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# Equivalence Classes of Uninorms

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**Abstract.** In this paper, some properties of an order induced by uninorms are studied. The set of incomparable elements with respect to the $U$-partial order for any uninorm on bounded lattices is investigated. Also, an equivalence relation on the class of uninorms induced by a $U$-partial order is investigated and discussed. Finally, the relationships between an order induced by uninorms and distributivity property for uninorms are investigated.

## 1. Introduction

Uninorms were introduced by Yager and Rybalov [29]. Uninorms are special aggregation operators which have proven to be useful in many applications like fuzzy logic, expert systems, neural networks, fuzzy system modeling [14, 16, 28].

In [19], uninorms on bounded lattices were studied. Also, the smallest and the greatest uninorm with neutral element $e \in L \setminus \{0, 1\}$ on $L$ were obtained.

In [24], a natural order for semigroups was defined. Similarly, in [18], a partial order defined by means of t-norms on a bounded lattice was introduced.

In [15], an order induced by uninorms on bounded lattice was defined and some properties of such an order were investigated. The uninorms, t-norms (t-conorms) and the order induced by uninorm (nullnorm) were also studied by many other authors in other papers [1, 3, 8–13, 17, 26, 27, 29].

The present paper is organized as follows. We shortly recall some basic notions in Section 2. In Section 3, we survey that the set ${\mathcal{I}_U}^{L(x)}$, denoting the set of all incomparable elements with arbitrary but fixed $x \in L \setminus \{0, 1\}$ according to the $\preceq_U$. Also, we investigate the set of incomparable elements with respect to the $U$-partial order for any uninorm on $(L, \leq, 0, 1)$ and we determine the sets of incomparable elements w.r.t. $U$-partial order of the greatest and weakest uninorm on $(L, \leq, 0, 1)$. In Section 4, we investigate an equivalence relation on the class of uninorms on the unit interval $[0, 1]$ and we determine the equivalence classes of some special uninorms on $[0, 1]$. In Section 5, the relationship between an order induced by uninorms and distributiity property for uninorms on the unit interval $[0, 1]$ are investigated. In Section 6, concluding remarks are given.

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## 2. Preliminaries

Let us now recall all necessary basic notions. A bounded lattice $(L, \leq)$ is a lattice which has the top and bottom elements, which are written as 1 and 0, respectively, that is there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$, for all $x \in L$.

**Definition 2.1.** ([6, 19]) Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $U : L^2 \to L$ is called a uninorm on $L$, if it is commutative, associative, monotone and has a neutral element $e \in L$.

We denote by $U(e)$ the set of all uninorms on $L$ with the neutral element $e \in L$. In this paper, to make it short, the set $(0, e] \times [e, 1) \cup [e, 1) \times (0, e]$ for $e \in L \setminus \{0, 1\}$ will be denoted by $A(e)$, that is $A(e) = (0, e] \times [e, 1) \cup [e, 1) \times (0, e]$ for $e \in L \setminus \{0, 1\}$. Clearly, $U$ is a t-norm (t-conorm) if $e = 1$ ($e = 0$).

**Example 2.2.** ([22]) The four basic t-norms $T_M$, $T_P$, $T_L$ and $T_D$ on $[0, 1]$ are given by, respectively,

$T_M(x, y) = \min(x, y)$,

$T_P(x, y) = xy$,

$T_L(x, y) = \max(x + y - 1, 0)$,

$$T_D(x, y) = \begin{cases} 0, & \text{if } (x, y) \in [0, 1)^2 \\ \min(x, y), & \text{otherwise.} \end{cases}$$

**Example 2.3.** ([22]) The four basic t-conorms $S_M$, $S_P$, $S_L$ and $S_D$ on $[0, 1]$ are given by, respectively,

$S_M(x, y) = \max(x, y)$,

$S_P(x, y) = x + y - xy$,

$S_L(x, y) = \min(x + y, 1)$,

$$S_D(x, y) = \begin{cases} 1, & \text{if } (x, y) \in (0, 1)^2 \\ \max(x, y), & \text{otherwise.} \end{cases}$$

The t-norms $T_\wedge$ and $T_W$ on $L$ are defined as follows, respectively:

$T_\wedge(x, y) = x \wedge y$

$$T_W(x, y) = \begin{cases} x, & if\ y = 1 \\ y, & if\ x = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the t-conorms $S_\vee$ and $S_W$ can be defined as above.

In particular, we have obtained $T_W = T_D$ and $T_\wedge = T_M$ for $L = [0, 1]$.

**Example 2.4.** ([22]) The t-norm $T^{nM}$ on $[0, 1]$ is defined as follows:

$$T^{nM}(x, y) = \begin{cases} 0, & x + y \leq 1 \\ \min(x, y), & \text{otherwise.} \end{cases}$$

$T^{nM}$ is called nilpotent minimum t-norm. This t-norm has been introduced by J. Fodor.

**Definition 2.5.** ([7]) A t-norm $T$ on $L$ is *divisible* if the following condition holds:

$$\forall\, x, y \in L \quad \text{with} \quad x \leq y \quad \text{there is a} \quad z \in L \quad \text{such that} \quad x = T(y, z).$$

The infimum t-norm $T_\wedge$ is divisible: $x \leq y$ is equivalent to $x \wedge y = x$. A basic example of a non-divisible t-norm on an arbitrary bounded lattice $L$ (i.e., card $L > 3$) is the t-norm $T_W$. Similarly, t-conorm $S_\vee$ is divisible. $S_W$ is a non-divisible t-conorm on an arbitrary bounded lattice $L$ (i.e., card $L > 3$).

