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Approximation Scheme for Handling Coupled Systems of Differential Equations within Reproducing Kernel Method

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Abstract. This paper proposes an efficient numerical method to obtain analytical-numerical solutions for a class of system of boundary value problems. This new algorithm is based on a reproducing kernel Hilbert space method. The analytical solution is calculated in the form of series in reproducing kernel space with easily computable components. In addition, convergence analysis for this method is discussed. In this sense, some numerical examples are given to show the effectiveness and performance of the proposed method. The results reveal that the method is quite accurate, simple, straightforward, and convenient to handle a various range of differential equations.

1. Introduction

Systems of boundary value problems arise naturally in several branches, not only in mathematics, but also in physics as physical differential equations and in scientific and engineering applications including potential theory, electrostatics, fluid mechanics, astronomy, relaxation processes and so on. Actually, many real life problems are often stated as boundary value problems, such as Sturm-Liouville forms, wave and Laplace's equations, different electromagnetic applications, and even some situations of black holes appear as systematic treatments via boundary value problems. To get more information about BVPs, we refer to [11, 13, 18, 29, 30]. Mostly, it is difficult to find the exact solutions of nonlinear and non-homogeneous BVPs, so a lot of attention of many authors has been made to find their analytical and numerical approximate solutions. Because of the significant difficulty to get a closed form solution of various nonlinear BVPs, several iterative techniques can be applied to approach the results of many numerical experiments that confirm the efficiency of the reduction to the boundary value problems. For instance, see [5, 7, 8, 10, 23, 25, 26, 28, 31, 32] and the references therein. Anyhow, coupled of fourth and second-order differential systems with homogeneous boundary conditions constitute a very interesting class for many realism matters, they are actually found to be a powerful tool to describe certain physical problems.

In this paper, iterative form of reproducing kernel method is proposed for solving coupled differential systems of fourth and second-order BVPs in the appropriate Hilbert space. The key point is to construct

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the direct sum of the RKHSs that satisfying the boundary conditions of coupled differential systems in order for determining their exact and numerical solutions. The exact and numerical solutions are proposed very accurately in series formula with easily computable coefficients. However, coupled differential system of fourth and second-order BVPs has been studied systematically in this approach for development and implementation of reliable method. To be more precisely, consider the set of ordinary differential equations in the form [33]:

$$\begin{aligned} L_1 u_1(x) &= f_1\left(x, u_1(x), u_2(x), u_1'(x), u_2'(x)\right), \\ L_2 u_2(x) &= f_2\left(x, u_1(x), u_2(x), u_1'(x), u_2'(x)\right), \end{aligned} \tag{1}$$

with the boundary conditions

$$\begin{aligned} &u_1(0) = u_1''(0) = u_1(1) = u_1''(1) = 0, \\ &u_2(0) = u_2(1) = 0, \end{aligned} \tag{2}$$

where the linear differential operators L_1 and L_2 are given by

$$\begin{aligned} &L_1 : W_2^5[0,1] \to W_2^1[0,1] \text{ such that } L_1 u_1(x) = u_1''''(x), \\ &L_2 : W_2^3[0,1] \to W_2^1[0,1] \text{ such that } L_2 u_2(x) = u_2''(x), \end{aligned} \tag{3}$$

$x \in [0,1]$, $u_1 \in W_2^5[0,1]$ and $u_2 \in W_2^3[0,1]$ are unknown functions to be determined, $f_i(x, v_1, w_1, v_2, w_2), i = 1, 2$, is continuous function in $W_2^1[0,1]$ as $v_i = v_i(x) \in W_2^5[0,1]$ and $w_i = w_i(x) \in W_2^3[0,1]$, $0 \le x \le 1$, $-\infty < v_i, w_i < \infty$, $i = 1, 2$, and $W_2^1[0,1], W_2^3[0,1], W_2^5[0,1]$ are RKHSs. Without loss of generality, we assume that system. (1) to (3) has a unique analytical solution on $[0,1]$.

In 1907, the reproducing kernel was introduced by Stanislaw Zaremba. In the mid of 20th century, Nachman Aronszajn developed the reproducing kernels, systematically. The RKHS theory has many applications in quantum mechanics, computational processing, complex and harmonic analysis [6, 14, 16, 17]. Many recent papers in both differential and integral equations apply a method based on the theory to solve related problems. To understand the fundamentals and the properties of reproducing kernel Hilbert spaces, the reader is kindly requested to go over the references [1–4, 9, 12, 15, 19–22, 24, 27, 34]. On the other hand, there is generally a drive to find new more advantageous ways to make the analyze problems using practice methods. In our procedure, the approximate solution is obtained by n-th term intercept of the analytical solution, whereas the error is proved to converge to zero in the sense of space norm. Besides that, we have uniformly convergence of approximate solution to analytical solution together with its derivatives. In addition, we show from the presented examples that the RKHS approach is capable to handle wide scale of applications of BVPs. Finally, it is worth it to mention that we do not take care about transforming or preserving a continuous-time system, so it does not matter at what time we make our calculations.

The structure of this article is organized as follows. In Section 2, we construct two useful direct sum reproducing kernel spaces and obtain two extended reproducing kernel functions. Afterwards, in Section 3, there are more theoretical details written in a logical order based upon the reproducing kernel theory. The main practical point is to describe iterative technique to handle non-linearity case of the proposed system, as well as error analysis of the solutions are also presented in Section 4. The mentioned sections are very important to build methodology of the presented method before passing to the numerical examples in Section 5. After all, some remarkable concluded points are pointed out in Section 6. This paper ends in Appendices with two parts about the kernel functions of the inner product spaces $W_2^3[0,1]$ and $W_2^5[0,1]$.

2. Building Appropriate Inner Product Spaces

The reproducing kernel approach builds on a Hilbert space H, which requires that all Dirac evaluation functional in H are bounded and continuous. In this section, two essential RKHSs $W_2^5[0,1] \bigoplus W_2^3[0,1]$ and $W_2^1[0,1] \bigoplus W_2^1[0,1]$ are constructed. Then, we utilize the reproducing kernel concept to obtain the reproducing kernel functions $\left(R_x^{\{5\}}(y), R_x^{\{3\}}(y)\right)^T$ and $\left(R_x^{\{1\}}(y), R_x^{\{1\}}(y)\right)^T$ in order to formulate the solutions in the mentioned spaces, in which every function satisfies the boundary conditions of Eq. (2). Before the construction, it is necessary to present some preliminary facts upon the reproducing kernel theory that will be used further in the remainder of the paper. Throughout this analysis, the symbol $\mathbb{C}$ indicates the set of complex numbers while $L^2[0,1] = \{u \mid \int_0^1 u^2(x)\,dx < \infty\}$ and $l^2 = \{A \mid \sum_{i=1}^{\infty} A_i^2 < \infty\}$.

Definition 2.1. ([6]) Let H be a Hilbert space of function $\theta : \Omega \to H$ on a set Ω. A function $R : \Omega \times \Omega \to \mathbb{C}$ is a reproducing kernel of H if the following conditions are satisfied. Firstly, $R(\cdot, x) \in H$ for each $x \in \Omega$. Secondly, $\langle \theta(\cdot), R(\cdot, x)\rangle = \theta(x)$ for each $\theta \in H$ and each $x \in \Omega$.

The last condition is called "the reproducing property" which means that, the value of the function θ at the point x is reproducing by the inner product of θ with $R(\cdot, x)$. Indeed, a Hilbert space which possesses a reproducing kernel is called a RKHS.

