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Reliable Numerical Algorithm for Handling Fuzzy Integral Equations of Second Kind in Hilbert Spaces

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Abstract. Integral equations under uncertainty are utilized to describe different formulations of physical phenomena in nature. This paper aims to obtain analytical and approximate solutions for a class of integral equations under uncertainty. The scheme presented here is based upon the reproducing kernel theory and the fuzzy real-valued mappings. The solution methodology transforms the linear fuzzy integral equation to crisp linear system of integral equations. Several reproducing kernel spaces are defined to investigate the approximate solutions, convergence and the error estimate in terms of uniform continuity. An iterative procedure has been given based on generating the orthonormal bases that rely on Gram-Schmidt process. Effectiveness of the proposed method is demonstrated using numerical experiments. The gained results reveal that the reproducing kernel is a systematic technique in obtaining a feasible solution for many fuzzy problems.

1. Introduction

Fuzzy integral equations are an influential part of the theory of uncertainty analysis that have become indispensable mathematical tools to describe and analyze real-world problems [14, 22, 25, 29, 40, 41]. They have a wide proportion of applications in different fields of science and engineering, which have been utilized to introduce uncertain parameter values in modelling many physical processes in nature [4, 26, 35, 42]. In many applications, some of parameters are typically given in uncertainty, so it is necessary to develop a suitable model and a viable algorithm to address these issues. Uncertainty during the modeling stages comes from many sources such as the errors and approximations in estimation values, data collection, initial data measurement, model parameters, model structure, and so on. However, it is too complicated to insert an explicit formula for these equations, thereby it is need reliable approximations to treat the complexities of uncertain data. Usually, by using parametrization of the fuzzy numbers, the fuzzy integral equation may be converted to a crisp system of integral equations that can be solved approximately.

Numerical simulations for fuzzy integral equations have continued to grow and attracted the attention of mathematicians and physicists. Over the past decades, various techniques have been proposed to deal with the uncertain integral equations. Based on this trend, the iterative procedure based on quadrature formula is described in [27, 37], while the successive approximations procedure is presented in [19, 20]. The numerical-analytical procedures to solve fuzzy integral equations including fuzzy Laplace transform method [36],

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homotopy analysis method [39], fuzzy differential transform method [32], Adomian decomposition method [38], finite differences method [15] and variational iteration method [34] are also applied. Other numerical techniques developed for fuzzy integral equations based on fuzzy Haar wavelets, Legendre wavelets, Legendre and Chebyshev interpolations, Galerkin, triangular and block-pulse functions can be found in [16, 18, 21, 28, 31].

This paper aims to extend the application of the reproducing kernel method (RKM) in finding approximate solutions for a class of linear fuzzy Fredholm-Volterra integral equations in the following form

$$\mathcal{F}(\tau) = g(\tau) + \lambda_1 \int_a^b h(\tau,\rho)\mathcal{F}(\rho)d\rho + \lambda_2 \int_a^\tau k(\tau,\rho)\mathcal{F}(\rho)d\rho, \quad a \le \rho < \tau \le b, \tag{1}$$

where λ_1, λ_2 are positive parameters, $g : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is continuous fuzzy-valued function, $h(\tau,\rho), k(\tau,\rho)$ are positive crisp kernel functions over the square $a \le \rho < \tau \le b$, and a, b are real finite constants in which $\mathbb{R}_{\mathcal{F}}$ refers to the set of fuzzy numbers on $\mathbb{R}$.

The theory of the RKM has a wide range of applications in applied mathematics, numerical analysis, financial mathematics, machine learning, statistical learning, control theory, global optimization and so on. Recently, the reproducing kernel theory has been used to treat many nonlinear problems such that nonlinear boundary value problems, nonlinear partial differential equations, nonlinear oscillator models, nonlinear differential-difference equations, nonlinear integral and integro-differential equations, nonlinear fractional differential equations, and nonlinear fuzzy models [1, 5, 6, 12, 17, 30]. The RKM possess several advantages: it is accurate and need less effort to achieve the results, it has excellent properties in error estimation reflects high accuracy and reliability, provides uniform convergence rates to the exact solution, it is possible to pick any point in the interval of integration and as well the approximate solution will be applicable, and offers a straightforward assumption to deal with various boundary conditions [2, 3, 7, 9–11, 13, 23].

The organization of this work is as follows. In the next section, preliminaries about fuzzy calculus theory are given. Section 3 contains the formulation of model problem and some necessary concepts of the reproducing kernel theory. Section 4 is devoted to computational algorithm and to error estimates. Simulation results are presented in Section 5 to guarantee the RK procedure and to confirm its performance. Finally, the last section includes concluding observations.

2. Preliminaries

In this section, we briefly recall some definitions and results related to fuzzy analysis needed throughout the paper. We start with the basic definition of fuzzy numbers.

Definition 2.1. ([25]) A fuzzy number is a mapping $v : \mathbb{R} \to [0,1]$ that satisfies normal, fuzzy convex, upper semicontinuous and compactly supported.

An equivalent parametric definition is given by Kaleva [29] and Friedman et al. [22] as follows:

Definition 2.2. A fuzzy number u in parametric form is a pair (v_-^r, v_+^r) of functions v_-^r, v_+^r, $0 \le r \le 1$, which satisfy the following properties:

- v_-^r is bounded left continuous non-decreasing function over $[0,1]$,
- v_+^r is bounded left continuous non-increasing function over $[0,1]$,
- $v_-^r \le v_+^r \;\; 0 \le r \le 1$.

For $0 < r \le 1$, the r-level set of a fuzzy number v is defined as $[v]^r = \{\alpha \in \mathbb{R} : v(\alpha) \ge r\}$, which is a closed and bounded interval $[v]^r = [v_{1r}, v_{2r}]$, where v_-^r and v_+^r denote the left-hand and the right-hand endpoint of the r-level set, respectively. Also, $[v]^0 = \overline{\{\alpha \in \mathbb{R} : v(\alpha) > 0\}}$ is a closure of the support of v, which is a compact set. The set of all normal, convex, and upper semi-continuous fuzzy numbers with bounded r

level intervals is denoted by $\mathbb{R}_{\mathcal{F}}$. For arbitrary Any real number $\mu \in \mathbb{R}$ can be interpreted as a fuzzy number $\widetilde{\mu} = \chi_{\{\mu\}}$, where $\chi_{\{\mu\}}$ is the characteristic function at μ, therefore $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$.

For arbitrary fuzzy numbers $v = (v_{1r}, v_{2r})$, $w = (w_{1r}, w_{2r})$ and real number μ, the addition $v \oplus w$ and scalar multiplication $v \odot w$ are defined as follows:

- $v = w$ if and only if $v_{1r} = w_{1r}$ and $v_{2r} = w_{2r}$.
- $v \oplus w = (v_{1r} + w_{1r}, v_{2r} + w_{2r})$.
- $$\mu \odot v = \begin{cases} (\mu v_{1r}, \mu v_{2r}), & \mu \geq 0 \\ (\mu v_{2r}, \mu v_{1r}), & \mu < 0 \end{cases}$$

Supremum metric structure on $\mathbb{R}_{\mathcal{F}}$ is the most commonly metric used and is given by the following definition:

Definition 2.3. ([41]) For arbitrary fuzzy numbers $v = (v_{1r}, v_{2r})$, $w = (w_{1r}, w_{2r})$, the quantity $d(v, w) = sup_{r\in[0,1]} max\Big\{|v_{1r} - w_{1r}|, |v_{2r} - w_{2r}|\Big\}$ is the distance between v and w. It is demonstrated that $(d, \mathbb{R}_{\mathcal{F}})$ is a complete metric space [29] and the following properties are realized:

- $d(v \oplus u, w \oplus u) = d(v, w)$, $\forall v, w, u \in \mathbb{R}_{\mathcal{F}}$.
- $d(v \oplus u, w \oplus u) \leq d(v, w) + d(u, z)$, $\forall v, w, u, z \in \mathbb{R}_{\mathcal{F}}$.
- $d(\mu \odot v, \mu \odot w) = |\mu| d(v, w)$, $\forall \mu \in \mathbb{R}$, $\forall v, w \in \mathbb{R}_{\mathcal{F}}$.
- $d(\mu_1 \odot v, \mu_2 \odot w) = |\mu_1 - \mu_1| d(v, \widetilde{0})$, $\forall \mu_1, \mu_2 \in \mathbb{R}$, $\mu_1\mu_2 \geq 0$, $\forall v \in \mathbb{R}_{\mathcal{F}}$.

Definition 2.4. ([14]) A fuzzy-valued function $g : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is said to be continuous at $\tau_0 \in [a, b]$ if for each fixed $\varepsilon > 0$, there is $\delta > 0$ such that $d(g(\tau), g(\tau_0)) < \varepsilon$ whenever $|\tau - \tau_0| < \delta$. If g is continuous for each $\tau_0 \in [a, b]$, then g is said to be continuous on $[a, b]$.

A fuzzy number v is upper bound for $g : [a, b] \to \mathbb{R}_{\mathcal{F}}$ if $g_{1r}(\tau) \leq v_{1r}$ and $g_{2r}(\tau) \leq v_{2r}$ for all $\tau \in [a, b]$, $r \in [0, 1]$, and v is lower bound for $g : [a, b] \to \mathbb{R}_{\mathcal{F}}$ if $v_{1r} \leq g_{1r}(\tau)$ and $g_{2r}(\tau) \leq v_{2r}$ for all $\tau \in [a, b]$, $r \in [0, 1]$. Further, a fuzzy-valued function $g : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is said to be bounded if it has a lower and an upper bound. The following definition is presented for fuzzy integral based on Riemann integral concept. It is worth noting that all the different concepts of fuzzy integral yield the same results in the sense of continuity of the fuzzy-valued function. The concept defined in this paper is more appropriate for numerical computations.