Note: ([5, 10]) Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, if $a$ and $b$ are incomparable, then we use the notation $a \parallel b$ in this case.

**Definition 2.6.** ([5]) Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, $a \leq b$, a subinterval $[a, b]$ of $L$ is defined as

$[a, b] = \{x \in L \mid a \leq x \leq b\}$

Similarly, $[a, b) = \{x \in L \mid a \leq x < b\}$, $(a, b] = \{x \in L \mid a < x \leq b\}$ and $(a, b) = \{x \in L \mid a < x < b\}$.

**Definition 2.7.** ([18]) Let $L$ be a bounded lattice and $T$ be a t-norm on $L$. The order defined as follows is called a $T-$ *partial order* (triangular order) for t-norm $T$:

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L.$$

**Definition 2.8.** ([15]) Let $L$ be a bounded lattice and $S$ be a t-conorm on $L$. The order defined as follows is called a $S-$ *partial order* for t-conorm $S$:

$$x \preceq_S y :\Leftrightarrow S(k, x) = y \text{ for some } k \in L.$$

**Definition 2.9.** ([15]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $U$ be a uninorm with neutral element $e$ on $L$. Define the following relation, for $x, y \in L$, as

$$x \preceq_U y :\Leftrightarrow \begin{cases} \text{if } x, y \in [0, e] \text{ and there exist } k \in [0, e] \text{ such that } U(k, y) = x \text{ or,} \\ \text{if } x, y \in [e, 1] \text{ and there exist } \ell \in [e, 1] \text{ such that } U(x, \ell) = y \text{ or,} \\ \text{if } x, y \in L^* \text{ and } x \leq y, \end{cases} \quad (1)$$

where $I_e = \{x \in L \mid x \parallel e\}$ and $L^* = [0, e] \times [e, 1] \cup [0, e] \times I_e \cup [e, 1] \times I_e \cup [e, 1] \times [0, e] \cup I_e \times [0, e] \cup I_e \times [e, 1] \cup I_e \times I_e$.

**Proposition 2.10.** ([15]) *The relation $\preceq_U$ defined in (1) is a partial order on L.*

Note: The partial order $\preceq_U$ in (1) is called the $U$-partial order on $L$.

**Definition 2.11.** ([2]) Let $U$ be a nullnorm on $[0, 1]$ and let $K_U$ be defined by

$K_U = \{x \in (0, 1) \mid \text{for some } y \in (0, 1),\ [x < y \text{ and } x \npreceq_U y] \text{ or } [y < x \text{ and } y \npreceq_U x]\}$.

**Definition 2.12.** ([2]) Define a relation $\beta$ on the class of all uninorms on $[0, 1]$ by $U_1 \beta U_2$,

$U_1 \beta U_2 :\Leftrightarrow K_{U_1} = K_{U_2}$.

## 3. Regarding the Set $K_U^L$ on any Bounded Lattices

In this section, we investigate the set of all incomparable elements with arbitrary but fixed $x \in L \setminus \{0, 1\}$ according to the $U$-partial order. Also, we determine above introduced the sets of the smallest and the greatest uninorms on $(L, \leq, 0, 1)$. Thus, we conclude for the some basic t-norms and t-conorms in Corollary 3.13, Corollary 3.14, Corollary 3.16, Corollary 3.17.

**Definition 3.1.** ([20]) Let $U$ be a uninorm on $(L, \leq, 0, 1)$ with neutral element $e$ and let $I_U^{L(x)}$ be defined by

$I_U^{L(x)} = \{y_x \in L \backslash \{0, 1\} \mid [x < y_x \text{ and } x \npreceq_U y_x] \text{ or } [y_x < x \text{ and } y_x \npreceq_U x] \text{ or } x \parallel y_x\}$.

In the following, the notation $I_U^{L(x)}$ is used to denote the set of all incomparable elements with $x \in L \setminus \{0, 1\}$ according to $\preceq_U$. Clearly, $\mathcal{I}_U^{L(x)} = \emptyset$ for $x = 0$ and 1. By the definition of $\mathcal{I}_U^{L(x)}$, for any $x \in L \setminus \{0, 1\}$, the set $\mathcal{I}_U^{L(x)}$ does not contain 0 and 1.

**Definition 3.2.** Let $(L, \leq, 0, 1)$ be a bounded lattice. The set $I_x^L$ for $x \in L \setminus \{0, 1\}$ is defined by

$I_x^L = \{y \in L \backslash \{0, 1\} \mid x \parallel y\}$.

Note: For any uninorm on $(L, \leq, 0, 1)$, we have that $I_x^L \subseteq I_U^{L(x)}$ for $x \in L$.

**Lemma 3.3.** *Let $(L, \leq, 0, 1)$ be a bounded lattice. For all uninorms $U$ and all $x \in L$ it holds that $0 \preceq_U x$, $x \preceq_U x$ and $x \preceq_U 1$.*

**Proposition 3.4.** *Let $(L, \leq, 0, 1)$ be a bounded lattice. Consider the function on $L$ defined as follows:*

$$U_{T_\wedge}(x,y) = \begin{cases} x \wedge y, & (x,y) \in [0,e]^2 \\ x \vee y, & (x,y) \in [0,e] \times (e,1] \cup (e,1] \times [0,e] \\ y, & x \in [0,e], y \parallel e \\ x, & y \in [0,e], x \parallel e \\ 1, & \text{otherwise.} \end{cases}$$