Definition 2.2. ([27]) The kernel space $W_2^1[0,1]$ is defined as $W_2^1[0,1] = \{z : z$ is absolutely continuous function on $[0,1]$ and $z' \in L^2[0,1]\}$. The inner product in $W_2^1[0,1]$ is given by

$$\langle z_1(x), z_2(x)\rangle_{W_2^1} = z_1(0)\,z_2(0) + \int_0^1 z_1'(x)z_2'(x)dx, \tag{4}$$

and the norm is $\|z\|_{W_2^1} = \sqrt{\langle z(x), z(x)\rangle_{W_2^1}}$, where $z_1, z_2 \in W_2^1[0,1]$.

Definition 2.3. The kernel space $W_2^3[0,1]$ is defined as $W_2^3[0,1] = \{z : z, z', z''$ are absolutely continuous functions on $[0,1]$, $z''' \in L^2[0,1]$, and $z(0) = 0, z(1) = 0\}$. The inner product in $W_2^3[0,1]$ is given by

$$\langle z_1(x), z_2(x)\rangle_{W_2^3} = \sum_{i=0}^{2} z_1^{(i)}(0)\,z_2^{(i)}(0) + \int_0^1 z_1'''(x)z_2'''(x)dx, \tag{5}$$

and the norm is $\|z\|_{W_2^3} = \sqrt{\langle z(x), z(x)\rangle_{W_2^3}}$, where $z_1, z_2 \in W_2^3[0,1]$.

Definition 2.4. The kernel space $W_2^5[0,1]$ is defined as $W_2^5[0,1] = \{z : z^{(i)}, i = 0,1,2,3,4$ are absolutely continuous functions on $[0,1]$, $z^{(5)} \in L^2[0,1]$, and $z(0) = z''(0) = z(1) = z''(1) = 0\}$. The inner product in $W_2^5[0,1]$ is given by

$$\langle z_1(x), z_2(x)\rangle_{W_2^5} = \sum_{i=0}^{2} z_1^{(i)}(0)\,z_2^{(i)}(0) + \sum_{i=0}^{1} z_1^{(i)}(1)\,z_2^{(i)}(1) + \int_0^1 z_1^{(5)}(x)z_2^{(5)}(x)dx, \tag{6}$$

and the norm is $\|z\|_{W_2^5} = \sqrt{\langle z(x), z(x)\rangle_{W_2^5}}$, where $z_1, z_2 \in W_2^5[0,1]$.

An important subsets of RKHSs are those associated to continuous kernels. These spaces have wide applications, including complex analysis, quantum mechanics, statistics, machine learning and harmonic process [6, 14, 16, 17]. Before any further discussion, we need to obtain the reproducing kernels functions of the spaces $W_2^1[0,1]$, $W_2^3[0,1]$, and $W_2^5[0,1]$, respectively, as follows.

Theorem 2.1. ([27]) *The Hilbert space $W_2^1[0,1]$ is a complete reproducing kernel with the reproducing kernel function*

$$R_x^{\{1\}}(y) = \begin{cases} R_{x,1}^{\{1\}}(y) = 1 + y, & y \le x, \\ R_{x,2}^{\{1\}}(y) = 1 + x, & y > x. \end{cases} \tag{7}$$

Theorem 2.2. *(a) The Hilbert space* $W_2^3[0,1]$ *is a complete reproducing kernel with the reproducing kernel function*

$$R_x^{\{3\}}(y) = \begin{cases} R_{x,1}^{\{3\}}(y) = \sum_{i=0}^{5} a_i(x) y^i, & y \le x, \\ R_{x,2}^{\{3\}}(y) = \sum_{i=0}^{5} b_i(x) y^i, & y > x, \end{cases} \tag{8}$$

where $a_i(x)$ *and* $b_i(x)$*,* $i = 0, 1, ..., 5$*, are unknown coefficients of* $R_x^{\{3\}}(y)$*.*

(b) The Hilbert space $W_2^5[0,1]$ *is a complete reproducing kernel with the reproducing kernel function*

$$R_x^{\{5\}}(y) = \begin{cases} R_{x,1}^{\{5\}}(y) = \sum_{i=0}^{9} c_i(x) y^i, & y \le x, \\ R_{x,2}^{\{5\}}(y) = \sum_{i=0}^{9} d_i(x) y^i, & y > x, \end{cases} \tag{9}$$

where $p_i(x)$ *and* $q_i(x)$*,* $i = 0, 1, ..., 9$ *are unknown coefficients of* $R_x^{\{5\}}(y)$*.*

Proof. The proof with the coefficients $a_i(x), b_i(x), i = 0, 1, ...5$, of $R_x^{\{3\}}(y)$ and $p_i(x), q_i(x), i = 0, 1, ..., 9$, of $R_x^{\{5\}}(y)$ are given in Appendices. □

Henceforth and not to conflict unless stated otherwise, we denote $W[0,1] = W_2^5[0,1] \bigoplus W_2^3[0,1]$, $H[0,1] = W_2^1[0,1] \bigoplus W_2^1[0,1]$, and $R_x(y) = \left(R_x^{\{5\}}(y), R_x^{\{3\}}(y)\right)^T$, $r_x(y) = \left(R_x^{\{1\}}(y), R_x^{\{1\}}(y)\right)^T$.

Definition 2.5. (a) The Hilbert space $H[0,1]$ is defined as $H[0,1] = \{z = (z_1, z_2)^T : z_1, z_2 \in W_2^1[0,1]\}$. The inner product in $H[0,1]$ is building as $\langle z(x), w(x) \rangle_H = \sum_{j=1}^{2} \left\langle z_j(x), w_j(x) \right\rangle_{W_2^1}$ and the norm is $\|z\|_H = \sqrt{\sum_{j=1}^{2} \left\|z_j\right\|_{W_2^1}^2}$, where $z, w \in H[0,1]$.

(b) The Hilbert space $W[0,1]$ is defined as $W[0,1] = \{z = (z_1, z_2)^T : z_1 \in W_2^5[0,1]$ and $z_2 \in W_2^3[0,1]\}$. The inner product in $W[0,1]$ is building as $\langle z(x), w(x) \rangle_W = \langle z_1(x), w_1(x) \rangle_{W_2^5} + \langle z_2(x), w_2(x) \rangle_{W_2^3}$ and the norm is $\|z\|_W = \sqrt{\|z_1\|_{W_2^5}^2 + \|z_2\|_{W_2^3}^2}$, where $z, w \in W[0,1]$.

3. Exact and Numerical Solutions

In this section, we introduce many useful properties together with family of hypotheses relating to RKHSs to solve the coupled differential system of Eqs. (1) - (3). Anyhow, formulation and implementation method of exact and numerical solutions are given in the RKHSs $W[0,1]$ and $H[0,1]$. Meanwhile, we construct an orthogonal function basis of the space $W[0,1]$ based on the use of Gram-Schmidt orthogonalization process.

For conduct of proceedings in the algorithm construction, set $f = (f_1, f_2)^T$, $u = (u_1, u_2)^T$, $u' = \left(u_1', u_2'\right)^T$, and $L = \text{diag}(L_1, L_2)$, where

$$L : W[0,1] \to H[0,1]. \tag{10}$$

Thus, the coupled differential systems of Eqs. (1) - (3) can be written as follows:

$$Lu(x) = f(x, u(x), u'(x)), \tag{11}$$

with boundary conditions:

$$u(0) = u(1) = \left(e_1^T u''(0)\right) e_1 = \left(e_1^T u''(1)\right) e_1 = (0,0)^T, \tag{12}$$

where $u \in W[0,1]$ and $f \in H[0,1]$. Here, $\left(e_1^T u''(0)\right) e_1 = \left(u_1''(0), 0\right)^T$ and $\left(e_1^T u''(1)\right) e_1 = \left(u_1''(1), 0\right)^T$.