Definition 2.5. ([29]) Let $\mathcal{F} : \Omega \to \mathbb{R}_{\mathcal{F}}$. For a partition $\mathcal{P}_\tau^n = \{\tau_0, \tau_1, ..., \tau_n\}$ of Ω and for arbitrary points $\xi_i \in [\tau_{i-1}, \tau_i]$, $i = 1, 2, ..., n$. Let $\varrho_p = \sum_{i=1}^n \Big(\mathcal{F}(\xi_i)(\tau_i - \tau_{i-1})\Big)$ and $\Delta_\tau = \max_{1\leq i\leq n} |\tau_i - \tau_{i-1}|$, then the definite integral of $\mathcal{F}(\tau)$ over Ω is defined by $\int_\Omega \mathcal{F}(\tau) d\tau = \lim_{\Delta_\tau \to 0} \varrho_p$ provided the limit exists in the metric space $(\mathbb{R}_{\mathcal{F}}, d)$.

Theorem 2.1. ([29]) *Let $\mathcal{F} : \Omega \to \mathbb{R}_{\mathcal{F}}$ be continuous fuzzy-valued function and put $[F(\tau)]^r = [\mathcal{F}_{1r}(\tau), \mathcal{F}_{2r}(\tau)]$ for each $r \in [0, 1]$. Then $\int_\Omega \mathcal{F} d\tau \in \mathbb{R}_{\mathcal{F}}$ exist, $\mathcal{F}_{1r}(\tau), \mathcal{F}_{2r}(\tau)$ are integrable functions on Ω, and $[\int_\Omega \mathcal{F}(\tau) d\tau]^r = \int_\Omega [\mathcal{F}(\tau)]^r d\tau = [\int_\Omega \mathcal{F}_{1r}(\tau) d\tau, \int_\Omega \mathcal{F}_{2r}(\tau) d\tau]$.*

3. Fuzzy Integral Equation in Reproducing Kernel Spaces

In this section, the original fuzzy Fredholm-Volterra integral equation is converted equivalently into crisp Fredholm-Volterra integral equations system based on the use of Riemann integrability. However, to do this, consider the standard form of Fredholm-Volterra integral equation as follows:

$$\mathcal{F}(\tau) = g(\tau) + \lambda_1 \int_a^b h(\tau,\rho)\mathcal{F}(\rho)d\rho + \lambda_2 \int_a^\tau k(\tau,\rho)\mathcal{F}(\rho)d\rho, \quad a \le \rho < \tau \le b, \tag{2}$$

where $g : [a,b] \to \mathbb{R}$ is continuous real-valued function and $h(\tau,\rho), k(\tau,\rho)$ are arbitrary crisp kernel functions over the square $a \le \rho < \tau \le b$, $(a, b \in \mathbb{R})$, such that $h(\tau,\rho)$ is non-negative for $a \le \rho \le c_1$ and is non-positive for $c_1 \le \rho \le b$; and $k(\tau,\rho)$ is non-negative for $a \le \rho \le c_2$ and is non-positive for $c_2 \le \rho \le \tau$. Consequently, if $g(\tau)$ is a crisp function, then the solution $F(\tau)$ of Eq. (2) is a crisp. Otherwise if $g(\tau)$ is a fuzzy function, then Eq. (2) may possess only fuzzy solution $\mathcal{F}(\tau)$. Now, we assume that $g(\tau)$ is a fuzzy function, $\mathtt{H}(\tau,\rho,\mathcal{F}(\rho)) = h(\tau,\rho)\mathcal{F}(\rho)$ and $\mathtt{K}(\tau,\rho,\mathcal{F}(\rho)) = k(\tau,\rho)\mathcal{F}(\rho)$. Let $[g_{1r}(\tau), g_{2r}(\tau)]$ and $[\mathcal{F}_{1r}(\tau), \mathcal{F}_{2r}(\tau)]$ be the parametric forms of $g(\tau)$ and $\mathcal{F}(\tau)$, respectively; then the parametric form of Eq. (2) is given by

$$[\mathcal{F}(\tau)]^r = [g(\tau)]^r + \lambda_1 \int_a^b [\mathtt{H}(\tau,\rho,\mathcal{F}(\rho))]^r d\rho + \lambda_2 \int_a^b [\mathtt{K}(\tau,\rho,\mathcal{F}(\rho))]^r d\rho, \tag{3}$$

where

$$[\mathtt{H}(\tau,\rho,\mathcal{F}(\rho))]^r = \Big[\mathtt{H}_{1r}\big(\tau,\rho,\mathcal{F}_{1r}(\rho),\mathcal{F}_{2r}(\rho)\big), \mathtt{H}_{2r}\big(\tau,\rho,\mathcal{F}_{1r}(\rho),\mathcal{F}_{2r}(\rho)\big)\Big]$$

$$[\mathtt{K}(\tau,\rho,\mathcal{F}(\rho))]^r = \Big[\mathtt{K}_{1r}\big(\tau,\rho,\mathcal{F}_{1r}(\rho),\mathcal{F}_{2r}(\rho)\big), \mathtt{K}_{2r}\big(\tau,\rho,\mathcal{F}_{1r}(\rho),\mathcal{F}_{2r}(\rho)\big)\Big]$$

such that

$$\mathtt{H}_{1r}\big(\tau,\rho,\mathcal{F}_{1r}(\rho),\mathcal{F}_{2r}(\rho)\big) = \min\big\{h(\tau,\rho)\mathcal{F}_{1r}(\rho), h(\tau,\rho)\mathcal{F}_{2r}(\rho)\big\},$$

$$\mathtt{H}_{2r}\big(\tau,\rho,\mathcal{F}_{1r}(\rho),\mathcal{F}_{2r}(\rho)\big) = \max\big\{h(\tau,\rho)\mathcal{F}_{1r}(\rho), h(\tau,\rho)\mathcal{F}_{2r}(\rho)\big\},$$

$$\mathtt{K}_{1r}\big(\tau,\rho,\mathcal{F}_{1r}(\rho),\mathcal{F}_{2r}(\rho)\big) = \min\big\{k(\tau,\rho)\mathcal{F}_{1r}(\rho), k(\tau,\rho)\mathcal{F}_{2r}(\rho)\big\},$$

$$\mathtt{K}_{2r}\big(\tau,\rho,\mathcal{F}_{1r}(\rho),\mathcal{F}_{2r}(\rho)\big) = \max\big\{k(\tau,\rho)\mathcal{F}_{1r}(\rho), k(\tau,\rho)\mathcal{F}_{2r}(\rho)\big\}.$$

Thereby the parametric forms of the fuzzy Fredholm-Volterra integral equations system can be converted as follows:

$$\mathcal{F}_{1r}(\tau) = g(\tau)_{1r} + \lambda_1\Big(\int_a^{c_1} h(\tau,\rho)\mathcal{F}_{1r}(\rho))d\rho + \int_{c_1}^b h(\tau,\rho)\mathcal{F}_{2r}(\rho))d\rho\Big) + \lambda_2\Big(\int_a^{c_2} k(\tau,\rho)\mathcal{F}_{1r}(\rho))d\rho + \int_{c_2}^\tau k(\tau,\rho)\mathcal{F}_{2r}(\rho))d\rho\Big), \tag{4}$$

$$\mathcal{F}_{2r}(\tau) = g(\tau)_{2r} + \lambda_1\Big(\int_a^{c_1} h(\tau,\rho)\mathcal{F}_{2r}(\rho))d\rho + \int_{c_1}^b h(\tau,\rho)\mathcal{F}_{1r}(\rho))d\rho\Big) + \lambda_2\Big(\int_a^{c_2} k(\tau,\rho)\mathcal{F}_{2r}(\rho))d\rho + \int_{c_2}^\tau k(\tau,\rho)\mathcal{F}_{1r}(\rho))d\rho\Big). \tag{5}$$

Definition 3.1. ([5]) Let $\mathcal{S}$ be a nonempty abstract set. A function $\varphi : \mathcal{S} \times \mathcal{S} \to \mathbb{C}$ is a reproducing kernel of the Hilbert space $\mathcal{H}$ iff (i) $\forall \tau \in \mathcal{S}, \varphi(\cdot,\tau) \in \mathcal{H}$ and (ii) $\forall \tau \in \mathcal{S}, \forall \psi \in \mathcal{H}, \langle \psi(\cdot), \varphi(\cdot,\tau)\rangle = \psi(\tau)$. The condition (ii) is "the reproducing property". If there exists a reproducing kernel φ of $\mathcal{H}$, then $\mathcal{H}$ is called a reproducing kernel Hilbert space (RKHS).

Next, a complete reproducing kernel space $\mathcal{W}^m[a,b] = \mathcal{H}_2^m[a,b] \oplus \mathcal{H}_2^m[a,b]$ is constructed to represent the solution of Eq. (2).

Definition 3.2. The inner product space $\mathcal{W}_2^m[a,b]$ is defined as $\mathcal{W}_2^m[a,b] = \{f|f, f', ..., f^{(m-1)}$ are absolutely continuous functions,$f^{(m)} \in L^2[a,b]\}$ where $L^2[a,b] = \{f| \int_a^b f^2(\rho)d\rho < \infty\}$. The inner product and norm are given, respectively, by $\langle f(\tau), g(\tau)\rangle_{\mathcal{W}_2^m[a,b]} = \sum_{k=0}^{m-1} f^{(k)}(a)g^{(k)}(a) + \int_a^b f^{(m)}g^{(m)}(\rho)d\rho$ and $\|f\|_{\mathcal{W}_2^m} = \sqrt{\langle f, f\rangle_{\mathcal{W}_2^m}}$, $f, g \in \mathcal{W}_2^m[a,b]$.