*$U_{T_\wedge}$ is the greatest uninorm on $L$ with neutral element $e$ [19]. Then*

a) $I_{U_{T_\wedge}}^{L(x)} = \{y_x \in (e,1) \mid x \neq y_x\} \cup I_x^L$ *for $x \in (e,1)$.*

b) $I_{U_{T_\wedge}}^{L(x)} = I_x^L$ *for $x \in (0,e)$ or $x \parallel e$.*

*Proof.* a) Let $y_x \in I_{U_{T_\wedge}}^{L(x)}$ be arbitrary for $x \in (e,1)$. Based on Lemma 3.3, it must be $x \neq y_x$. So, we will show that $y_x \in (e,1)$ or $y_x \in I_x^L$. Suppose that $y_x \notin (e,1)$ and $y_x \notin I_x^L$. Since $y_x \in I_{U_{T_\wedge}}^{L(x)}$, we have that $y_x < x$ and $y_x \npreceq_{U_{T_\wedge}} x$ or $x < y_x$ and $x \npreceq_{U_{T_\wedge}} y_x$ or $x \parallel y_x$.

Let $y_x < x$ and $y_x \npreceq_{U_{T_\wedge}} x$.

Since $y_x \notin (e,1)$, we have that $y_x = 1$, $y_x \in [0,e]$ or $y_x \parallel e$. It can not be $y_x = 1$ by Lemma 3.3. Let $y_x \in [0,e]$. Since $y_x \in [0,e]$ and $x \in (e,1)$, it is obtained that $y_x \preceq_{U_{T_\wedge}} x$, by the definition of $\preceq_U$. This is a contradiction. Let $y_x \parallel e$. Since $y_x < x$ and $y_x \parallel e$, then it is obtained that $y_x \preceq_{U_{T_\wedge}} x$, a contradiction by the definition of $\preceq_U$.

Let $x < y_x$ and $x \npreceq_{U_{T_\wedge}} y_x$.

If $y_x = 1$, then we have $x \preceq_{U_{T_\wedge}} 1$, which is a contradiction.

Let $y_x \in [0,e]$. Since $x < y_x \leq e$, then we have that $U_{T_\wedge}(x, y_x) = x \wedge y_x = x$. So, it is obtained that $x \preceq_{U_{T_\wedge}} y_x$, a contradiction. Let $y_x \parallel e$. Since $x < y_x$ and $y_x \parallel e$, then it is obtained that $x \preceq_{U_{T_\wedge}} y_x$, a contradiction by the definition of $\preceq_U$.

Finally since $y_x \notin I_x^L$, it can not be $x \parallel y_x$. So, we have that $y_x \in (e,1)$ or $y_x \in I_x^L$.

Thus, we have $I_{U_{T_\wedge}}^{L(x)} \subseteq \{y_x \in (e,1) \mid x \neq y_x\} \cup I_x^L$ for $x \in (e,1)$.

Conversely, let $y_x \in (e,1)$ or $y_x \in I_x^L$ such that $x \neq y_x$ for $x \in (e,1)$. We want to show that $y_x \in I_{U_{T_\wedge}}^{L(x)}$. Suppose that $y_x \notin I_{U_{T_\wedge}}^{L(x)}$. In this case, $y_x < x$ and $y_x \preceq_{U_{T_\wedge}} x$ or $x < y_x$ and $x \preceq_{U_{T_\wedge}} y_x$.

Let $y_x \in (e,1)$ and $x \neq y_x$ for $x \in (e,1)$.

• Let $y_x < x$ and $y_x \preceq_{U_{T_\wedge}} x$. Then, there exists an element $k \in [e,1]$ such that $U_{T_\wedge}(y_x, k) = x$. If $k = 1$, then we have $x = y_x$, which is a contradiction. Since $k \in [e,1)$, it is obtained that

$$U_{T_\wedge}(y_x, k) = x = 1$$

a contradiction by the definition of $U_{T_\wedge}$. So, it must be $y_x \npreceq_{U_{T_\wedge}} x$.

• Let $x < y_x$ and $x \preceq_{U_{T_\wedge}} y_x$. Similar arguments are suggested for this case.

So, $\{y_x \in (e,1) \mid x \neq y_x\} \subseteq I_{U_{T_\wedge}}^{L(x)}$ for all $x \in (e,1)$.

Let $y_x \in I_x^L$ for $x \in (e,1)$. By the definition of $I_U^{L(x)}$, we have that $I_x^L \subseteq I_U^{L(x)}$.

Thus, we have $I_{U_{T_\wedge}}^{L(x)} = \{y_x \in (e,1) \mid x \neq y_x\} \cup I_x^L$ for all $x \in (e,1)$.

b) Let $x \in (0,e)$. It is clear that $I_x^L \subseteq I_U^{L(x)}$ for every uninorm on $L$. Conversely, let $y_x \in I_{U_{T_\wedge}}^{L(x)}$. We need to show that $y_x \in I_x^L$. We suppose that $y_x \notin I_x^L$. In this case $x < y_x$ or $y_x < x$. Let $x < y_x$. If $x < y_x < e$, then we have

$$x = y_x \wedge x = U_{T_\wedge}(y_x, x).$$

So we have that $x \preceq_{U_{T_\wedge}} y_x$, a contradiction. If $x < e < y_x$, then it is obtained that $x \preceq_{U_{T_\wedge}} y_x$, a contradiction by the definition of $\preceq_U$.
Let $y_x < x$. Since $y_x < x < e$, we have

$$y_x = y_x \wedge x = U_{T_\wedge}(y_x, x).$$

So, it is obtained that $y_x \preceq_{U_{T_\wedge}} x$, a contradiction.
So, $I^{L(x)}_{U_{T_\wedge}} \subseteq I^L_x$ for $x \in (0, e)$. Consequently, we have $I^{L(x)}_{U_{T_\wedge}} = I^L_x$ for $x \in (0, e)$.
If $x \parallel e$, then similarly it can be shown that $I^{L(x)}_{U_{T_\wedge}} = I^L_x$. □