Lemma 3.1. *The operators* $L_1 : W_2^5[0,1] \to W_2^1[0,1]$ *and* $L_2 : W_2^3[0,1] \to W_2^1[0,1]$ *are linear bounded operators.*

Proof. Clearly, L_1 and L_2 are linear operators. Thus, it is enough to show that they are bounded. For L_1, we need to prove that $\|L_1u\|^2_{W_2^1} \le M\|u\|^2_{W_2^5}$, where M is positive constant. From Equation (4), we have $\|L_1u\|^2_{W_2^1} = \langle L_1u(x), L_1u(x)\rangle_{W_2^1} = [L_1u(0)]^2 + \int_0^1 [(L_1u_1)'(x)]^2 dx$. By reproducing property of $R_x^{\{5\}}(y)$, we have $u(x) = \left\langle u(y), R_x^{\{5\}}(y)\right\rangle_{W_2^5}$, $(L_1u)(x) = \left\langle u(y), (L_1R_x^{\{5\}})(y)\right\rangle_{W_2^5}$ and $(L_1u)'(x) = \left\langle u(y), (L_1R_x^{\{5\}})'(y)\right\rangle_{W_2^5}$. By Schwarz inequality, we get

$$\begin{aligned} |(L_1u)(x)| &= \left|\left\langle u(x), (L_1R_x^{\{5\}})(x)\right\rangle_{W_2^5}\right| \le \left\|(L_1R_x^{\{5\}})(x)\right\|_{W_2^1}\|u\|_{W_2^5} \le M_1\|u\|_{W_2^5}, \\ \left|(L_1u)'(x)\right| &= \left|\left\langle u(x), (L_1R_x^{\{5\}})'(x)\right\rangle_{W_2^5}\right| \le \left\|(L_1R_x^{\{5\}})'(x)\right\|_{W_2^1}\|u\|_{W_2^5} \le M_2\|u_1\|_{W_2^5}, \end{aligned} \tag{13}$$

where $M_1, M_2 > 0$. Thus, $\|L_1u\|^2_{W_2^1} = [L_1u(0)]^2 + \int_0^1 [(L_1u)'(x)]^2 dx \le \left(M_1^2 + M_2^2\right)\|u\|^2_{W_2^5}$, where $M = \left(M_1^2 + M_2^2\right) > 0$. Similarly for L_2, one can prove that $\|L_2u\|^2_{W_2^1} \le C\|u\|^2_{W_2^3}$, $C > 0$ using the reproducing property of $R_x^{\{3\}}(y)$. □

Theorem 3.1. *The operator* $L : W[0,1] \to H[0,1]$ *is bounded linear operator.*

Proof. Clearly, L is a linear operator. For each $u \in W[0,1]$, using Definition 5, we have

$$\begin{aligned} \|Lu\|_H &= \sqrt{\sum_{j=1}^{2}\left\|L_ju_j\right\|^2_{W_2^1}} = \sqrt{\|L_1u_1\|^2_{W_2^1} + \|L_2u_2\|^2_{W_2^1}} \le \sqrt{\left(\|L_1\|\,\|u_1\|_{W_2^5}\right)^2 + \left(\|L_2\|\,\|u_2\|_{W_2^3}\right)^2} \\ &\le \left(\left(\sum_{j=1}^{2}\left\|L_j\right\|^2\right)\left(\|u_1\|^2_{W_2^5} + \|u_2\|^2_{W_2^3}\right)\right)^{1/2} \le \left(\sum_{j=1}^{2}\left\|L_j\right\|^2\right)^{1/2}\|u\|_W. \end{aligned}$$

The boundedness of L_j for $j = 1,2$, implies that L is bounded. □

To construct an orthogonal function systems of the space $W[0,1]$, we set $\varphi_{ij}(x) = R_{x_i}^{\{1\}}(x)e_j$ and $\psi_{ij}(x) = L^*\varphi_{ij}(x)$, $j = 1,2$, $i = 1,2,3,...$, on a dense subset $\{x_i\}_{i=1}^{\infty}$ of $[0,1]$, where $e_1 = (1,0)^T$, $e_2 = (0,1)^T$ and $L^* = \text{diag}\left(L_1^*, L_2^*\right)$ is the adjoint operator of L.

The orthonormal system $\left\{\overline{\psi}_{ij}(x)\right\}_{(i,j)=(1,1)}^{(\infty,2)}$ of $W[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\left\{\psi_{ij}(x)\right\}_{(i,j)=(1,1)}^{(\infty,2)}$ as follows

$$\overline{\psi}_{ij}(x) = \sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\psi_{lk}(x),\ \beta_{lk}^{ij} > 0,\ i = 1,2,3,...,\ j = 1,2,$$

where the orthogonalization coefficients β_{lk}^{ij} are described in the following algorithm:

Algorithm 3.1. *To determine the orthonormal function system* $\left\{\overline{\psi}_{ij}(x)\right\}_{(i,j)=(1,1)}^{(\infty,2)}$ *of the space* $W[0,1]$, *do the following:*

Stage 1: For $i = 1,2,3,...$, and $j = 1,2$, we have

If $l = i = 1$, then set $\beta_{lk}^{ij} = \frac{1}{\|\psi_{1k}\|_W}$, If $l = i \neq 1$, then set $\beta_{lk}^{ij} = \frac{1}{d_{lk}^{ij}}$,,

If $l < i$, then set $\beta_{lk}^{ij} = \frac{-1}{d_{lk}^{ij}} \sum\limits_{p=i}^{l-1} c_{lk}^{pj} \beta_{pk}^{ij}$,

where $d_{lk}^{ij} = \sqrt{\left\|\psi_{lk}\right\|_W^2 - \sum\limits_{p=1}^{l-1} \left(c_{lk}^{pj}\right)^2}$, $c_{lk}^{pj} = \left\langle \psi_{lk}(x), \overline{\psi}_{pk}(x) \right\rangle_W^2$.

Stage 2: For $i = 1, 2, 3, ...$ and $j = 1, 2$ set that

$$\overline{\psi}_{ij}(x) = \sum_{l=1}^{i} \sum_{k=1}^{j} \beta_{lk}^{ij} \psi_{lk}(x); \tag{14}$$

Output: the orthogonalization coefficients β_{lk}^{ij} of the orthonormal systems $\overline{\psi}_{ij}(x)$, and then the orthonormal function systems $\left\{\overline{\psi}_{ij}(x)\right\}_{(i,j)=(1,1)}^{(\infty,2)}$.

Frankly, $\psi_{ij}(x) = L^*\varphi_{ij}(x) = \left\langle L^*\varphi_{ij}(y), R_x(y)\right\rangle_W = \left\langle \varphi_{ij}(y), L_y R_x(y)\right\rangle_H = L_y R_x(y)\big|_{y=x_i} \in W[0,1]$. Hence, $\psi_{ij}(x)$ can be expressed in the form of $\psi_{ij}(x) = L_y R_x(s)\big|_{y=x_i}$. Here, L_y indicates that the operator L applies to the function of y.

$$\begin{aligned}
\beta_{ij} &= \frac{1}{\left\|\psi_1\right\|}, \text{ for } i = j = 1, \\
\beta_{ij} &= \frac{1}{d_{ik}}, \text{ for } i = j \neq 1, \\
\beta_{ij} &= -\frac{1}{d_{ik}} \sum_{k=j}^{i-1} c_{ik}\beta_{kj}, \text{ for } i > j,
\end{aligned}$$

such that $d_{ik} = \sqrt{\left\|\psi_i\right\|^2 - \sum\limits_{k=1}^{i-1} (c_{ik})^2}$, $c_{ik} = \left\langle \psi_i, \overline{\psi}_k \right\rangle_{W_2^3}$, and $\{\psi_i(x)\}_{i=1}^{\infty}$ is the orthogonal system in $W_2^3[0,1]$.