Theorem 3.1. *The space $\mathcal{W}_2^m[a,b]$ is a **RKHS**. The reproducing kernel function $Q_\tau^m(s)$ can be written as follows*

$$Q_\tau^m(s) = \begin{cases} \sum_{i=0}^{m-1} \frac{1}{(i!)^2}(\tau-a)^i(s-a)^i + \frac{1}{((m-1)!)^2}\int_a^s(\tau-\rho)^{m-1}(s-\rho)^{m-1}(\rho)d\rho, \ s \le \tau \\ \sum_{i=0}^{m-1} \frac{1}{(i!)^2}(\tau-a)^i(s-a)^i + \frac{1}{((m-1)!)^2}\int_a^\tau(\tau-\rho)^{m-1}(s-\rho)^{m-1}(\rho)d\rho, \ s > \tau \end{cases}$$

Proof. Let $\{\xi_n\}_{n=1}^\infty$ be a Cauchy sequence in the space $\mathcal{W}_2^m[a,b]$, then $\|\xi_{n+1} - \xi_n\|^2_{\mathcal{W}_2^m} = \sum_{i=0}^{m-1}\left(\xi_{n+1}^i(a) - \xi_{n+1}^i(a)\right)^2 + \int_a^b\left(\xi_{n+1}^{(m)}(\rho) - \xi_{n+1}^{(m)}(\rho)\right)^2 d\rho \to 0$ as soon as $n \to \infty$. Anyhow, $\{\xi_n(a)\}_{n=1}^\infty$,$\{\xi_n'(a)\}_{n=1}^\infty$,..., $\left\{\xi_n^{(m-1)}(a)\right\}_{n=1}^\infty$ be the all Cauchy sequence of real number and $\left\{\xi_n^{(m)}(\tau)\right\}_{n=1}^\infty$ be a Cauchy sequence in $L^2[a,b]$. Thus, there exist a real number η_i, $i = 0, 1, ..., m-1$ and a real-valued function $\xi \in L^2[a,b]$ such that $\xi_n(a) = \eta_0$, $\xi_n'(a) = \eta_1$,...,$\xi_n^{(m-1)}(a) = \eta_{m-1}$, and $\int_a^b\left(\xi_{n+1}^{(m)}(\rho) - \xi(\rho)\right)^2 d\rho \to 0$ as $n \to \infty$. Indeed, let $v^{(m)}(\tau) = \xi(\tau) \in L^2[a,b]$, then

$$v^{(m)}(\tau) = \eta_0 + \eta_1(\tau - a) + ... + \eta_{m-1}\frac{(\tau-a)^{m-1}}{(m-1)!} + \int_a^\tau \frac{(\tau-\rho)^{m-1}}{(m-1)!}\xi(\rho)d\rho. \tag{6}$$

Clearly, $v^{(i)}(t)$, $i = 0, 1, , m-1$ are absolutely continuous functions such that $v(t) \in \mathcal{W}_2^m[a,b]$, $v^{(i)}(t) = \eta_i$, $i = 0, 1, , m-1$, and $\|\xi_n - v\|^2_{\mathcal{W}_2^m} \to 0$ as $n \to \infty$. So, $\mathcal{W}_2^m[a,b]$ is complete Hilbert space.

For the conduct of proceedings in the proof, for any $f \in \mathcal{W}_2^m[a,b]$, it yields that

$$f(\tau) = \sum_{i=0}^{m-1}\frac{f^{(i)}(a)(\tau-a)^{(m-1)}}{i!} + \frac{1}{(m-1)!}\int_a^\tau(\tau-\rho)^{m-1}f^{(m)}(\rho)d\rho \tag{7}$$

Consequently, from Cauchy-Schwarz inequality, one can obtain that

$$|f(\tau)| \le \left|\sum_{i=0}^{m-1}\frac{f^{(i)}(a)(\tau-a)^{(m-1)}}{i!}\right| + \frac{1}{(m-1)!}\left|\int_a^\tau(\tau-\rho)^{m-1}f^{(m)}(\rho)d\rho\right|$$

$$\le \left(\sum_{i=0}^{m-1}\frac{(\tau-a)^{2i}}{(i!)^2}\right)^{\frac{1}{2}}\left(\sum_{i=0}^{m-1}\left(f^{(i)}(a)\right)^2\right)^{\frac{1}{2}} + \frac{1}{(m-1)!}\left(\int_a^b(b-\rho)^{2(m-1)}d\rho\right)^{\frac{1}{2}}\left(\int_a^b\left(f^{(m)}(\rho)\right)^2 d\rho\right)^{\frac{1}{2}} \tag{8}$$

$$\le \left(\left(\sum_{i=0}^{m-1}\frac{(b-a)^{2i}}{(i!)^2}\right)^{\frac{1}{2}} + \frac{(b-a)^{m-\frac{1}{2}}}{(m-1)!}\right)\|f\|_{\mathcal{W}_2^m}.$$

Hence, for any $f \in \mathcal{W}_2^m[a,b]$, one can write

$$\langle f(s), Q_\tau^m(s)\rangle = \sum_{i=0}^{m-1} f^i(a)(Q_\tau^m)^{(i)}(a) + \int_a^\tau f^m(s)(Q_\tau^m)^{(m)}(s)ds$$

$$= \sum_{i=0}^{m-1} \frac{f^{(i)}(a)(\tau-a)^i}{i!}(s-a)^i + \frac{1}{(m-1)!}\int_a^\tau (\tau-s)^{m-1} f^{(m)}(s)ds \tag{9}$$

So, the proof of the theorem is completely constructed. □

Corollary 3.1. *The reproducing kernel functions $Q_\tau^2(s)$ and $Q_\tau^1(s)$ on the interval $[a,b]$ can be written, respectively, by*

$$Q_\tau^2(s) = \begin{cases} 1+(\tau-a)(s-a)-\frac{1}{6}(\tau-a)^2(2a+\tau-3s), \ s\le\tau \\ 1+(\tau-a)(s-a)-\frac{1}{6}(s-a)^2(2a+s-3\tau), \ s>\tau \end{cases} \tag{10}$$

and

$$Q_\tau^1(s) = \begin{cases} 1-a+\tau, \ s\le\tau \\ 1-a+s, \ s>\tau \end{cases} \tag{11}$$

The lower and upper bounds of the kernel functions Q^1 and Q^2 on $[a,b]$ can be given as in the following theorem:

Theorem 3.2. *The The image of the set $[a,b]$ under the mapping $Q^2 : [a,b]\times[a,b]\to\mathbb{R}$ and $Q^1 : [a,b]\times[a,b]\to\mathbb{R}$ are obtained as $Q^2_{[a,b]}([a,b]) = 1+(b-a)^2\left[\frac{-1}{6}(b-a), 1+\frac{1}{2}(b-a)\right]$ and $Q^1_{[a,b]}([a,b]) = 1+[0,b-a]$, respectively.*

Proof. From the interval analysis theory; the replacement of the two variables s and τ by the corresponding interval variables will gives the following natural inclusion function at $[a,b]$:

$$\begin{aligned} Q^2_{[a,b]}([a,b]) &= 1+([a,b]-a)([a,b]-a)-\frac{1}{6}([a,b]-a)^2(2a+[a,b]-3[a,b]) \\ &= 1+[0,b-a][0,b-a]-\frac{1}{6}[0,b-a]^2[3(a-b),b-a] \\ &= 1+[0,(b-a)^2]-\frac{1}{6}[0,(b-a)^2][3(a-b),b-a] \\ &= 1+[0,(b-a)^2]-\frac{1}{6}[(b-a)^2 3(a-b),(b-a)^2(b-a)] \\ &= 1+[0,(b-a)^2]-\frac{1}{6}[-3(b-a)^3,(b-a)^3] \\ &= 1+[0,(b-a)^2]+\frac{1}{6}[-(b-a)^3,3(b-a)^3] \\ &= 1+\left[0-\frac{1}{6}(b-a)^3,[(b-a)^2+3/6(b-a)]^3\right] \\ &= 1+\left[-\frac{1}{6}(b-a)^3,[(b-a)^2+\frac{1}{2}(b-a)]^3\right] \\ &= 1+(b-a)^2\left[-\frac{1}{6}(b-a),1+\frac{1}{2}(b-a)\right]. \end{aligned} \tag{12}$$

In similar fashion, it follows that $Q^1_{[a,b]}([a,b]) = 1+[0,b-a]$. □

Definition 3.3. The inner product space $\mathcal{W}_2^m[a,b]$ is defined as $\mathcal{W}_2^m[a,b] = \left\{f = (f_1, f_2)^T : f_1, f_2 \in \mathcal{W}_2^m[a,b]\right\}$

The inner product and norm in $\mathcal{W}_2^m[a,b]$ are given, respectively, by $\langle f(\tau), g(\tau)\rangle_{\mathcal{W}^m[a,b]} = \sum_{i=1}^{2}\langle f_i(\tau), g_i(\tau)\rangle_{\mathcal{W}_2^m[a,b]}$ and $\|f\|_{\mathcal{W}^m} = \sqrt{\sum_{i=1}^{2}\|f\|^2_{\mathcal{W}_2^m}}$, $f, g \in \mathcal{W}^m[a,b]$.

