130}, w_{130}) = (16620657195672643875956$

$20839613920483911723740125085355030801429665220366155075897997802501222942737, 117525794$

$10837112794654569776915329804975338083278242707657531918087582917385550732789385474618908 28).$

Using Mathematica 9.0, we checked $\sigma_{od}(2w_m^2) \neq \sigma_{od}(x_m^2)$ satisfying $x_m^2 - 2w_m^2 = 1$ with $m = 1, ..., 130$.

Remark 4.4. Let $n \leq 153168087149$ be an odd non-square-free positive integer. Then, using Appendix, there does not exist n satisfying $\sigma_{od}(n) = \sigma_{od}(n+1)$. First case is $\sigma_{od}(153168087150) = \sigma_{od}(153168087151)$ with $153168087151 = 672^2 \times 1481 \times 23039$.

Theorem 4.5. *There does not exist n satisfying* $\sigma_{od}(n) = \sigma_{od}(n+1) = \sigma_{od}(n+2) \equiv 1 \pmod 2$.

Proof. We assume that there exist n satisfying $\sigma_{od}(n) = \sigma_{od}(n+1) = \sigma_{od}(n+2) \equiv 1 \pmod 2$. By Table 4 and Lemma 4.1 the possible case of n is $n \equiv 0 \pmod 4$. We have $x_m^2 - 2y_m^2 = -1$ and $x'^2_l - 2w_l^2 = 1$ by Remark 4.3. By assumption there exist m and l satisfying $x_m = x'_l$. But we cannot find positive integers y_m and w_l satisfying $2y_m^2 - 2w_l^2 = 2$. This is the proof of Theorem 4.5. □

Remark 4.6. Sierpiński has asked if $\sigma(n) = \sigma(n+1)$ infinitely often. Jud McCranie found 832 solutions of

$$\sigma(n) = \sigma(n+1) \text{ for } n < 4.25 \times 10^9 ;$$

(see [11, p. 103]).
Erdös [7] made the much stronger conjecture that for every integer $k \geq 1$ there is an n such that $\sigma(n) = \sigma(n+1) = \cdots = \sigma(n+k)$ has infinitely many solutions for each k. Haukkanen [14] observed that for no $n \leq 2 \cdot 10^8$ such that $\sigma(n) = \sigma(n+1) = \sigma(n+2)$. Let $\sigma^*(n) = \sum_{d|n,\ \frac{n}{d} \text{ odd}} d$. By [17, Table 11] and the Appendix, we shall compare the above problem for $\sigma(n)$, $\sigma^*(n)$ and $\sigma_{od}(n)$ as follows.

n	$\sigma^*(n) = \sigma^*(n+1)$	$\sigma(n) = \sigma(n+1)$	$\sigma_{od}(n) = \sigma_{od}(n+1)$
$n < 200$	$3, 6, 7, 10, 22, 31, 58, 82, 106, 140, 154, 160, 166, 180$	14	1
$n < 4.25 \times 10^9$	$\sharp\{n \mid \sigma^*(n) = \sigma^*(n+1)\} = 1870$	$\sharp\{n \mid \sigma(n) = \sigma(n+1)\} = 832$	$\sharp\{n \mid \sigma_{od}(n) = \sigma_{od}(n+1)\} = 64$
n	$\sigma^*(n) = \sigma^*(n+1) = \sigma^*(n+2)$	$\sigma(n) = \sigma(n+1) = \sigma(n+2)$	$\sigma_{od}(n) = \sigma_{od}(n+1) = \sigma_{od}(n+2)$
$n < 4.25 \times 10^9$	6	no	no

TABLE 5. $\sigma^*(n)$, $\sigma(n)$ and $\sigma_{od}(n)$.

The equation $\sigma_{od}(n) = \sigma_{od}(n+1) = \sigma_{od}(n+2)$ has no solution for $n \leq 2^{40}$ (see the Appendix, https://drive.google.com/open?id=1zuZ6DbgKUg7ueMMtbC6SVRhP9W8Exxgc).

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