**Corollary 3.5.** *Let $(L, \leq, 0, 1)$ be a bounded lattice and card$(L) > 3$. For the drastic product t-conorm $S_W$ on $L$, $I^{L(x)}_{S_W} = L \setminus \{0, 1\}$ for $x \in L \setminus \{0, 1\}$.*

**Corollary 3.6.** *Let $(L, \leq, 0, 1)$ be a bounded lattice. For the infimum t-norm $T_\wedge$ on $L$, $I^{L(x)}_{T_\wedge} = I^L_x$ for $x \in L$.*

*Proof.* In Proposition 3.4, if we put a neutral element $e = 0$ and $e = 1$, then we obtain drastic product t-conorm $S_W$ and infimum t-norm $T_\wedge$ on $L$. □

**Proposition 3.7.** *Let $(L, \leq, 0, 1)$ be a bounded lattice. Consider the function on L defined as follows:*

$$U_{S_\vee}(x, y) = \begin{cases} x \vee y, & (x, y) \in [e, 1]^2 \\ x \wedge y, & (x, y) \in [0, e) \times [e, 1] \cup [e, 1] \times [0, e) \\ y, & x \in [e, 1], y \parallel e \\ x, & y \in [e, 1], x \parallel e \\ 0, & \text{otherwise.} \end{cases}$$

*$U_{S_\vee}$ is the smallest uninorm on L with neutral element e [19]. Then*
a) $I^{L(x)}_{U_{S_\vee}} = \{y_x \in (0, e) \mid x \neq y_x\} \cup I^L_x$ *for* $x \in (0, e)$.
b) $I^{L(x)}_{U_{S_\vee}} = I^L_x$ *for* $x \in (e, 1)$ *or* $x \parallel e$.

The proof of this proposition is similar to the proof of Proposition 3.4.

**Corollary 3.8.** *Let $(L, \leq, 0, 1)$ be a bounded lattice. For the t-conorm $S_\vee$ on $L$, $I^{L(x)}_{S_\vee} = I^L_x$ for $x \in L$.*

**Corollary 3.9.** *Let $(L, \leq, 0, 1)$ be a bounded lattice and card$(L) > 3$. For the weakest t-norm $T_W$ on $L$, $I^{L(x)}_{T_W} = L \setminus \{0, 1\}$ for $x \in L \setminus \{0, 1\}$.*

*Proof.* In Proposition 3.7, if we put a neutral element $e = 0$ and $e = 1$, then we get that a t-conorm $S_\vee$ and a t-norm $T_W$ on $L$, respectively. □

Now, we study on the set of all incomparable elements with respect to the $U$ partial order with some uninorm $U$ on a bounded lattice $(L, \leq, 0, 1)$.

**Definition 3.10.** ([20]) Let $U$ be a nullnorm on $(L, \leq, 0, 1)$ with neutral element $e$ and let $K^L_U$ be defined by

$$K^L_U = \{x \in L\backslash\{0, 1\} \mid \text{for some } y \in L\backslash\{0, 1\},\ [x < y \text{ implies } x \npreceq_U y] \text{ or } [y < x \text{ implies } y \npreceq_U x] \text{ or } x \parallel y\}.$$

**Definition 3.11.** ([4]) Let $(L, \leq, 0, 1)$ be a bounded lattice. The set $I_L$ is defined by

$$I_L = \{x \in L \mid \exists\, y \in L \text{ such that } x \parallel y\}.$$

**Proposition 3.12.** *Let $(L, \leq, 0, 1)$ be a bounded lattice and card$([e, 1]) > 3$. Consider the greatest uninorm $U_{T_\wedge}$ with neutral element e in Proposition 3.4. Then, we have that $K^L_{U_{T_\wedge}} = (e, 1) \cup I_L$.*

*Proof.* Let $x \in (e,1) \cup I_L$. Then, we have that $x \in (e,1)$ or $x \in I_L$. Let us show that $x \in K^L_{U_{T_\wedge}}$.

Let $x \in (e,1)$ and $y \in (e,1)$ such that $x < y$. Then, it must be the case that $x \npreceq_{U_{T_\wedge}} y$. Suppose that $x \preceq_{U_{T_\wedge}} y$. Then, there exists an element $k \in [e,1]$ such that

$$U_{T_\wedge}(x,k) = y.$$

If $k = 1$, we have that $x = y$, which is a contradiction.
If $k \in [e,1)$, it is obtained that $U_{T_\wedge}(x,k) = y = 1$, which is a contradiction. Since for any $x \in (e,1)$, there exists an element $y \in (e,1)$, $x < y$ such that $x \npreceq_{U_{T_\wedge}} y$. That is $x \in K^L_{U_{T_\wedge}}$. So, $(e,1) \subseteq K^L_{U_{T_\wedge}}$.

Let $x \in I_L$. Then, there exists $y \in L$ such that $x \parallel y$. Thus, we have that $x \in K^L_{U_{T_\wedge}}$, by the definition of $K^L_{U_{T_\wedge}}$. So, $I_L \subseteq K^L_{U_{T_\wedge}}$. So, it is obtained that $(e,1) \cup I_L \subseteq K^L_{U_{T_\wedge}}$.