4. Analysis of Reproducing Kernel Method

In this section, the solution formulation of model problem (2) is acquired in the $\mathcal{W}^2[a,b]$. First of all, the linear operator $\mathcal{L} : \mathcal{W}^2[a,b] \to \mathcal{W}^1[a,b]$ is described by

$$\mathcal{L}\mathcal{F}_r(\tau) = \mathcal{F}_r(\tau) - \lambda_1 \int_a^\tau h(\tau,\rho)\mathcal{F}_r(\rho)d\rho - \lambda_2 \int_a^b k(\tau,\rho)\mathcal{F}_r(\rho)d\rho = G_r(\tau) \tag{13}$$

where

$$\mathcal{F}_r = \begin{bmatrix} f_{1r} \\ f_{2r} \end{bmatrix}, G_r = \begin{bmatrix} g_{1r} \\ g_{2r} \end{bmatrix}, \mathcal{L} = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}, l_{ij} : \mathcal{W}_2^2[a,b] \to \mathcal{W}_2^1[a,b], i,j = 1,2, \textit{ so that} \tag{14}$$

$$l_{i,j}f(\tau) = \begin{cases} f(\tau) - \int_a^{c_1} h(\tau,\rho)f(\rho)d\rho - \int_a^{c_2} k(\tau,\rho)f(\rho)d\rho, \ i = j \\ \\ -\int_{c_1}^{b} h(\tau,\rho)f(\rho)d\rho - \int_{c_2}^{\tau} k(\tau,\rho)f(\rho)d\rho, \qquad i \neq j \end{cases} \tag{15}$$

From definition of the inner product and norm of the $\mathcal{W}_2^2[a,b]$ and $\mathcal{W}_2^1[a,b]$ as well as by using Schwarz inequality, it is easy to show that $l_{ij} : \mathcal{W}_2^2[a,b] \to \mathcal{W}_2^1[a,b]$, $i,j = 1,2$, is a bounded linear operator.

Lemma 4.1. *The $\mathcal{L} : \mathcal{W}^2[a,b] \to \mathcal{W}^1[a,b]$ is a bounded linear operator.*

Proof. It is clear that $\mathcal{L}$ is a linear operator since l_{ij} is linear for $i,j = 1,2$. For boundedness of $\mathcal{L}$, it follows that

$\|\mathcal{L}\mathcal{F}_r\|_{W^1} = \sqrt{\sum_{i=1}^{2}\|\sum_{i=1}^{2} l_{i,j}f_{j,r}\|^2_{\mathcal{W}_2^1}} \le \sqrt{\sum_{i=1}^{2}\left(\sum_{i=1}^{2}\|l_{i,j}\|\,\|f_{jr}\|_{\mathcal{W}_2^2}\right)^2} \le \sqrt{\sum_{i=1}^{2}\left(\sum_{i=1}^{2}\|l_{i,j}\|^2\right)\left(\sum_{i=1}^{2}\|f_{jr}\|^2_{\mathcal{W}_2^2}\right)}$
$\sqrt{\sum_{i=1}^{2}\sum_{i=1}^{2}\|l_{i,j}\|^2}\|\mathcal{F}_r\|_{\mathcal{W}^2}$, $\mathcal{F}_r \in \mathcal{W}^2[a,b]$ The boundedness of l_{ij}, $i,j = 1,2$ implies that $\mathcal{L}$ is bounded. □

Lemma 4.2. *Let $\{\tau_k\}_{k=1}^\infty$ be a dense subset of $[a,b]$, then $[l_{ij}^* Q_{\tau_k}^1(\eta)] = [l_{ij}Q_\tau^2(s)]\big|_{s=\tau_k}$, $i,j = 1,2$.*

Proof. From the reproducing property of $\mathcal{W}^2[a,b]$, we have $[l_{ij}^* Q_{\tau_k}^1(\eta)] = \langle [l_{ij}^* Q_{\tau_k}^1(s)], Q_\eta^2(s)\rangle_{\mathcal{W}^2} = \langle Q_{\tau k}^1(s), [l_{ij}Q_\eta^2(s)]\rangle_{\mathcal{W}^2} = [l_{ij}Q_\eta^2(\tau_k)] = [l_{ij}Q_\tau^2(s)]\big|_{s=\tau_k}$, where l_{ij}^* is an adjoint operator of l_{ij}.

Let $\{\tau_k\}_{k=1}^\infty$ be a countable dense set on $[a,b]$, put $\psi_{ij}(\tau) = \begin{bmatrix} l_{1j}Q_\tau^2(s)|_{s=t_i} \\ l_{2ij}Q_\tau^2(s)|_{s=t_i} \end{bmatrix}$, $i = 1,2,...,j = 1,2,...$ where the subscript s of l means that the operator$l$ acts on the function of s in which $\psi_{ij} \in \mathcal{W}^2[a,b]$. □

Theorem 4.1. *Let $\{\tau_k\}_{k=1}^\infty$ be be a dense subset of $[a,b]$, then $\left\{\psi_{ij}(\tau)\right\}_{(1,1)}^{(\infty,2)}$ is linearly independent in the $\mathcal{W}^2[a,b]$.*

Proof. Assume that $\{\psi_{i_1j_1}(\tau), \psi_{i_2j_2}(\tau), , \psi_{i_kj_k}(\tau)\}$. Let $\alpha = max\{i_1, i_2, ..., i_k\}$ and put $a_i(s) = Q_{\tau_i}^1(s)$. Since $\{\psi_{ij}(t)\}_{(1,1)}^{(\alpha,2)}$ is linearly dependent, then there exists c_{ij} $(j = 1,2, i = 1,2,...,\alpha)$ of α, are not all zero such that $\sum_{j=1}^{2}\sum_{j=1}^{\alpha} c_{ij}\psi_{ij} = 0$. Thus, for any $\mathcal{F}_r \in \mathcal{W}^2[a,b]$, we have

$\langle \mathcal{L}\mathcal{R}_r(s), \begin{bmatrix} \sum_{i=1}^{\alpha} c_{i1}a_i(s) \\ \sum_{i=1}^{\alpha} c_{i2}a_i(s) \end{bmatrix}\rangle_{\mathcal{W}^1} = \langle \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}\begin{bmatrix} f_{1r}(s) \\ f_{2r}(s) \end{bmatrix}, \begin{bmatrix} \sum_{i=1}^{\alpha} c_{i1}a_i(s) \\ \sum_{i=1}^{\alpha} c_{i2}a_i(s) \end{bmatrix}\rangle_{\mathcal{W}^1} = \langle \begin{bmatrix} \sum_{i=1}^{2} l_{1j}f_{jr}(s) \\ \sum_{i=1}^{2} l_{2j}f_{jr}(s) \end{bmatrix}, \begin{bmatrix} \sum_{i=1}^{\alpha} c_{i1}a_i(s) \\ \sum_{i=1}^{\alpha} c_{i2}a_i(s) \end{bmatrix}\rangle_{\mathcal{W}^1}$

$$= \langle \sum_{j=1}^{2} l_{1j} f_{jr}(s), \sum_{i=1}^{\alpha} c_{i1} a_i(s) \rangle_{W_2^1} + \langle \sum_{j=1}^{2} l_{2ij} f_{jr}(s), \sum_{i=1}^{\alpha} c_{i2} a_i(s) \rangle_{W_2^1}$$

$$= \sum_{j=1}^{2} \sum_{i=1}^{\alpha} \langle l_{1j} f_{jr}(s), c_{i1} a_i(s) \rangle_{W_2^1} + \langle l_{2ij} f_{jr}(s), c_{i2} a_i(s) \rangle_{W_2^1}$$

$$= \sum_{j=1}^{2} \sum_{i=1}^{\alpha} \langle f_{jr}(s), l_{1j}^{*} c_{i1} a_i(s) \rangle_{W_2^2} + \sum_{j=1}^{2} \sum_{i=1}^{\alpha} \langle f_{jr}(s), l_{2j}^{*} c_{i2} a_i(s) \rangle_{W_2^2} = \langle \begin{bmatrix} \sum_{i=1}^{2} f_{jr}(s) \\ \sum_{i=1}^{2} f_{jr}(s) \end{bmatrix}, \begin{bmatrix} \sum_{i=1}^{\alpha} l_{1j}^{*} c_{i1} a_i(s) \\ \sum_{i=1}^{\alpha} l_{2j}^{*} c_{i2} a_i(s) \end{bmatrix} \rangle_{W^2}$$

$$= \langle \begin{bmatrix} f_{1r}(s) \\ f_{2r}(s) \end{bmatrix}, \begin{bmatrix} \sum_{j=1}^{2} \sum_{i=1}^{\alpha} l_{1j}^{*} c_{i1} a_i(s) \\ \sum_{j=1}^{2} \sum_{i=1}^{\alpha} l_{2j}^{*} c_{i2} a_i(s) \end{bmatrix} \rangle_{W^2} = \langle \mathcal{F}_r(s), \sum_{j=1}^{2} \sum_{i=1}^{\alpha} c_{1j} \psi_{ij} \rangle_{W^2} = 0.$$

Consequently, $\sum_{j=1}^{2} \sum_{i=1}^{\alpha} c_{1j} a_i(s) = 0$. Now, let the function $\varphi_k(\tau) \in W_2^1$ be defined as,

$$\varphi_k(\tau) = \begin{cases} 1, & \tau = \tau_k \\ 0, & \tau = \tau_i, i \neq k, i = 1, 2, ..., \alpha, \\ \mu_k(\tau), & \tau = \tau_i, i \neq k, i > \alpha, \end{cases} \tag{16}$$

where $\mu_k(\tau)$ is non-zero linear function. Thus, $c_{kj} = \langle \varphi_k(s), \sum_{i=1}^{\alpha} c_{ij} a_i(s) \rangle_{W_2^1} = 0$, $j = 1, 2, k = 1, 2, 3, ..., \alpha$. Which means a contradiction; $\{\psi_{ij}(t)\}_{(1,1)}^{(\infty,2)}$ is linearly dependent in the $W^2[a, b]$. □