Theorem 3.2. *For Eqs. (11) and (12). Let $\{x_i\}_{i=1}^{\infty}$ be dense subset on $[0,1]$, then $\left\{\psi_{ij}(x)\right\}_{(i,j)=(1,1)}^{(\infty,2)}$ is the complete function systems of the space $W[0,1]$.*

Proof. For each fixed $u \in W[0,1]$, let $\left\langle u(x), \psi_{ij}(x)\right\rangle_W = 0$, this means, $\left\langle u(x), \psi_{ij}(x)\right\rangle_W = \left\langle u(x), L^*\varphi_{ij}(x)\right\rangle_W = \left\langle Lu(x), \varphi_{ij}(x)\right\rangle_H = Lu(x_i) = 0$. Whilst on the other hand, $u(x) = \sum\limits_{j=1}^{2} u_j(x) e_j = \sum\limits_{j=1}^{2} \left\langle u(\cdot), R_x(\cdot) e_j\right\rangle_W e_j$; thus, $Lu(x_i) = \sum\limits_{j=1}^{2} \left\langle Lu(x), \varphi_{ij}(x)\right\rangle_H e_j = 0$. Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $Lu(x) = 0$. From the existence of inverse operator L^{-1} it's concluded that $u(x) = 0$. □

Theorem 3.3. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and the solution of Eqs. (11) and (12) is unique, then the exact solution satisfies the infinite expansion form*

$$u(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{2} \sum_{l=1}^{i} \sum_{k=1}^{j} \beta_{ik}^{ij} f_k\left(x_l, u(x_l), u'(x_l)\right) \bar{\psi}_{ij}(x). \tag{15}$$

Proof. Applying Theorem 3.2, one can easy to see that $\left\{\bar{\psi}_{ij}(x)\right\}_{(i,j)=(1,1)}^{(\infty,2)}$ is the complete orthonormal basis of $W[0,1]$. Remark that $\left\langle u(x), \varphi_{ij}(x)\right\rangle = u_j(x_i)$ for each $u \in W[0,1]$, and $\sum_{i=1}^{\infty}\sum_{j=1}^{2}\left\langle u(x), \bar{\psi}_{ij}(x)\right\rangle_W \bar{\psi}_{ij}(x)$ is the Fourier series expansion about $\left\{\bar{\psi}_{ij}(x)\right\}_{(i,j)=(1,1)}^{(\infty,2)}$. Then the series $\sum_{i=1}^{\infty}\sum_{j=1}^{2}\left\langle y(x), \bar{\psi}_{ij}(x)\right\rangle_W \bar{\psi}_{ij}(x)$ is convergent in the sense of $\|\cdot\|_W$. So we have

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\left\langle u(x), \bar{\psi}_{ij}(x)\right\rangle_W \bar{\psi}_{ij}(x)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\langle u(x), \sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\psi_{lk}(x)\rangle_W \bar{\psi}_{ij}(x)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\langle u(x), L^*\varphi_{lk}(x)\rangle_W \bar{\psi}_{ij}(x)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\langle Lu(x), \varphi_{lk}(x)\rangle_H \bar{\psi}_{ij}(x)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\langle f_k(x, u(x), u'(x)), \varphi_{lk}(x)\rangle_H \bar{\psi}_{ij}(x)\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij} f_k(x_l, u(x_l), u'(x_l))\bar{\psi}_{ij}(x).
\end{aligned} \tag{16}$$

The proof is complete. □

Anyhow, the numerical solution $u_n(x)$ of $u(x)$ for Eqs. (11) and (12) can be obtained directly by taking finitely many terms in the series representation form of $u(x)$ for Eq. (15) as follows:

$$u_n(x) = \sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij} f_k(x_l, u(x_l), u'(x_l))\bar{\psi}_{ij}(x). \tag{17}$$

Remark 3.1. According to the basic motivation for the RKHS method in solving Eqs. (11) and (12), we notice the following two cases:

Case 1: If Eq. (11) is linear, then the approximate solution can be obtained directly from Eqs. (15).

Case 2: If Eq. (11) is nonlinear, then the approximate solution can be obtained by the following iterative process.

From Eq. (15), the representation form of the exact solution of Eqs. (11) and (12) can be written as

$$u(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{2} A_{ij}\bar{\psi}_{ij}(x), \tag{18}$$

where $A_{ij} = \sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{ik}^{ij} f_k(x_l, u(x_l), u'(x_l))$. Prior to apply of the proceedings, set $x_1 = 0$, that is, $u(x_1)$ is known from the boundary conditions of Eq. (12), which implies that the exact value of $f(x_1, u(x_1), u'(x_1))$ is also known. From a different viewpoint, for numerical computations, we define initial data $u_0(x_1) = u(x_1) = 0$ (or choose any fixed $u_0(x)$ in $W[0,1]$) and the n-term approximation to $u(x)$ by

$$u_n(x) = \sum_{i=1}^{n}\sum_{j=1}^{2} B_{ij}\bar{\psi}_{ij}(x), \tag{19}$$

where the coefficients B_{ij} of $\bar{\psi}_{ij}(x)$ are given by

$$B_{ij} = \sum_{l=1}^{i} \sum_{k=1}^{j} \beta_{lk}^{ij} f_k \left(x_l, u_{l-1}(x_l), u'_{l-1}(x_l) \right). \tag{20}$$

Hence, Eq. (18) is obtained by substituting the coefficients of Eq. (19) in (20), which satisfies the boundary conditions of Eq. (12).

4. Convergence Analysis of iterative Technique

In this section, we will show that $u_n(x)$ in the above iterative formula is converge to the exact solution $u(x)$ of Eqs. (11) and (12).

Lemma 4.1. *If $z \in W[0,1]$, then there exist positive numbers K_i such that $\left|z^{(i)}(x)\right| \leq K_i \|z\|_W$, where $i = 0,1,2$.*

Proof. For each $z_1 \in W_2^5[0,1]$, then $\left|z_1^{(i)}(x)\right| = \left|\left\langle z_1(x), \partial_x^i R_x(x)\right\rangle_{W_2^5}\right| \leq \left\|\partial_x^i R_x(x)\right\|_{W_2^5} \|z_1\|_{W_2^5} \leq M_i \|z_1\|_{W_2^5}$, where $i = 0,1,2,3,4$. Similarly, for each $z_2 \in W_2^3[0,1]$, one can get $\left|z_2^{(i)}(x)\right| = \left|\left\langle z_2(x), \partial_x^i R_x(x)\right\rangle_{W_2^3}\right| \leq \left\|\partial_x^i R_x(x)\right\|_{W_2^3} \|z_2\|_{W_2^3} \leq N_i \|z_2\|_{W_2^3}$, where $i = 0,1,2$. On the other aspect as well, if $z \in W[0,1]$, then $z(x) = (z_1(x), z_2(x))^T$ with $z_1 \in W_2^5[0,1]$ and $z_2 \in W_2^3[0,1]$. Thus, we have that

$$\begin{aligned} \left|z^{(i)}(x)\right| &= \sqrt{\left|z_1^{(i)}(x)\right|^2 + \left|z_2^{(i)}(x)\right|^2} \\ &\leq \sqrt{M_i^2 \|z_1\|_{W_2^5}^2 + N_i^2 \|z_2\|_{W_2^3}^2} \\ &\leq \sqrt{\max\left\{M_i^2, N_i^2\right\}\left(\|z_1\|_{W_2^5}^2 + \|z_2\|_{W_2^3}^2\right)} \\ &\leq K_i \|z\|_W, K_i = \sqrt{\max\left\{N_i^2, M_i^2\right\}}, i = 0,1,2. \end{aligned} \tag{21}$$