Conversely, let $x \in K^L_{U_{T_\wedge}}$. We need to show that $x \in (e,1) \cup I_L$. Suppose that $x \notin (e,1) \cup I_L$. That is, $x \notin (e,1)$ and $x \notin I_L$. Since $x \in K^L_{U_{T_\wedge}}$, there exists an element $y \in L \setminus \{0,1\}$ such that $x < y$ and $x \npreceq_{U_{T_\wedge}} y$ or $y < x$ and $y \npreceq_{U_{T_\wedge}} x$ or $x \parallel y$.

Let $x < y$ and $x \npreceq_{U_{T_\wedge}} y$. Since $x \notin (e,1)$, it must be $x = 1$, $x \in [0,e]$ or $x \parallel e$.
It can not be $x = 1$ by Lemma 3.3
Let $x \in [0,e]$. In this case, $e < y$, $y < e$ or $y \parallel e$. If $y = e$, then we have that $x \preceq_{U_{T_\wedge}} e = y$, a contradiction.
If $x < y < e$, then we have that

$$U_{T_\wedge}(x,y) = x \wedge y = x.$$

So, we have that $x \preceq_{U_{T_\wedge}} y$, which is a contradiction.
If $x \leq e < y$, it is obtained that $x \preceq_{U_{T_\wedge}} y$, a contradiction, by the definition of $\preceq_U$.
If $y \parallel e$, since $x < y$, we have that $x \preceq_{U_{T_\wedge}} y$, a contradiction, by the definition of $\preceq_U$.

Let $y < x$ and $y \npreceq_{U_{T_\wedge}} x$.
If $x = 1$, then we have $y \preceq_{U_{T_\wedge}} 1$, which is a contradiction.
Let $x \in [0,e]$. Since $y < x$, we have that

$$U_{T_\wedge}(x,y) = x \wedge y = x.$$

So, it is obtained that $y \preceq_{U_{T_\wedge}} x$, which is a contradiction.

Finally, since $x \notin I_L$, it can not be $x \parallel y$. Thus, we have that $K^L_{U_{T_\wedge}} \subseteq (e,1) \cup I_L$.
Consequently, we showed $K^L_{U_{T_\wedge}} = (e,1) \cup I_L$. □

**Corollary 3.13.** ([4]) *Let $(L, \leq, 0, 1)$ be a bounded lattice. For the infimum t-norm $T_\wedge$ on L, $K^L_{T_\wedge} = I_L$.*

**Corollary 3.14.** *Let $(L, \leq, 0, 1)$ be a bounded lattice. For the drastic product t-conorm $S_W$ on L,*

$$K^L_{S_W} = \begin{cases} \emptyset, & \text{if } \mathrm{card}(L) \leq 3 \\ L \setminus \{0,1\}, & \text{otherwise.} \end{cases}$$

*Proof.* In Proposition 3.12, if we put a neutral element $e = 0$, then we have $K^L_{S_W} = L \setminus \{0,1\}$ for $card(L) > 3$ . □

**Proposition 3.15.** *Let $(L, \leq, 0, 1)$ be a bounded lattice and card$([0,e]) > 3$. Consider the smallest uninorm $U_{S_\vee}$ with neutral element e in Proposition 3.7. Then, we have that $K^L_{U_{S_\vee}} = (0,e) \cup I_L$.*

The proof of this proposition is similar to the proof of Proposition 3.12.

**Corollary 3.16.** ([4]) *Let $(L, \leq, 0, 1)$ be a bounded lattice. For the weakest t-norm $T_W$ on L,*

$$K^L_{T_W} = \begin{cases} \emptyset, & \text{if } \mathrm{card}(L) \leq 3 \\ L \setminus \{0,1\}, & \text{otherwise.} \end{cases}$$

**Corollary 3.17.** *Let $(L, \leq, 0, 1)$ be a bounded lattice. For t-conorm $S_\vee$ on $L$, $K^L_{S_\vee} = I_L$.*

*Proof.* In Proposition 3.15, if we put a neutral element $e = 0$, then we have $K^L_{S_\vee} = I_L$. □

**Remark 3.18.** Let $(L, \leq, 0, 1)$ be a chain. For any uninorm $U$ with neutral element $e \in L \setminus \{0, 1\}$, if $| L | \leq 4$, then it is obtained that $K^L_U = \emptyset$. If $(L, \leq, 0, 1)$ is not a chain, then it may not be true. For example, let $L = \{0, e, x, 1\}$ whose lattice diagram is displayed in Figure 1.

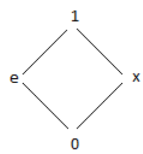


Figure 1: The order $\leq$ on $L$

It is clear that $K^L_U \neq \emptyset$ for every uninorm $U$ with neutral element $e$.

**Proposition 3.19.** ([19]) *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0, 1\}$ and $U$ be a uninorm with neutral element $e$ on $L$. Then,*

(i) $T^* = U |_{[0,e]^2} : [0, e]^2 \to [0, e]$ *is a t-norm on* $[0, e]$.
(ii) $S^* = U |_{[e,1]^2} : [e, 1]^2 \to [e, 1]$ *is a t-conorm on* $[e, 1]$.