Theorem 4.2. *The set $\left\{\psi_{ij}(\tau)\right\}_{(1,1)}^{(\infty,2)}$ is a complete function system in the $W^2[a, b]$.*

Proof. For each $\mathcal{F}_r \in W^2[a, b]$, we have $\langle \mathcal{F}_r(\tau), \psi_{ij}(\tau) \rangle_{W^2} = \langle \begin{bmatrix} f_{1r}(\tau) \\ f_{2r}(\tau) \end{bmatrix}, \begin{bmatrix} l_{j1} Q_t^2(s)|_{s=t_k} \\ l_{j2} Q_t^2(s)|_{s=t_k} \end{bmatrix} \rangle_{W^2} =$

$$= \langle f_{1r}(\tau), l_{j1} Q_t^2(s)|_{s=t_k} \rangle_{W_2^2} + \langle f_{2r}(\tau), l_{j2} Q_t^2(s)|_{s=t_k} \rangle_{W_2^2} \tag{17}$$

$$= \langle f_{1r}(\tau), [l_{j1}^{*} Q_{t_i}^1(\tau)] \rangle_{W_2^2} + \langle f_{2r}(\tau), [l_{j2}^{*} Q_{\tau_i}^1(\tau)] \rangle_{W_2^2} = l_{j1} f_{1r}(\tau_i) + l_{j1} f_{2r}(\tau_i).$$

Since $\{\tau_i\}_{i=1}^{\infty}$, then $l_{11} f_{1r}(\tau) + l_{12} f_{2r}(\tau) = 0$ and $l_{21} f_{2r}(\tau) + l_{22} f_{2r}(\tau) = 0$.Hence, $\mathcal{F}_r(\tau) = \begin{bmatrix} f_{1r}(\tau) \\ f_{2r}(\tau) \end{bmatrix} = 0$

Let $\xi_{2(i-1)+j}(\tau) = \psi_{ij}(\tau)$ for $i = 1, 2, ..., j = 1, 2$, then the sequence $\left\{\psi_{ij}(\tau)\right\}_{(1,1)}^{(\infty,2)}$ can be arranged as $\left\{\psi_{11}(\tau), \psi_{12}(\tau), \psi_{21}(\tau), \psi_{22}(\tau), ...\right\} \equiv \left\{\xi_1(\tau), \xi_2(\tau), \xi_3(\tau), ...\right\}$. Anyhow, the orthonormal system $\left\{\overline{\xi_i}(\tau)\right\}_{i-1}^{\infty}$ can be derived from Gram-Schmidt process of $\left\{\xi_i(\tau)\right\}_{i-1}^{\infty}$ by

$$\overline{\xi_i}(\tau) = \sum_{k=1}^{i} \beta_{ik} \xi_k(\tau), \qquad i = 1, 2, ..., \tag{18}$$

where β_{ik} are orthogonalization coefficients given by $\beta_{11} = \|\xi_1\|_{W^2}^{-1}$, $\beta_{ij} = (\Delta_i)^{-\frac{1}{2}} \sum_{k=1}^{i} (-b_{ik} \beta_{kj})$ for $i > j$, $\Delta_i = \left(\|\xi_i\|_{W^2}^2 - \sum_{k=1}^{i-1} b_{ik}^2\right)$ and $b_{ik} = \langle \xi_i, \overline{\xi}_k \rangle_{W^2}$. □

Theorem 4.3. *If $\mathcal{F}_r(\tau)$ is the solution of Eq.* (2) *is unique, then*

$$\mathcal{F}_r(\tau) = \sum_{i=1}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik} \sigma_k \right) \overline{\xi}_k(\tau) \tag{19}$$

where $\left\{\tau_i\right\}_{i=1}^{\infty}$ is dense in $[a, b]$ and $\sigma_k = \langle \mathcal{F}_r(\tau), \xi_k(\tau) \rangle_{W^2} = \begin{cases} f_{1r}(\tau_{\gamma_o}), & \gamma_o = \frac{k+1}{2}; k \text{ is odd}, \\ f_{2r}(\tau_{\gamma_e}), & \gamma_e = \frac{k}{2}; k \text{ is even}, \end{cases}$

Proof. From Theorem 4.2, we acquire that $\{\overline{\xi}_i\}_{i=1}^{\infty}$ is the complete orthonormal basis of $\mathcal{W}^2[a,b]$. Let $\mathcal{F}_r(\tau)$ be the exact solution of Eq. (2), then $\mathcal{F}_r(\tau)$ can be expanded as a Fourier series in terms of the complete orthonormal basis $\{\overline{\xi}_i\}_{i=1}^{\infty}$ of the space $\mathcal{W}^2[a,b]$ as,

$$\mathcal{F}_r(\tau) = \sum_{i=1}^{\infty} \langle \mathcal{F}_r(\tau), \xi_k(\tau)\rangle_{\mathcal{W}^2} \overline{\xi}_k(\tau) \tag{20}$$

Based on Eq. (18), $\langle \mathcal{F}_r(\tau), \xi_k(\tau)\rangle_{\mathcal{W}^2}$ can be written by

$$\langle \mathcal{F}_r(\tau), \xi_k(\tau)\rangle_{\mathcal{W}^2} = \langle \mathcal{F}_r(\tau), \sum_{k=1}^{i} \beta_{ik}\xi_k(\tau)\rangle_{\mathcal{W}^2} = \sum_{k=1}^{i} \beta_{ik} \langle \mathcal{F}_r(\tau), \xi_i(\tau)\rangle_{\mathcal{W}^2} = \sum_{k=1}^{i} \beta_{ik}\alpha_k$$

$$= \sum_{k \text{ is odd}}^{i} \beta_{ik} f_{1r}(\tau_{\gamma_o}) + \sum_{k \text{ is even}}^{i} \beta_{ik} f_{2r}(\tau_{\gamma_e}).$$

□

Remark 4.1. The approximate solution $\mathcal{F}_r^n(\tau) = \begin{bmatrix} f_{1r}^n(\tau) \\ f_{2r}^n(\tau) \end{bmatrix}$ of Eq. (2) can be obtained directly by truncating the infinite series representation form of $\mathcal{F}_r(\tau) = \begin{bmatrix} f_{1r}(\tau) \\ f_{2r}(\tau) \end{bmatrix}$ as follows

$$\mathcal{F}_r^n(\tau) = \sum_{i=1}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik}\sigma_k \right) \overline{\xi}_k(\tau). \tag{21}$$

Theorem 4.4. *The approximate solution $\mathcal{F}_r^n(\tau)$ of Eq. (2) is converges uniformly to the exact solution $\mathcal{F}_r(\tau)$ as $n \to \infty$.*

Proof. By the expression form of the reproducing kernel function $Q_\tau^2(s)$, one can write

$$|f_{ir}^n(\tau) - f_{ir}(\tau) = |\langle f_{ir}^n(s) - f_{ir}(s), Q_\tau^2(s)\rangle| \leq \|f_{ir}^n - f_{ir}\|_{\mathcal{W}_2^2} \|Q_\tau^2\|_{\mathcal{W}_2^2} \leq \sqrt{Q_\tau^2(\tau)} \|f_{ir}^n - f_{ir}\|_{\mathcal{W}_2^2} \leq M_0 \|f_{ir}^n - f_{ir}\|_{\mathcal{W}_2^2}, \tag{22}$$

where $M_0 \in \mathbb{R}$.

Thereby $f_{ir}^n(\tau)$ converges uniformly to $f_{ir}(\tau)$, $i = 1, 2$ on $[a,b]$ as $n \to \infty$. Hence, $\mathcal{F}_r^n(\tau)$ converges uniformly to $\mathcal{F}_r(\tau)$ as $n \to \infty$. □

Corollary 4.1. *Let $e_n = \|\mathcal{F}_r - \mathcal{F}_r^n\|_{\mathcal{W}^2[a,b]}$. Then, the sequence $\{e_n\}$ is decreasing in the sense of the $\|.\|_{\mathcal{W}^2[a,b]}$ such that $e_n \to 0$ as soon as $n \to \infty$.*

Proof. From Eq. (19) and Eq. (21), we have

$$e_n^2 = \left\| \sum_{i=n+1}^{\infty} \langle \mathcal{F}_r(\tau), \overline{\xi}_i(\tau)\rangle_{\mathcal{W}^2} \overline{\xi}_i(\tau) \right\|_{\mathcal{W}^2[a,b]}^2 = \sum_{i=n+1}^{\infty} \langle \mathcal{F}_r(\tau), \overline{\xi}_i(\tau)\rangle_{\mathcal{W}^2[a,b]}^2,$$

$$e_{n-1}^2 = \left\| \sum_{i=n}^{\infty} \langle \mathcal{F}_r(\tau), \overline{\xi}_i(\tau)\rangle_{\mathcal{W}^2} \overline{\xi}_i(\tau) \right\|_{\mathcal{W}^2[a,b]}^2 = \sum_{i=n}^{\infty} \langle \mathcal{F}_r(\tau), \overline{\xi}_i(\tau)\rangle_{\mathcal{W}^2[a,b]}^2. \tag{23}$$

Consequently, it obvious that $e_{n-1} \geq e_n$ and the sequence $\{e_n\}$ is decreasing in the sense of $\|.\|_{\mathcal{W}^2[a,b]}$. On the other hand, since $\| \sum_{i=1}^{\infty} \langle \mathcal{F}_r(\tau), \overline{\xi}_i(\tau)\rangle_{\mathcal{W}^2} \overline{\xi}_i(\tau)$ is convergent, then $e_n^2 = \sum_{i=n+1}^{\infty} \langle \mathcal{F}_r(\tau), \overline{\xi}_i(\tau)\rangle_{\mathcal{W}^2[a,b]}^2 \to 0$ as soon as $n \to \infty$. □

5. Numerical Results

In this section, the RKHS method is applied for solving the linear fuzzy integral equation. For the purpose of comparison, the method is compared systematically with the methods used in [8, 24, 33]. The results are very encouraging and are found outperforms in terms of accuracy and efficiency. In the process of computation, all the symbolic and numerical computations are performed by using MAPLE 13 software package.