□

Theorem 4.1. *If $\|u_{n-1} - u\|_W \to 0$, $x_n \to y$ $(n \to \infty)$, $\|u_n\|_W$ is bounded, and $f(x, u(x), u'(x))$ is continuous for $x \in [0,1]$, then $f(x_n, u_{n-1}(x_n), u'_{n-1}(x_n)) \to f(y, u(y), u'(y))$ as $n \to \infty$.*

Proof. First of all, we note that

$$\begin{aligned} \left|u_{n-1}(x_n) - u(y)\right| &= \left|u_{n-1}(x_n) - u_{n-1}(y) + u_{n-1}(y) - u(y)\right| \\ &\leq \left|u_{n-1}(x_n) - u_{n-1}(y)\right| + \left|u_{n-1}(y) - u(y)\right| \\ &\leq \left|u'_{n-1}(\xi_1)\right|\left|x_n - y\right| + \left|u_{n-1}(y) - u(y)\right|, \xi_1 \text{ lies between } x_n \text{ and } y. \end{aligned} \tag{22}$$

By Lemma 4.1, it is known that $\left|u_{n-1}(y) - u(y)\right| \leq K_0 \|u_{n-1} - u\|_W$, which yields $|u_{n-1}(s) - u(s)| \to 0$ as $n \to \infty$, and $\left|u'_{n-1}(\xi_1)\right| \leq K_1 \|u_{n-1}\|_W$. By boundedness of $\|u_{n-1}(x)\|_W$, one gets that $\left|u_{n-1}(x_n) - u(y)\right| \to 0$ as $n \to \infty$. Now, we will show that $u'_{n-1}(x_n) \to u'(y)$ as follows

$$\begin{aligned} \left|u'_{n-1}(x_n) - u'(y)\right| &= \left|u'_{n-1}(x_n) - u'_{n-1}(y) + u'_{n-1}(y) - y'(y)\right| \\ &\leq \left|u'_{n-1}(x_n) - u'_{n-1}(y)\right| + \left|u'_{n-1}(y) - u'(y)\right| \\ &\leq \left|u''_{n-1}(\xi_2)\right|\left|x_n - y\right| + \left|u'_{n-1}(y) - u'(y)\right|, \xi_2 \text{ lies between } x_n \text{ and } y. \end{aligned} \tag{23}$$

Similarly, it is known that $\left|u'_{n-1}(y) - u'(y)\right| \leq K_1 \|u_{n-1} - u\|_W$, which yields $\left|u'_{n-1}(y) - u'(y)\right| \to 0$ as $n \to \infty$, and $\left|u''_{n-1}(\xi_2)\right| \leq K_2 \|u_{n-1}\|_W$. Thus, $\left|u'_{n-1}(x_n) - u'(y)\right| \to 0$ as $n \to \infty$. From the continuation of $f(x, u(x), u'(x))$, it is implies that $f(x_n, u_{n-1}(x_n), u'_{n-1}(x_n)) \to f(y, u(y), u'(y))$ as $n \to \infty$. □

Theorem 4.2. *Suppose that $\|u_n\|_W$ is bounded in Eq. (19). If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then the n-term approximate solution $u_n(x)$ in the iterative formula of Eq. (20) converges to the exact solution $u(x)$ of Eqs. (11) and (12) in $W[0,1]$.*

Proof. First of all, we will prove the convergence of $u_n(x)$. From Eq. (19), we get that $u_{n+1}(x) = u_n(x) + \sum_{j=1}^{2} B_{(n+1)j}\bar{\psi}_{(n+1)j}(x)$. By the orthogonality of $\left\{\bar{\psi}_{ij}(x)\right\}_{(i,j)=(1,1)}^{(\infty,2)}$, it follows that $\|u_{n+1}\|_W^2 = \|u_n\|_W^2 + \sum_{j=1}^{2} B_{(n+1)j}^2 = \|u_{n-1}\|_W^2 + \sum_{j=1}^{2} B_{nj}^2 + \sum_{j=1}^{2} B_{(n+1)j}^2 = ... = \|u_0\|_W^2 + \sum_{i=1}^{n+1}\sum_{j=1}^{2} B_{ij}^2$. That is, $\|u_{n+1}\|_W \geq \|u_n\|_W$. Due to the condition that $\|u_n\|_W$ is bounded, $\|u_n\|_W$ is convergent as $n \to \infty$. Then, there exists a constant α such that $\sum_{i=1}^{\infty}\sum_{j=1}^{2} B_{ij}^2 = \alpha$. It implies that $\sum_{j=1}^{2} B_{ij}^2 \in l^2, i = 1,2,3,....$ Since $(u_m(x) - u_{m-1}(x)) \perp (u_{m-1}(x) - u_{m-2}(x)) \perp ... \perp (u_{n+1}(x) - u_n(x))$, for $m > n$, one get that

$$\begin{aligned}\|u_m - u_n\|_W^2 &= \|u_m - u_{m-1} + u_{m-1} - ... + u_{n+1} - u_n\|_W^2 \\ &\leq \|u_m - u_{m-1}\|^2 + \|u_{m-1} - u_{m-2}\|^2 + ... + \|u_{n+1} - u_n\|_W^2 = \sum_{l=n+1}^{m}\sum_{j=1}^{2} B_{lj}^2.\end{aligned} \tag{24}$$

Consequently, as $n,m \to \infty$, we have $\|u_m - u_{m-1}\|_W^2 \to 0$ as soon as $\sum_{l=n+1}^{m}\sum_{j=1}^{2} B_{lj}^2 \to 0$. Considering the completeness of $W[0,1]$, there exist a $u(x) \in W[0,1]$ such that $u_n(x) \to u(x)$ as $n \to \infty$ in the sense of the norm of $W[0,1]$.

Secondly, we will show that $u(x)$ is the solution of Eqs. (11) and (12). From Eq. (19), we have $(Lu)(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{2} B_{ij}L\bar{\psi}_{ij}(x)$, and then $(Lu)_k(x_l) = \sum_{i=1}^{\infty}\sum_{j=1}^{2} B_{ij}\left\langle L\bar{\psi}_{ij}(x), \varphi_{lk}(x)\right\rangle_H = \sum_{i=1}^{\infty}\sum_{j=1}^{2} B_{ij}\left\langle \bar{\psi}_{ij}(x), L^*\varphi_{lk}(x)\right\rangle_W = \sum_{i=1}^{\infty}\sum_{j=1}^{2} B_{ij}\left\langle \bar{\psi}_{ij}(x), \psi_{lk}(x)\right\rangle_W$. Thus,