**Proposition 3.20.** ([18]) *Let $(L, \leq, 0, 1)$ be a bounded lattice and $U$ be a uninorm with neutral element $e$ on $L$. If $([0, e] \cup [e, 1], \preceq_U)$ is a chain, then $T^*$ and $S^*$ are divisible on $[0, e]$ and $[e, 1]$, respectively.*

**Remark 3.21.** The converse of the above Proposition 3.20 may not be true. Consider the lattice $(L = \{0, a, b, c, d, e, f, 1\}, \leq, 0, 1)$ whose lattice diagram is displayed in Figure 2.

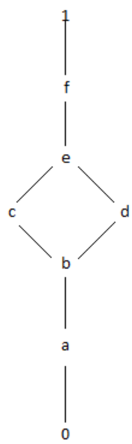


Figure 2: The order $\leq$ on $L$

Consider the uninorm $U : L^2 \to L$ with neutral element $e$ defined as follows:

$$U(x,y) = \begin{cases} x \wedge y, & (x,y) \in [0,e]^2 \\ x \vee y, & \text{otherwise.} \end{cases}$$

$T^*(x,y) = U\mid_{[0,e]^2}(x,y) = x \wedge y$ and $S^*(x,y) = U\mid_{[e,1]^2}(x,y) = x \vee y$ are divisible t-norm and t-conorm for $x, y \in [0,e]$ and $x, y \in [e,1]$, respectively. It is clear that $(L, \preceq_U)$ is not a chain.

## 4. The Equivalence Classes Obtained From *U*-Partial Order

$U$-partial order introduced above allows us to introduce the next equivalence relation on the class of all uninorms on the unit interval $[0,1]$. In this section, we investigate the equivalence relation on the class of all uninorms on the unit interval $[0,1]$. We determine the equivalence classes of the smallest and greatest uninorms on $[0,1]$. In this way, we obtain the equivalence classes of the some basic t-norms and t-conorms in Corollary 4.7, Corollary 4.6, Corollary 4.10 and Corollary 4.11.

**Definition 4.1.** ([20]) Define a relation $\sim$ on the class of all uninorms on the unit interval $[0,1]$ by $U_1 \sim U_2$ if and only if the $U_1$-partial order coincides with the $U_2$-partial order.

**Lemma 4.2.** ([20]) *The relation $\sim$ is an equivalence relation.*

**Definition 4.3.** ([20]) For a given uninorm $U$ on a bounded lattice $(L, \leq, 0, 1)$, we denote the $\sim$ equivalence class linked to $U$ by $\overline{U}$, i.e,

$$\overline{U} = \{U' \mid \quad U' \sim U\}.$$

**Proposition 4.4.** *Consider the smallest uninorm $\underline{U_e} : [0,1]^2 \to [0,1]$ with neutral element $e \in (0,1)$ defined by*

$$\underline{U_e}(x,y) = \begin{cases} 0, & (x,y) \in [0,e)^2 \\ \max(x,y), & (x,y) \in [e,1]^2 \\ \min(x,y), & \text{otherwise.} \end{cases}$$

*Then, the equivalence class of the t-conorm $\underline{U_e}\mid_{[e,1]^2}$ is the set of all divisible t-conorms on $[e,1]$, and the equivalence class of the t-norm $\underline{U_e}\mid_{[0,e]^2}$ consists only the t-norm $\underline{U_e}\mid_{[0,e]^2}$.*

*Proof.* Let $S'$ be a t-conorm on $[e,1]$. Let $S' \in \overline{\underline{U_e}}\mid_{[e,1]^2}$ and $x \leq y$ for $x, y \in [e,1]$. Since $x \leq y$, then we have that $\underline{U_e}\mid_{[e,1]^2}(x,y) = max(x,y) = y$. So, it is obtained that $x \preceq_{\underline{U_e}|_{[e,1]^2}} y$. Since $S' \in \overline{\underline{U_e}}\mid_{[e,1]^2}$, then we have $x \preceq_{S'} y$. Then there exists an element $k \in [e,1]$ such that $S'(x,k) = y$. So, $S'$ is a divisible t-conorm on $[e,1]$.

Conversely, let $S'$ is a divisible t-conorm on $[e,1]$. Let $x \preceq_{\underline{U_e}|_{[e,1]^2}} y$ for $x, y \in [e,1]$. Since $x \preceq_{\underline{U_e}|_{[e,1]^2}} y$, we have that $x \leq y$. Since $S'$ is a divisible t-conorm, there exists an element $\ell \in [e,1]$ such that $S'(x,\ell) = y$. So, we have that $x \preceq_{S'} y$. Conversely, let $x \preceq_{S'} y$. Similarly it can be shown that $x \preceq_{\underline{U_e}|_{[e,1]^2}} y$. So, $\preceq_{\underline{U_e}|_{[e,1]^2}} = \preceq_{S'}$.

The equivalence class of the t-norm $\underline{U_e}\mid_{[0,e]^2}$ consists only the t-norm $\underline{U_e}\mid_{[0,e]^2}$ by [21]. □

**Remark 4.5.** In Proposition 4.4, if a t-conorm is not divisible t-conorm on $[e,1]$, then $\preceq_{\underline{U_e}|_{[e,1]^2}} \neq \preceq_S$. Consider the t-norm $S_D$ on $[e,1]$ defined by

$$S_D(x,y) = \begin{cases} y, & x = e \\ x, & y = e \\ 1, & \text{otherwise.} \end{cases}$$

It is clear that $S_D$ is not divisible t-conorm. We claim that $\preceq_{\underline{U_e}|_{[e,1]^2}} \neq \preceq_{S_D}$.