Algorithm 5.1 To approximate the solution $\mathcal{F}^n(\tau)$ of Eq. (2), we do the following main steps:

Input: The dependent interval [a,b], the unit truth interval $[0,1]$, the integers n,m, the kernel functions $Q^1_\tau(s)$, $Q^2_\tau(s)$, the linear operator L, the crisp kernel functions $h(\tau,\rho)$, $k(\tau,\rho)$, and the nonhomogeneous term $g(\tau)$.

Output: The RKHS solution $\mathcal{F}^n_r(\tau)$ of $\mathcal{F}_r(\tau)$ for Eq. (2).

Step 1: Set $\mathcal{F}_r(\tau) = \begin{bmatrix} f_{1r}(\tau) \\ f_{2r}(\tau) \end{bmatrix}$ and $\mathcal{F}^n_r(\tau) = \begin{bmatrix} f^n_{1r}(\tau) \\ f^n_{2r}(\tau) \end{bmatrix}$.

Step 2: Fixed τ *in* $[a,b]$ and set $s \in [a,b]$;
If $s \leq \tau$, then let $Q^2_\tau(s) = 1 + (\tau - a)(s - a) - \frac{1}{6}(\tau - a)^2(2a + \tau - 3s)$;
Else let $Q^2_\tau(s) = 1 + (\tau - a)(s - a) - \frac{1}{6}(s - a)^2(2a + s - 3\tau)$;
For $i = 1, 2, ..., n$, $h = 1, 2, ..., m$, and $j = 1, 2$, do the following:

$$Set \ \ \tau_i = \frac{i-1}{n-1};$$

$$Set \ \ r_h = \frac{h-1}{m-1};$$

$$Set \ \ \psi_{ij}(\tau_i) = \begin{bmatrix} [l_{1j}Q^2_\tau(s)]|_{s=t_i} \\ [l_{2j}Q^2_\tau(s)]|_{s=t_i} \end{bmatrix};$$

Output: the complete orthogonal system $\psi_{ij}(\tau_i)$.

Step 3: For $l = 2, 3..., n$ and $k = 1, 2..., l$, do the following:.

$$Set \ \ \eta_l = \sqrt{||\xi_l||^2_{\mathcal{W}^2} - \sum_{p=1}^{l-1} c^2_{lp}};$$

$$Set \ \ c_{lk} = \langle \xi_l, \overline{\xi}_l \rangle_{\mathcal{W}^2};$$

$$If \ \ k \neq l, \ \ then \ set \ \ \beta_{lk} = -\frac{1}{\eta_l} \sum_{p=k}^{i-1} c_{lp}\beta_{pk};$$

$$Else \ \ set \ \ \beta_{ll} = \eta_l^{-1};$$

$$Else \ \ set \ \ \beta_{11} = ||\xi_1||^{-1}_{\mathcal{W}^2};$$

Output: the orthogonalization coefficients β_{lk}.

Step 4: For $l = 2, 3..., n$ and $k = 1, 2..., l$, do the following:

$$Set \ \ \overline{\xi}_i(\tau_i) = \sum_{k=1}^{l} \beta_{lk}\xi_k(\tau_i);$$

Output: the orthonormal function system $\overline{\xi}_l(\tau_i)$.

Step 5: Set $\mathcal{F}^0_{r_h}(\tau_1) = 0$; For $i = 1, 2, ..., n$, do the following:

$$Set \quad \sigma_k = \begin{cases} f_{1r}(\tau_{\gamma_o}), & \gamma_o = \frac{k+1}{2}; k \ \ is \ \ odd \\ f_{2r}(\tau_{\gamma_e}), & \gamma_e = \frac{k}{2}; k \ \ is \ \ even, \end{cases}$$

$$Set \quad \mathcal{F}^n_{r_h}(\tau_i) = \sum_{i=1}^{n} \Big(\sum_{k=1}^{i} \beta_{ik}\sigma_k \Big) \overline{\xi}_i(\tau_i);$$

Output: the RKHS solution $\mathcal{F}^n_{r_h}(\tau)$ of $\mathcal{F}_r(\tau)$ for Eq. (2).

Example 5.1. ([8]) Consider the following linear fuzzy integral equation of Volterra type:

$$\mathcal{F}(\tau) = (\gamma + 1)\tau + \tau^2\varphi(\tau) + \int_0^\tau \tau^2(1 - 2\rho)\mathcal{F}(\rho)d\rho, 0 \le \rho < \tau \le 1, \tag{24}$$

where $\gamma \in \mathcal{R}_F$, $\varphi(\tau)$ is continuous fuzzy-valued function such that

$$[\varphi(\tau)]^r = \Big[-\frac{2}{3}r\tau^3 + \frac{4}{3}\tau^3 + \frac{1}{2}r\tau^2 - \tau^2 + \frac{1}{12} - \frac{1}{12}r, \frac{2}{3}r\tau^3 - \frac{1}{2}r\tau^2 + \frac{1}{12}r - \frac{1}{12}\Big], \ \ r \in (0, 1)$$

The corresponding exact solution is $\mathcal{F}(\tau) = (\gamma + 1)\tau$, where $\gamma(s) = max(0, 1 - |s|)$.
If we set $r = 1 - |s|$, that is, $s = r - 1$ or $s = 1 - r$, then $[\gamma]^r = [r - 1, 1 - r]$. The crisp kernel function $k(\tau, \rho) = \tau^2(1 - 2\rho)$ is nonnegative on $0 \le \rho \le \frac{1}{2}$, and nonpositive on $\frac{1}{2} \le \rho \le 1$. Anyhow, at the full truth degree, $r = 1$, the model (24) will be transformed into crisp integral equation

$$\mathcal{F}(\tau) = \frac{2}{3}\tau^5 - \frac{1}{2}\tau^4 + \tau + \int_0^\tau \tau^2(1 - 2\rho)\mathcal{F}(\rho)d\rho$$

which has the exact solution $\mathcal{F}(\tau) = \tau$. Consequently, the corresponding parametric form of Eq. (24) can be written by the following system of crisp integral equations

$$\mathcal{F}(\tau) = r\tau + \tau^2\bigg(-\frac{2}{3}r\tau^3 + \frac{4}{3}\tau^3 + \frac{1}{2}r\tau^2 - \tau^2 + \frac{1}{12} - \frac{1}{12}r \bigg) + \int_0^{\frac{1}{2}} \tau^2(1-2\rho)f_{1r}(\rho)d\rho + \int_{\frac{1}{2}}^{t} \tau^2(1-2\rho)f_{2r}(\rho)d\rho \tag{25}$$

$$\mathcal{F}(\tau) = (2 - r)\tau + \tau^2\bigg(\frac{2}{3}r\tau^3 - \frac{1}{2}r\tau^2 + \frac{1}{12}r - \frac{1}{12} \bigg) + \int_0^{\frac{1}{2}} \tau^2(1 - 2\rho)f_{2r}(\rho)d\rho + \int_{\frac{1}{2}}^{t} \tau^2(1 - 2\rho)f_{1r}(\rho)d\rho \tag{26}$$

Using RKHS method, taking $\tau_i = \frac{i-1}{n-1}$ $i = 1, 2, , n$ and $r_h = \frac{h-1}{m-1}, h = 1, 2, , m$.
Our next goal is to illustrate numerical results of the RKHS algorithm. Table 1 present the approximate solutions of problem (24) for $\tau = 1$ and different values of r. From this table, one can see that the results obtained at $\tau = 1$ satisfy the convex symmetric triangular fuzzy number due to fuzzy coefficients appeared in problem (24), as well as the branch solutions f_{1r}, f_{2r} satisfy the inequality $f_{1r}(1) < f_{2r}(1)$ for $r \in (0, 1)$ and equality hold for $r = 1$. Anyhow, a homotopy perturbation HP [13] and analysis HA [7] methods are applied to problem (24). Our results are better than the results obtained by these methods. Table 2 shows a numerical comparison between the errors for approximating $[\mathcal{F}(\tau)]^r = [f_{1r}(\tau), f_{2r}(\tau)]$ with the HP method [40] and the HA method [41] in the sense of the metric $d_\infty(\mathcal{F}(\tau), \mathcal{F}^n(\tau))$. For the HA and HP solutions, one can conclude that, the accuracy of certain node is inversely proportional to its distance from the endpoint of $[0, 1]$, while on the other hand, a steady state accuracy is notable in the RKHS solution at all nodes of $[0, 1]$.

Table 1: The fuzzy exact and approximate solutions of Example 5.1 at $\tau = 1$.

r_h	Exact solution	RKHS solution
0	$[0.0, 0.2]$	$[0.000000000899646, 2.000000000820322]$
0	$[0.2, 1.8]$	$[0.200000000855965, 1.800000000880739]$
0	$[0.4, 1.6]$	$[0.400000000781364, 1.600000000809896]$
0	$[0.6, 1.4]$	$[0.600000000870276, 1.400000000722733]$
0	$[0.8, 1.2]$	$[0.800000000865688, 1.200000000812860]$
0	$[1.0, 1.0]$	$[1.000000000821667, 1.000000000821667]$

Table 2: Numerical comparison using the d_∞ metric errors of Example 5.1.