$\sum_{l'=1}^{l}\sum_{k'=1}^{k} B_{l'k'}^{lk}(Lu)_{k'}(x_{l'}) = \sum_{i=1}^{\infty}\sum_{j=1}^{2} B_{ij}\langle\bar{\psi}_{ij}(x), \sum_{l'=1}^{l}\sum_{k'=1}^{k} B_{l'k'}^{lk}\psi_{l'k'}(x)\rangle_W = \sum_{i=1}^{\infty}\sum_{j=1}^{2} B_{ij}\left\langle \bar{\psi}_{ij}(x), \bar{\psi}_{l'k'}(x)\right\rangle_W = B_{lk}$. For $l = 1$, we have $(Lu)_j(x_1) = f_j\left(x_1, u_0(x_1), u_0'(x_1)\right)$, $j = 1,2$, that is, $Lu(x_1) = f\left(x_1, u_0(x_1), u_0'(x_1)\right)$. Also, for $l = 2$, we have $(Lu)_j(x_2) = f_j\left(x_2, u_1(x_2), u_1'(x_2)\right)$, $j = 1,2$, that is, $Lu(x_2) = f\left(x_2, u_1(x_2), u_1'(x_2)\right)$. Hence, the general pattern formula can be written as $Lu(x_n) = f\left(x_n, u_{n-1}(x_n), u_{n-1}'(x_n)\right)$. Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, for every $y \in [0,1]$, there exists subsequence $\left\{x_{n_j}\right\}_{j=1}^{\infty}$ such that $x_{n_j} \to y$ as $j \to \infty$. Here, it is easy to see that $Lu\left(x_{n_j}\right) = f\left(x_{n_j}, u_{n_j-1}\left(x_{n_j}\right), u_{n_j-1}'\left(x_{n_j}\right)\right)$. Therefore, let $j \to \infty$, then by Theorem 4.1 and the continuity of f, one gets that $Lu(y) = f(y, u(y), u'(y))$, that is, $u(x)$ is solution of Eq. (11). Since $\bar{\psi}_{ij}(x) \in W[0,1]$, $u(x)$ satisfies the boundary conditions in Eq. (12). To put in another way, $u(x)$ is the solution of Eqs. (11) and (12), where $u(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{2} B_{ij}\bar{\psi}_{ij}(x)$. The proof is complete. □

Theorem 4.3. *Assume that $u(x)$ is the solution of Eqs. (11) and (12) and ε_n is the error between the approximate solution $u_n(x)$ in Eq. (17) and the exact solution $u(x)$. Then, the error sequence $\{\varepsilon_n\}$ is monotone decreasing with regards to the norm of $W[0,1]$ and $\varepsilon_n \to 0$ $(n \to \infty)$.*

Proof. Suppose that $u(x)$ and $u_n(x)$ are given by Eqs. (15) and (17), respectively. Then, we have

$\varepsilon_n^2 = \sum_{i=n+1}^{\infty}\sum_{j=1}^{2}\left\langle u(x), \bar{\psi}_{ij}(x)\right\rangle_W^2$ *and* $\varepsilon_{n-1}^2 = \sum_{i=n}^{\infty}\sum_{j=1}^{2}\left\langle u(x), \bar{\psi}_{ij}(x)\right\rangle_W^2$, *which implies that* $\varepsilon_{n-1} \geq \varepsilon_n$ *and shows the error* ε_n *is monotone decreasing with regards to* $\|\cdot\|_W$. *But on the other aspect as well, from Theorem 3.3, we know that* $\sum_{i=1}^{\infty}\sum_{j=1}^{2}\left\langle u(x), \bar{\psi}_{ij}(x)\right\rangle_W \bar{\psi}_{ij}(x)$ *is convergent. Therefore,* $\varepsilon_n^2 = \sum_{i=n+1}^{\infty}\sum_{j=1}^{2}\left\langle u(x), \bar{\psi}_{ij}(x)\right\rangle_W^2 \to 0.$ □

5. Numerical Outcomes

In this section, the proposed method is applied to demonstrate the simplicity and effectiveness for some systems of BVPs. The method is implemented in a direct way without using transformation, linearization or restrictive assumptions. Numerical results indicate that the present approach is very convenient for solving such systems. Anyhow, we all know that the algorithm is a finite sequence of rules for performing computations on a computer such that at each instant the rules determine exactly what the computer has to do next. Next algorithm is utilized to implement a procedure to solve the coupled differential system of Eqs. (1) - (3) numerically in terms of its grid points based on RKHS method.

Algorithm 5.1. *To obtain the approximate solution* $u_n(x)$ *for Eqs. (11) and (12), we do the following stages:*

Input: The interval $[0, 1]$, integer n, kernel functions $R_x^{\{i\}}(y)$, differential operator L, and function f.

Output: Approximate solution $u_n(x)$ at each point in the independent compact interval $[0, 1]$.

Stage 1: Fixed x in $[0, 1]$ and set $y \in [0, 1]$;

If $y \leq x$, set $R_x(y) = \left(R_{x,1}^{\{5\}}(y), R_{x,1}^{\{3\}}(y)\right)^T$;

Else set $R_x(y) = \left(R_{x,2}^{\{5\}}(y), R_{x,2}^{\{3\}}(y)\right)^T$;

For $i = 1, 2, ..., n$ and $j = 1, 2$, do the following:

Set $x_i = \frac{i-1}{n-1}$;

Set $\psi_{i,j}(x) = L_y\left[R_x(y)\right]_{y=x_i}$;

Stage 2: For $l = 2, 3..., n$ and $k = 1, 2..., l$, do Algorithm 3.1 for l and k;

Stage 3: For $l = 2, 3..., n-1$ and $k = 1, 2..., l-1$, do the following:

Set $\overline{\psi}_{ij}(x) = \sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\psi_{lk}(x)$;

Stage 4: Set $u_0(x_1) = u(x_1) = 0$;

Set $B_{ij} = \sum_{l=1}^{i}\sum_{k=1}^{2}\beta_{lk}^{ij}f_k\left(x_l, u_{l-1}(x_l), u_{l-1}'(x_l)\right)$;

Set $u_i(x) = \sum_{i=1}^{i}\sum_{j=1}^{2}B_{ij}\bar{\psi}_{ij}(x)$, and then stop.

Using RKHS algorithms, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$ in $u_n(x_i)$ of Eq. (17), and applying Algorithms 3.1 and 5.1 throughout the numerical computations; some graphical results, tabulate data, and numerical comparison are presented and discussed quantitatively at some selected grid points on $[0, 1]$ to illustrate the approximate solution for the following coupled differential system of fourth and second-order BVPs.

Example 5.1. Consider the linear differential system in following form:

$$\begin{aligned} u_1^{(4)}(x) &= x^2 u_2'(x) - u(x) + e^x u_2(x) + f_1(x), \\ u_2''(x) &= x u_1(x) + \sin(x) u_1'(x) + x^3 u_2(x) + f_2(x), \end{aligned} \tag{25}$$

with the boundary conditions

$$\begin{aligned} u_1(0) &= u_1''(0) = u_1(1) = u_1''(1) = 0, \\ u_2(0) &= u_2(1) = 0, \end{aligned} \tag{26}$$

where $x \in [0,1]$ in which $f_1(x)$ and $f_2(x)$ are chosen such that the exact solutions are $u(x) = x(1-x)e^{x(1-x)}$ and $u_2(x) = \sinh(x(1-x))$.

Example 5.2. Consider the nonlinear differential system in following form:

$$\begin{aligned} u_1^{(4)}(x) &= u_1^2(x) + \left(u_1'(x)\right)^3 - \left(u_2'(x)\right)^2 + \sin(x) e^{u_2(x)} + f_1(x), \\ u_2''(x) &= \sinh(u_1(x)) - u_1'(x) u_2'(x) + u_1(x) u_2(x) + f_2(x), \end{aligned} \tag{27}$$

with the boundary conditions

$$\begin{aligned} u_1(0) &= u_1''(0) = u_1(1) = u_1''(1) = 0, \\ u_2(0) &= u_2(1) = 0, \end{aligned} \tag{28}$$

where $x \in [0,1]$ in which $f_1(x)$ and $f_2(x)$ are chosen such that the exact solutions are $u(x) = 1.5x^3(x-1)^3 \cosh\left(e^{x+1}\right)$ and $u_2(x) = x(x-1)\cos(x)$.