Let $e = \frac{1}{2}$. Since $\underline{U_e}\mid_{[\frac{1}{2},1]^2}(\frac{2}{3}, \frac{3}{4}) = \frac{3}{4}$, then it is obtained that $\frac{2}{3} \preceq_{\underline{U_e}|_{[\frac{1}{2},1]^2}} \frac{3}{4}$. But $\frac{2}{3} \npreceq_{S_D} \frac{3}{4}$. Suppose that $\frac{2}{3} \preceq_{S_D} \frac{3}{4}$. Then there exists an element $k \in [\frac{1}{2}, 1]$ such that $S_D(\frac{2}{3}, k) = \frac{3}{4}$.

If $k = \frac{1}{2}$, then we have that $\frac{3}{4} = \frac{2}{3}$, a contradiction. If $k \in (\frac{1}{2}, 1]$, then it is obtained that $\frac{3}{4} = 1$, a contradiction. So, $\preceq_{\underline{U_e}|_{[\frac{1}{2},1]^2}} \neq \preceq_{S_D}$.

**Corollary 4.6.** *The equivalence class of the smallest t-conorm $S_M$ on* $[0,1]$ *is the set of all divisible t-conorms on* $[0,1]$.

*Proof.* In Proposition 4.4, if we put a neutral element $e = 0$, then we have a smallest t-conorm $S_M$ on $[0,1]$. □

**Corollary 4.7.** ([21]) *The equivalence class of the smallest t-norm $T_D$ on* $[0,1]$ *consists only the t-norm $T_D$ on* $[0,1]$.

**Proposition 4.8.** *Consider the greatest uninorm $\overline{U_e} : [0,1]^2 \to [0,1]$ with neutral element $e \in (0,1)$ defined by*

$$\overline{U_e}(x,y) = \begin{cases} \min(x,y), & (x,y) \in [0,e]^2 \\ 1, & (x,y) \in (e,1]^2 \\ \max(x,y), & \text{otherwise.} \end{cases}$$

*Then, the equivalence class of the t-norm $\overline{U_e}\mid_{[0,e]^2}$ is the set of all divisible t-norms on $[0,e]$, and the equivalence class of the t-conorm $\overline{U_e}\mid_{[e,1]^2}$ consists only the t-conorm $\overline{U_e}\mid_{[e,1]^2}$.*

**Remark 4.9.** In Proposition 4.8, if a t-norm is not divisible on $[0,e]$, then $\preceq_{\overline{U_e}}|_{[0,e]^2} \neq \preceq_T$. Consider the t-conorm $T_D$ on $[0,e]$ defined by

$$T_D(x,y) = \begin{cases} y, & x = e \\ x, & y = e \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $T_D$ is not divisible t-norm. We claim that $\preceq_{\overline{U_e}}|_{[0,e]^2} \neq \preceq_{T_D}$.
Let $e = \frac{1}{2}$. Since $\overline{U_e}\mid_{[0,\frac{1}{2}]^2}(\frac{1}{5},\frac{1}{6}) = \frac{1}{6}$, then it is obtained that $\frac{1}{6} \preceq_{\overline{U_e}}|_{[0,\frac{1}{2}]^2} \frac{1}{5}$. But $\frac{1}{6} \npreceq_{T_D} \frac{1}{5}$. On the condition that $\frac{1}{6} \preceq_{T_D} \frac{1}{5}$, there exists an element $\ell \in [0,\frac{1}{2}]$ such that $T_D(\frac{1}{5},\ell) = \frac{1}{6}$.
If $\ell = \frac{1}{2}$, we have that $\frac{1}{5} = \frac{1}{6}$, a contradiction. If $\ell \in [0,\frac{1}{2})$, then it is obtained that $\frac{1}{6} = 0$, a contradiction. So, $\preceq_{\overline{U_e}}|_{[0,\frac{1}{2}]^2} \neq \preceq_{T_D}$.

**Corollary 4.10.** *The equivalence class of the greatest t-conorm $S_D$ on* $[0,1]$ *consists only the t-conorm $S_D$ on* $[0,1]$.

*Proof.* In Proposition 4.8, if we put a neutral element $e = 0$, then we have a greatest t-conorm $S_D$ on $[0,1]$. □

**Corollary 4.11.** ([21]) *The equivalence class of the greatest t-norm $T_M$ on* $[0,1]$ *is the set of all divisible t-norms on* $[0,1]$.

## 5. Distributivity f0r Uninorms

In this section, we investigate the relationship between an order induced by uninorms and distributivity property for uninorms on the unit interval $[0,1]$. Thus, we give sufficiency condition for equivalent according to the $\beta$ in Corollary 5.6.

**Definition 5.1.** ([23]) Let $U_1$ and $U_2$ be uninorms on $[0,1]$. $U_1$ is distributive over $U_2$ if it is satisfies the following condition:

$$U_1(x, U_2(y,z)) = U_2(U_1(x,y), U_1(x,z)) \tag{2}$$

for all $x,y,z \in [0,1]$.

**Proposition 5.2.** *Let $U_1$ and $U_2$ be uninorms on* $[0,1]$ *with the same neutral elements. If $U_1$ is distributive over $U_2$, then $K_{U_2} \subseteq K_{U_1}$.*

*Proof.* Let $U_1$ and $U_2$ be uninorms on the unit interval $[0,1]$ and $U_1$ is distributive over $U_2$. Let $x \in K_{U_2}$. Then there exists an element $y \in (0,1)$ such that $x < y$ and $x \npreceq_{U_2} y$ or $y < x$ and $y \npreceq_{U_2} x$. Suppose that $x \notin K_{U_1}$. Then there exists an element $y \in (0,1)$ such that $x < y$ and $x \preceq_{U_1} y$ or $y < x$ and $y \preceq_{U_1} x$. Without loss of generality, we assume that $x < y$ and $x \preceq_{U_1} y$.