τ_i	HA method	HP method	RKHS method
0	0	0	2.8084×10^{-10}
0.1	1.7602×10^{-7}	1.7602×10^{-7}	3.0989×10^{-10}
0.2	6.3435×10^{-6}	6.3435×10^{-6}	4.6403×10^{-10}
0.3	1.1757×10^{-6}	1.1757×10^{-6}	5.4640×10^{-10}
0.4	1.7298×10^{-6}	1.7298×10^{-6}	5.9027×10^{-10}
0.5	2.5140×10^{-6}	2.5140×10^{-6}	6.0523×10^{-10}
0.6	4.1488×10^{-6}	4.1488×10^{-6}	7.7663×10^{-10}
0.7	1.2790×10^{-5}	1.2790×10^{-5}	7.8551×10^{-10}
0.8	6.8554×10^{-5}	6.8554×10^{-5}	8.0842×10^{-10}
0.9	4.0759×10^{-4}	4.0759×10^{-4}	1.2810×10^{-9}
1	2.6043×10^{-3}	2.6043×10^{-3}	3.5993×10^{-9}

Example 5.2. ([33]) Consider the following linear fuzzy integral equation of Fredholm-Volterra type:

$$\mathcal{F}(\tau) = -\frac{1}{3}\tau^3 + \tau^2 - \frac{1}{3}\tau - \frac{1}{4} + \gamma - 4\gamma\tau + \int_0^1 (\rho + \tau)\mathcal{F}(\rho)d\rho + \int_0^\tau \mathcal{F}(\rho)d\rho, 0 \leq \rho < \tau \leq 1, \tag{27}$$

The corresponding exact solution is $\mathcal{F}(\tau) = \tau^2 + 2\gamma$, where $\gamma(s) = \begin{cases} 10s - 9, & 0.9 \leq s \leq 1, \\ 5 - 4s, & 1 \leq s \leq 1.25, \\ 0, & o.w. \end{cases}$ Here, if we set $r = 10s - 9$, that is, $s = 0.9 + 0.1r$ and if we set $r = 5 - 4s$, that is, $s = 1.25 - 0.25r$, then $[\gamma]^r = [0.9 + 0.1r, 1.25 - 0.25r]$. For the conduct of proceedings in the RKHS solution, we have the following system of crisp integral equations

$$f_{1r}(t) = -\frac{1}{3}\tau^3 + \tau^2 - \frac{1}{3}r - \frac{1}{4} + (0.9 + 0.1r)\tau - 4(1.25 - 0.25r) + \int_0^1 (\rho + \tau)f_{1r}(\rho)d\rho + \int_0^t f_{1r}(\rho)d\rho,$$

$$f_{2r}(t) = -\frac{1}{3}\tau^3 + \tau^2 - \frac{1}{3}r - \frac{1}{4} + (1.25 - 0.25r)\tau - 4(0.9 + 0.1r) + \int_0^1 (\rho + \tau)f_{2r}(\rho)d\rho + \int_0^t f_{2r}(\rho)d\rho. \tag{28}$$

As we mentioned earlier, it is possible to pick any point in the independent interval of t and as well the RKHS solution will be applicable in the whole solution domain. Next, the absolute errors of numerically approximating $[\mathcal{F}(\tau)]^r$ for FIE (27) have been calculated for various τ and r as shown in Tables 3 and 4. In fact, results obtained by considering the corresponding crisp system of Eq. (28).

Table 2: The absolute errors of approximating $f_{1r}(t)$ of Example 5.2.

τ_i	HA method	HP method	RKHS method
0	0	0	2.8084×10^{-10}
0.1	1.7602×10^{-7}	1.7602×10^{-7}	3.0989×10^{-10}
0.2	6.3435×10^{-6}	6.3435×10^{-6}	4.6403×10^{-10}
0.3	1.1757×10^{-6}	1.1757×10^{-6}	5.4640×10^{-10}
0.4	1.7298×10^{-6}	1.7298×10^{-6}	5.9027×10^{-10}
0.5	2.5140×10^{-6}	2.5140×10^{-6}	6.0523×10^{-10}
0.6	4.1488×10^{-6}	4.1488×10^{-6}	7.7663×10^{-10}
0.7	1.2790×10^{-5}	1.2790×10^{-5}	7.8551×10^{-10}
0.8	6.8554×10^{-5}	6.8554×10^{-5}	8.0842×10^{-10}
0.9	4.0759×10^{-4}	4.0759×10^{-4}	1.2810×10^{-9}
1	2.6043×10^{-3}	2.6043×10^{-3}	3.5993×10^{-9}

Table 3. The absolute errors of approximating $f_{1r}(t)$ using RKHS method for Example 5.2.

τ_i	$r_h=0$	$r_h=0.25$	$r_h=0.5$	$r_h=0.75$	$r_h=1$
$\frac{1}{\sqrt{7}}$	2.3698×10^{-11}	1.6712×10^{-11}	1.3461×10^{-11}	2.6078×10^{-11}	2.6634×10^{-11}
$\frac{1}{\sqrt{5}}$	3.5145×10^{-11}	2.4452×10^{-10}	1.7444×10^{-11}	2.3188×10^{-11}	3.3969×10^{-11}
$\frac{1}{\sqrt{3}}$	5.2065×10^{-10}	3.5513×10^{-10}	2.1567×10^{-10}	5.6672×10^{-10}	5.9946×10^{-10}
$\frac{1}{\sqrt{2}}$	1.1364×10^{-9}	7.2207×10^{-9}	2.7781×10^{-9}	9.9933×10^{-10}	1.1044×10^{-10}

Table 4. The absolute errors of approximating $f_{2r}(t)$ using RKHS method for Example 5.2.

τ_i	$r_h=0$	$r_h=0.25$	$r_h=0.5$	$r_h=0.75$	$r_h=1$
$\frac{1}{\sqrt{7}}$	6.4841×10^{-11}	5.4277×10^{-11}	1.9599×10^{-11}	5.0528×10^{-11}	5.2267×10^{-11}
$\frac{1}{\sqrt{5}}$	1.3870×10^{-10}	2.8995×10^{-10}	8.5245×10^{-10}	5.7603×10^{-11}	1.6359×10^{-11}
$\frac{1}{\sqrt{3}}$	3.6264×10^{-10}	1.4222×10^{-10}	1.6899×10^{-10}	7.2636×10^{-10}	4.5594×10^{-10}
$\frac{1}{\sqrt{2}}$	4.1949×10^{-9}	2.2316×10^{-9}	1.9496×10^{-9}	3.2817×10^{-9}	7.8067×10^{-10}

Finally, the approximate solution set $[\mathcal{F}(\tau)]^r$ of Eq. (27) has been plotted in Figure 1.*a* at various values of τ and r. As the plots show, while the value of r increases on $[0,1]$, the set $[\mathcal{F}(\tau)]^r$ decreasing for each τ; that is, the r-cut representation form of the RKHS solution is valid level sets on the specific domain. Whilst on the other aspect as well, the geometric behavior of the support and the core of the RKHS solution is studied in Figure 1.*b*. Whereas the set of elements in $\mathbb{R}$ that have one membership in $\mathcal{F}(\tau)$ is called the core, denoted by $(\mathcal{F}(\tau)) = \{s \in \mathbb{R} : \mathcal{F}(s) = 1\}$, whilst the set of elements in $\mathbb{R}$ that have membership in between $(0,1]$ is called the support, denoted by $support\big(\mathcal{F}(\tau)\big) = \{s \in \mathbb{R} : \mathcal{F}(s) > 0\}$. Indeed, these sets associated with 1-cut and 0-cut representation form of $[\mathcal{F}(\tau)]^r$, respectively, that is, $core\big(\mathcal{F}(\tau)\big) = [\mathcal{F}(\tau)]^1$ and $support\big(\mathcal{F}(\tau)\big) = \big(min[\mathcal{F}(\tau)]^0, max[\mathcal{F}(\tau)]^0\big)$. The scenario of Figure 1.*b* is to plot the core and the support of the RKHS solution for various τ.

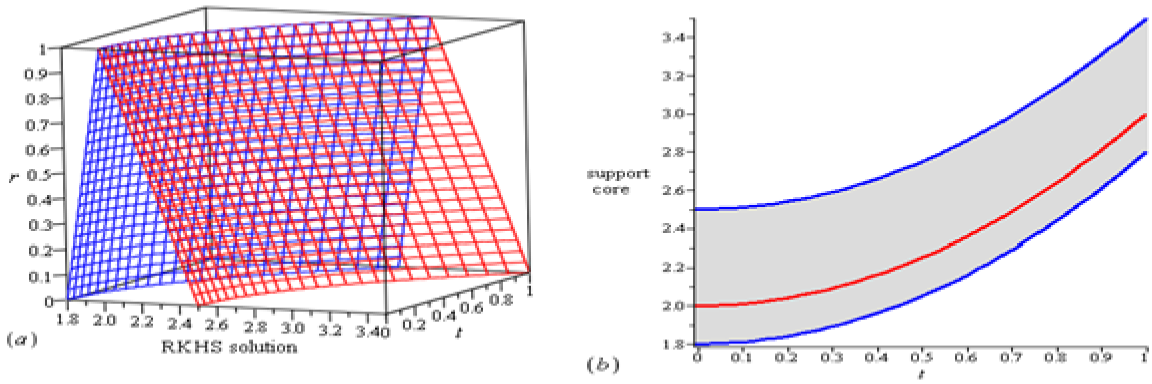

Figure 1: Geometric behavior for fuzzy approximate solution of Example 5.2 **(a)** The RK solution in terms of r-cut representation form (blue is the left branch solution; red is the right branch solution) **(b)** The support and the core of the RK solution (gray is support and red is core).