Example 5.3. Consider the nonlinear differential system in following form:

$$\begin{aligned} u_1^{(4)}(x) &= u_2'(x)^2 \sqrt{u_1(x)+1} - \cosh(x) u_2(x) + f_1(x), \\ u_2''(x) &= \cos\left(u_1'(x) + u_2'(x)\right) + \ln(u_1(x) u_2(x)) + f_2(x), \end{aligned} \tag{29}$$

with the boundary conditions

$$\begin{aligned} u_1(0) &= u_1''(0) = u_1(1) = u_1''(1) = 0, \\ u_2(0) &= u_2(1) = 0, \end{aligned} \tag{30}$$

where $x \in [0,1]$ in which $f_1(x)$ and $f_2(x)$ are chosen such that the exact solutions are $u_1(x) = \frac{x^3(3x-3)^3}{x+1}$ and $u_2(x) = \frac{x(1-x)}{x(1-x)+1}$.

The agreement between exact and approximate solution is investigated for Examples 5.1, 5.2, and 5.3 at several x values in $[0,1]$ by computing the absolute and relative errors and summarized in Tables 1, 2, 3, 4, 5, and 6, respectively. From the tables, it is clear that the approximate solutions are in close agreement with exact solutions for all examples, while the accuracy is in advanced by using only few tens of the RKHS iterations. Here, we can conclude that higher accuracy can be achieved by computing further RKHS iterations.

Table 1. The numerical results of $u_1(x)$ for Example 5.1.

x	Exact solution	Numerical solution	Absolute error	Relative error
0.1	0.098475685533469	0.098475624737515	$6.07959539 \times 10^{-8}$	$6.17370203 \times 10^{-7}$
0.2	0.187761739358690	0.187761660786022	$7.85726676 \times 10^{-8}$	$4.18470067 \times 10^{-7}$
0.3	0.259072392590916	0.259072343529560	$4.90613560 \times 10^{-8}$	$1.89373154 \times 10^{-7}$
0.4	0.305099796077137	0.305099655080952	$1.40996185 \times 10^{-7}$	$4.62131365 \times 10^{-7}$
0.5	0.321006354171935	0.321006152535450	$2.01636485 \times 10^{-7}$	$6.28138611 \times 10^{-7}$
0.6	0.305099796077137	0.305099565257299	$2.30819838 \times 10^{-7}$	$7.56538814 \times 10^{-7}$
0.7	0.259072392590916	0.259072338803208	$5.37877080 \times 10^{-8}$	$2.07616518 \times 10^{-7}$
0.8	0.187761739358690	0.187761728355463	$1.10032266 \times 10^{-8}$	$5.86020702 \times 10^{-8}$
0.9	0.098475685533469	0.098475593822828	$9.17106414 \times 10^{-8}$	$9.31302391 \times 10^{-7}$

Table 2. The numerical results of $u_2(x)$ for Example 5.1.

x	Exact solution	Numerical solution	Absolute error	Relative error
0.1	0.090121549216991	0.090121507174177	$4.20428137 \times 10^{-8}$	$4.66512327 \times 10^{-7}$
0.2	0.160683541012799	0.160683526872999	$1.41398004 \times 10^{-8}$	$8.79978145 \times 10^{-8}$
0.3	0.211546906993278	0.211545976182294	$9.30810984 \times 10^{-7}$	$4.40002171 \times 10^{-6}$
0.4	0.242310644627426	0.242310591478829	$5.31485966 \times 10^{-8}$	$2.19340742 \times 10^{-7}$
0.5	0.252612316808168	0.252611689867891	$6.26940277 \times 10^{-7}$	$2.19340742 \times 10^{-7}$
0.6	0.242310644627426	0.242310443690914	$2.00936512 \times 10^{-7}$	$8.29251691 \times 10^{-7}$
0.7	0.211546906993278	0.211546529489625	$3.77503653 \times 10^{-7}$	$1.78449148 \times 10^{-6}$
0.8	0.160683541012799	0.160683493094276	$4.79185229 \times 10^{-8}$	$2.98216747 \times 10^{-7}$
0.9	0.090121549216991	0.090121504120859	$4.50961319 \times 10^{-8}$	$5.00392328 \times 10^{-7}$

Table 3. The numerical results of $u_1(x)$ for Example 5.2.

x	Exact solution	Numerical solution	Absolute error	Relative error
0.1	−0.011054720961117	−0.011054739275356	$1.83142390 \times 10^{-8}$	$1.65668940 \times 10^{-6}$
0.2	−0.085093581239469	−0.085093620404585	$3.91651156 \times 10^{-8}$	$4.60259341 \times 10^{-7}$
0.3	−0.272619317970376	−0.272619488500633	$1.70530257 \times 10^{-7}$	$6.25525212 \times 10^{-7}$
0.4	−0.598379076348348	−0.598379157896445	$8.15480975 \times 10^{-8}$	$1.36281666 \times 10^{-7}$
0.5	−1.035880634370310	−1.035881249273993	$6.14903684 \times 10^{-7}$	$5.93604768 \times 10^{-7}$
0.6	−1.468220546701186	−1.468221327693474	$7.80992288 \times 10^{-7}$	$5.31931180 \times 10^{-7}$
0.7	−1.655891723299256	−1.655891757273746	$3.39744901 \times 10^{-8}$	$2.05173379 \times 10^{-8}$
0.8	−1.302423245272149	−1.302423268398874	$2.31267247 \times 10^{-8}$	$1.77566892 \times 10^{-8}$
0.9	−0.437962101925840	−0.437962142512359	$4.05865195 \times 10^{-8}$	$9.26713049 \times 10^{-8}$

Table 4. The numerical results of $u_2(x)$ for Example 5.2.

x	Exact solution	Numerical solution	Absolute error	Relative error
0.1	−0.089550374875022	−0.089550384925649	$1.00506270 \times 10^{-8}$	$1.12234338 \times 10^{-7}$
0.2	−0.156810652454599	−0.156811023357907	$3.70903308 \times 10^{-7}$	$2.36529408 \times 10^{-6}$
0.3	−0.200620662716377	−0.200620673090301	$1.03739239 \times 10^{-8}$	$5.17091500 \times 10^{-8}$
0.4	−0.221054638560692	−0.221054698735664	$6.01749720 \times 10^{-8}$	$2.72217640 \times 10^{-7}$
0.5	−0.219395640472593	−0.21939619957025	$5.59097657 \times 10^{-7}$	$2.54835354 \times 10^{-6}$
0.6	−0.198080547578323	−0.198080728440439	$1.80862116 \times 10^{-7}$	$9.13073589 \times 10^{-7}$
0.7	−0.160616859329743	−0.160616933305013	$7.39752702 \times 10^{-8}$	$4.60569772 \times 10^{-7}$
0.8	−0.111473073495546	−0.111473504764069	$4.31268523 \times 10^{-7}$	$3.86881343 \times 10^{-6}$
0.9	−0.055944897144360	−0.055944947730899	$5.05865386 \times 10^{-8}$	$9.04220781 \times 10^{-7}$

Table 5. The numerical results of $u_1(x)$ for Example 5.3.