Let $x, y \in [0, e]$. Then there exists an element $k \in [0, e]$ such that $U_1(y, k) = x$.

$$x = U_1(y, k) = U_1(y, U_2(k, e))$$

Since $U_1$ is distributive over $U_2$, then we get that

$$x = U_1(y, U_2(k, e)) = U_2(U_1(y, k), U_1(y, e)) = U_2(x, y).$$

So, it is obtained that $x \preceq_{U_2} y$, which is a contradiction.

Similar arguments are suggested for $x, y \in [e, 1]$. Since $x \in K_{U_2}$, it can not be $x, y \notin [0, e]$ and $x, y \notin [e, 1]$. Because if $x, y \notin [0, e]$ and $x, y \notin [e, 1]$, then we have that $x \preceq_U y$, by the definition of $\preceq_U$. □

**Proposition 5.3.** *Let $U_1$ and $U_2$ be uninorms on $[0,1]$. If $U_1$ is distributive over $U_2$ and $U_2$ is distributive over $U_1$, then $K_{U_1} = K_{U_2}$.*

**Remark 5.4.** The converse of the above Proposition 5.3 may not be true. Here is an example illustrating a such case.

**Example 5.5.** Consider the uninorms $U : [0,1]^2 \to [0,1]$ and $\underline{U_{\frac{1}{2}}} : [0,1]^2 \to [0,1]$ with neutral elements $\frac{1}{2}$ defined as follows:

$$U(x,y) = \begin{cases} 0, & (x,y) \in [0, \frac{1}{2}]^2 \text{ and } x + y \leq \frac{1}{2} \text{ and } (x,y) \neq (\frac{1}{4}, \frac{1}{4}), \\ \frac{1}{4}, & (x,y) = (\frac{1}{4}, \frac{1}{4}), \\ \max(x,y), & (x,y) \in [\frac{1}{2}, 1]^2, \\ \min(x,y), & \text{otherwise} \end{cases}$$

and

$$\underline{U_{\frac{1}{2}}}(x,y) = \begin{cases} 0, & (x,y) \in [0, \frac{1}{2})^2, \\ \max(x,y), & (x,y) \in [\frac{1}{2}, 1]^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

We have that $K_U = K_{\underline{U_{\frac{1}{2}}}} = (0, \frac{1}{2})$ (see [2]). But $U$ is not distributive over $\underline{U_{\frac{1}{2}}}$. Now, let us show that this claim.

$$U(\tfrac{1}{4}, \underline{U_{\frac{1}{2}}}(\tfrac{1}{4}, \tfrac{2}{3})) = U(\tfrac{1}{4}, \tfrac{1}{4}) = \tfrac{1}{4} \text{ and } \underline{U_{\frac{1}{2}}}(U(\tfrac{1}{4}, \tfrac{1}{4}), U(\tfrac{1}{4}, \tfrac{2}{3})) = \underline{U_{\frac{1}{2}}}(\tfrac{1}{4}, \tfrac{1}{4}) = 0.$$

Since $0 \neq \frac{1}{4}$, $U_1$ is not distributive over $U_2$.

**Corollary 5.6.** *Let $U_1$ and $U_2$ be uninorms on $[0,1]$. If $U_1$ is distributive over $U_2$ and $U_2$ is distributive over $U_1$, then $U_1$ and $U_2$ are equivalent according to the $\beta$.*

**Remark 5.7.** Let $U_1$ and $U_2$ be uninorms on $[0,1]$. If $U_1$ is distributive over $U_2$, then it can not be $K_{U_1} \subseteq K_{U_2}$. Consider the functions on $[0,1]$ defined as follows:

$$U_1(x,y) = \begin{cases} 0, & (x,y) \in [0, \frac{1}{2})^2, \\ 1, & (x,y) \in (\frac{1}{2}, 1]^2, \\ y, & x = \frac{1}{2}, \\ x, & y = \frac{1}{2}, \\ \min(x,y), & \text{otherwise,} \end{cases}$$

and

$$U_2(x,y) = \begin{cases} \min(x,y), & (x,y) \in [0,\frac{1}{2}]^2, \\ \max(x,y), & \text{otherwise.} \end{cases}$$

$U_1$ and $U_2$ are uninorms with neutral elements $\frac{1}{2}$. It is clear that $\preceq_{U_1} \subseteq \preceq_{U_2}$ and $U_1$ is distributive over $U_2$. It can be shown that $K_{U_1} = \{x \in (0,1) \mid x \neq e\}$ and $K_{U_2} = \emptyset$. Hence, we get that $K_{U_1} \nsubseteq K_{U_2}$.

## 6. Concluding Remarks

We have discussed and investigated some properties of $U$-partial order, denoted by $\preceq_U$. We have investigated that the set $\mathcal{I}_F^{(x)}$, denoting the set of all incomparable elements with arbitrary but fixed $x \in L \setminus \{0,1\}$ according to $\preceq_U$. We have determined the sets of incomparable elements w.r.t. $U$-partial order of the greatest and smallest uninorm on $L$. Also, we have investigated an equivalence relation on the class of uninorms on a bounded lattice $(L, \leq, 0, 1)$ and we have determined the equivalence classes of some special uninorms on the unit interval $[0,1]$. Finally, we have investigated the relationship between an order induced by uninorms and distributivity property for uninorms on the unit interval $[0,1]$.

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