6. Conclusion

In this paper, we studied a numerical algorithm to acquire solution for fractional fuzzy integral equation of Fredholm-Voltera type based on the reproducing kernel method. The analytic and numeric solutions are represented in series form in term of their parametric form in the space $\mathcal{W}^2[a,b]$. The convergence analysis and error estimation is presented to capture the behavior of s solution for the fuzzy-valued mappings. Numerical results, compared with other methods, indicate that the proposed method is of higher precision and is easy to implement to get these results.

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References

[1] O. Abu Arqub, M. Al-Smadi, Numerical algorithm for solving two-point, second-order periodic boundary value problems for mixed integro-differential equations, Appl. Math. Comput. 243 (2014) 911–922.
[2] O. Abu Arqub, M. Al-Smadi, Numerical algorithm for solving time-fractional partial integrodifferential equations subject to initial and Dirichlet boundary conditions, Numerical Meth. Partial Differential Eq. 34 (2017) 1577–1597.
[3] O. Abu Arqub, M. Al-Smadi, Atangana-Baleanu fractional approach to the solutions of Bagley-Torvik and Painlevé equations in Hilbert space, Chaos Solitons Fractals 117 (2018) 161–167.
[4] O. Abu Arqub, M. Al-Smadi, S. Momani, T. Hayat, Numerical Solutions of Fuzzy Differetial Equations using Reproducing Kernel Hilbert Space Method, Soft Comput. 20 (2016) 3283–3302.
[5] O. Abu Arqub, M. Al-Smadi, S. Momani, T. Hayat, Application of reproducing kernel algorithm for solving second-order, two-point fuzzy boundary value problems, Soft Comput. 21 (2017) 7191–7206.
[6] O. Abu Arqub, M. Al-Smadi, N. Shawagfeh, Solving Fredholm integro-differential equations using reproducing kernel Hilbert space method, Appl. Math. Comput. 219 (2013) 8938–8948.
[7] O. Abu Arqub, Z. Odibat, M. Al-Smadi, Numerical solutions of timefractional partial integrodifferential equations of Robin functions types in Hilbert space with error bounds and error estimates, Nonlinear Dynamics 94 (2018) 1819–1834.
[8] T. Allahviranloo, M. Khezerloo, M. Ghanbari, S. Khezerloo, The homotopy perturbation method for fuzzy Volterra integral equations, Internat. J. Comput. Cognition 8 (2010) 31–37.
[9] M. Al-Smadi, Simplified iterative reproducing kernel method for handling time-fractional BVPs with error estimation, Ain Shams Engin. J., in press (doi.org/10.1016/j.asej.2017.04.006).
[10] M. Al-Smadi, O. Abu Arqub, Computational algorithm for solving Fredholm time fractional partial integrodifferential equations of Dirichlet functions type with error estimates, Appl. Math. Comput. 342 (2019) 280–294.
[11] M. Al-Smadi, O. Abu Arqub, A. El-Ajou, A numerical iterative method for solving systems of first-order periodic boundary value problems, J. Appl. Math. 2014 (2014), Article ID 135465, 10 pages.

[12] M. Al-Smadi, O. Abu Arqub, N. Shawagfeh, S. Momani, Numerical investigations for systems of second-order periodic boundary value problems using reproducing kernel method, Appl. Math. Comput. 291 (2016) 137–148.

[13] M. Al-Smadi, A. Freihat, H. Khalil, S. Momani, R.A. Khan, Numerical multistep approach for solving fractional partial differential equations, Internat. J. Comput. Methods 14 (2017), 1750029.

[14] G.A. Anastassiou, Fuzzy Mathematics: Approximation Theory, Springer-Verlag, Berlin, Heidelberg, 2010.

[15] E. Babolian, H.Sadeghi Goghary, S. Abbasbandy, Numerical solution of linear Fredholm fuzzy integral equations of the second kind by Adomian method, Applied Mathematics and Computation, 161(3) (2005) 733–744.

[16] M. Barkhordari Ahmadi, M. Khezerloo, Fuzzy bivariate Chebyshev method for solving fuzzy Volterra-Fredholm integral equations, Internat. J. Industrial Math. 3(2) (2011) 67–77.

[17] M. Cui, Y. Lin, Nonlinear Numerical Analysis in the Reproducing Kernel Space, Nova Science, New York, NY, USA, 2009.

[18] R. Ezzati, S.M. Sadatrasou, On numerical solution of two-dimensional nonlinear Urysohn fuzzy integral equations based on fuzzy Haar wavelets, Fuzzy Sets Syst. 309 (2017) 145–164.

[19] R. Ezzati, S. Ziari, Numerical solution of nonlinear fuzzy Fredholm integral equations using iterative method, Appl. Math. Comput. 225 (2013) 33–42.

[20] O.S. Fard, M. Sanchooli, Two successive schemes for numerical solution of linear fuzzy Fredholm integral equations of the second kind, Australian J. Basic Appl. Sci. 4 (2010) 817–825.

[21] M.A. Fariborzi Araghi, N. Parandin, Numerical solution of fuzzy Fredholm integral equations by the Lagrange interpolation based on the extension principle, Soft Comput. 15 (2011) 2449–2456.

[22] M. Friedman, M. Ma, A. Kandel, Numerical solution of fuzzy differential and integral equations, Fuzzy Sets Syst. 106 (1999) 35–48.

[23] F. Geng, S.P. Qian, Reproducing kernel method for singularly perturbed turning point problems having twin boundary layers, Appl. Math. Letters 26 (2013) 998–1004.

[24] M. Ghanbari, Numerical solution of fuzzy linear Volterra integral equations of the second kind by homotopy analysis method, Internat. J. Industrial Math. 2 (2010) 73–87.

[25] R. Goetschel, W. Voxman, Elementary fuzzy calculus, Fuzzy Sets Syst. 18 (1986) 31–43.

[26] H.S. Goghary, M.S. Goghary, Two computational methods for solving linear Fredholm fuzzy integral equation of the second kind, Appl. Math. Comput. 182 (2006) 791–796.

[27] G. Gumah, K. Moaddy, M. Al-Smadi, I. Hashim, Solutions to uncertain Volterra integral equations by fitted reproducing kernel Hilbert space method, J. Function Spaces 2016 (2016), Article ID 2920463, 11 pages.

[28] J. Hamaydi, N. Qatanani, Computational Methods for Solving Linear Fuzzy Volterra Integral Equation, J. Appl. Math. 2017 (2017), Article ID 2417195, 12 pages.

[29] O. Kaleva, Fuzzy differential equations, Fuzzy Sets Syst. 24 (1987) 301–317.

[30] F. Mirzaee, Numerical solution of Fredholm fuzzy integral equations of the second kind using hybrid of block-pulse functions and Taylor series, Ain Shams Engin. J. 5 (2014) 631–636.

[31] F. Mirzaee, M.K. Yari, E. Hadadiyan, Numerical solution of two-dimensional fuzzy Fredholm integral equations of the second kind using triangular functions, Beni-Suef Univ. J. Basic Appl. Sci. 4 (2015) 109–118.

[32] A. Molabahrami, A. Shidfar, A. Ghyasi, An alalytical method for solving linear Fredholm fuzzy integral equations of the second kind, Computers and Mathematics with Applications 61 (2011) 2754–2761.

[33] Z. Mosayebi, Numerical solution of fuzzy linear Volterra-Fredholm-Hammerestein integral equations via collocation method based on radial basis functions, J. Fuzzy Set Valued Anal. 2014 (2014) 1–10.

[34] N. Parandin, M.A. Araghi, The numerical solution of linear fuzzy Fredholm integral equations of the second kind by using finite and divided differences methods, Soft Comput. 15 (2010) 729–741.

[35] R. Saadeh, M. Al-Smadi, G. Gumah, H. Khalil, R.A. Khan, Numerical investigation for solving two-point fuzzy boundary value problems by reproducing kernel approach, Appl. Math. Inform. Sci. 10 (2016) 2117–21293.

[36] S.M. Sadatrasoul, R. Ezzati, Iterative method for numerical solution of two-dimensional nonlinear fuzzy integral equations, Fuzzy Sets and Systems 280 (2015) 91–106.

[37] S.M. Sadatrasoul, R. Ezzati, Numerical solution of two-dimensional nonlinear Hammerstein fuzzy integral equations based on optimal fuzzy quadrature formula, J. Comput. Appl. Math. 292 (2016) 430–446.

[38] S. Salahshour, T. Allahviranloo, Application of fuzzy differential transform method for solving fuzzy Volterra integral equations, Appl. Math. Modelling 37 (2013) 1016–1027.

[39] S. Salahshour, M. Khezerloo, S. Hajighasemi, M. Khorasany, Solving fuzzy integral equations of the second kind by fuzzy Laplace transform method, Internat. J. Industrial Math. 4 (2012) 21–29.

[40] S. Seikkala, D. Vorobiev, Towards the theory of fuzzy differential equations, Fuzzy Sets Syst. 125 (2002) 231–237.

[41] C. Wu, Z. Gong, On Henstock integral of fuzzy-number-valued functions (I), Fuzzy Sets Syst. 120 (2001) 523–532.

[42] S. Ziari, A.M. Bica, New error estimate in the iterative numerical method for nonlinear fuzzy Hammerstein-Fredholm integral equations, Fuzzy Sets Syst. 295 (2016) 136–152.