x	Exact solution	Numerical solution	Absolute error	Relative error
0.1	0.017893636363636	0.017893625305815	$1.10578213 \times 10^{-8}$	$6.17975078 \times 10^{-7}$
0.2	0.092160000000000	0.092159955401006	$4.45989945 \times 10^{-8}$	$4.83930062 \times 10^{-7}$
0.3	0.192343846153846	0.192343763269042	$8.28848043 \times 10^{-8}$	$4.30919970 \times 10^{-7}$
0.4	0.266605714285714	0.266605676640272	$3.76454423 \times 10^{-8}$	$1.41202684 \times 10^{-7}$
0.5	0.281250000000000	0.281249947846163	$5.21538368 \times 10^{-8}$	$1.85435864 \times 10^{-7}$
0.6	0.233280000000000	0.233279921351801	$7.86481991 \times 10^{-8}$	$3.37140771 \times 10^{-7}$
0.7	0.147086470588235	0.147085787534663	$6.83053572 \times 10^{-7}$	$4.64389124 \times 10^{-6}$
0.8	0.061440000000000	0.061439878719195	$1.21280805 \times 10^{-7}$	$1.97397143 \times 10^{-6}$
0.9	0.010359473684211	0.010359443212143	$3.04720677 \times 10^{-8}$	$2.94146871 \times 10^{-6}$

Table 6. The numerical results of $u_2(x)$ for Example 5.3.

x	Exact solution	Numerical solution	Absolute error	Relative error
0.1	0.082568807339450	0.082568775755825	$3.15836246 \times 10^{-8}$	$3.82512787 \times 10^{-7}$
0.2	0.137931034482759	0.137930962508336	$7.19744231 \times 10^{-8}$	$5.21814568 \times 10^{-7}$
0.3	0.173553719008264	0.173553511174514	$2.07833750 \times 10^{-7}$	$1.19751828 \times 10^{-6}$
0.4	0.193548387096774	0.193548326700641	$6.03961325 \times 10^{-8}$	$3.12046685 \times 10^{-7}$
0.5	0.200000000000000	0.199999644964896	$3.55035104 \times 10^{-7}$	$1.77517552 \times 10^{-6}$
0.6	0.193548387096774	0.193548304849392	$8.22473822 \times 10^{-8}$	$4.24944808 \times 10^{-7}$
0.7	0.173553719008264	0.173553564235657	$1.54772607 \times 10^{-7}$	$8.91785022 \times 10^{-7}$
0.8	0.137931034482759	0.137930128016954	$9.06465805 \times 10^{-7}$	$6.57187709 \times 10^{-6}$
0.9	0.082568807339450	0.082568788849990	$1.84894604 \times 10^{-8}$	$2.23927909 \times 10^{-7}$

The numerical values of absolute errors for Example 5.3 have been plotted in Figures 1 and 2, respectively. As the plots show, the values of approximate solution various smoothly along the x-axis by satisfying their conditions of the corresponding systems. We recall that the accuracy and duration of simulation depend directly on size of the steps taken by the solver. Generally, decreasing step size increases accuracy of the results, while increasing the time required to simulate the problem.

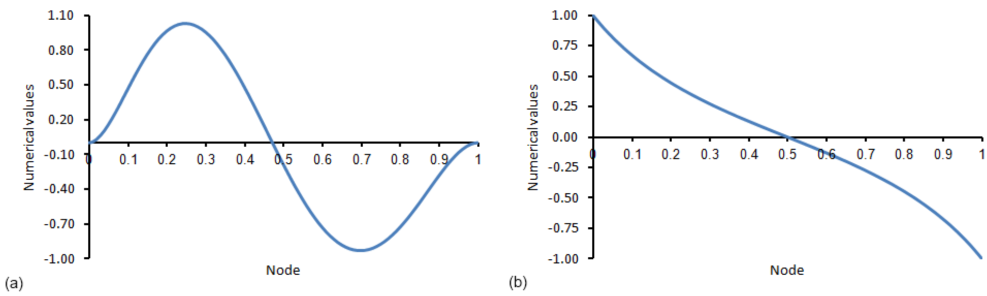

Figure 1: Graphical results for first derivatives of solutions for Example 3: (a) $u'_{1,n}(x_i)$ and (b) $u'_{2,n}(x_i)$

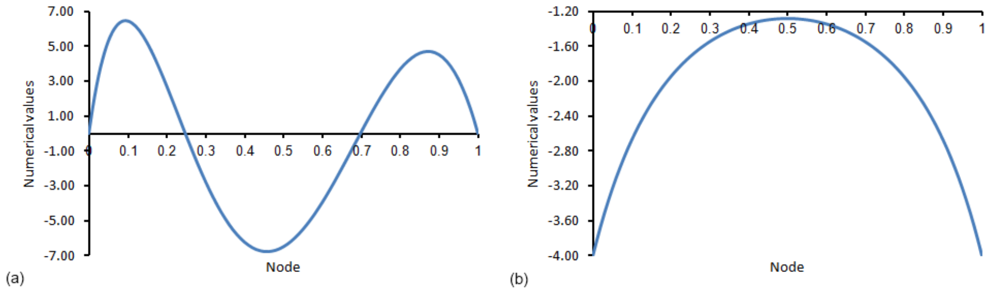

Figure 2: Graphical results for second derivatives of solutions for Example 3: (a) $u''_{1,n}(x_i)$ and (b) $u''_{2,n}(x_i)$

6. Concluding remarks

The reproducing kernel algorithm is practical and useful to solve not only the differential but also the integral equations. Recently, many authors take in their accounts the efficiency of this method. Meanwhile, this paper explores more large scale to apply RKHSs for solving differential systems of different orders. To do so, we construct appropriate Hilbert spaces, and we simplify the used algorithms and computations step by step. As a consequence, we come up with these results; firstly, the obtained solutions are smooth and uniformly convergent to the approximate ones; secondly, the efficient way to get the solution because that the error converges to zero in the norm space; thirdly, the capability of the process to handle different interesting numerical examples; fourthly, no time discretization is considered for computations. As a result, the current study shows how RKHSs method incorporates attractive features. In the future, we can handle more applications into our method. In process of computation, all the symbolic and numerical results are performed by using MAPLE 13 software package.

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7. Appendices

Proof. [Proof of Theorem 2.2] By using the tabular integration by parts of $\int_0^1 z'''(y)\,\partial_y^3 R_x^{\{3\}}(y)\,ds$, one can get that

$$\left\langle z(y), R_x^{\{3\}}(y)\right\rangle_{W_2^3} = \sum_{i=0}^{2} z^{(i)}(0)\,[\partial_y^i R_x^{\{3\}}(0) + (-1)^{i+1}\partial_y^{5-i} R_x^{\{3\}}(0)] + \sum_{i=0}^{2}(-1)^i u^{(i)}(1)\,\partial_y^{5-i} R_x^{\{3\}}(1) - \int_0^1 z(y)\,\partial_y^6 R_x^{\{3\}}(y)\,dy. \tag{31}$$

If $R_x^{\{3\}}(y) \in W_2^3[0,1]$, then $R_x^{\{3\}}(0) = R_x^{\{3\}}(1) = 0$ and if $z \in W_2^3[0,1]$, then $z(0) = z(1) = 0$. Hence, for each $x, y \in [0,1]$, assume $R_x^{\{3\}}(y)$ satisfy the following: $\partial_y^4 R_x^{\{3\}}(1) = 0$, $\partial_y^3 R_x^{\{3\}}(1) = 0$, $\partial_y^2 R_x^{\{3\}}(0) - \partial_y^3 R_x^{\{3\}}(0) = 0$, and $\partial_y R_x^{\{3\}}(0) + \partial_y^4 R_x^{\{3\}}(0) = 0$, then Eq. ((31)) becomes $\left\langle z(y), R_x^{\{3\}}(y)\right\rangle_{W_2^3} = \int_0^1 z(y)(-\partial_y^6 R_x^{\{3\}}(y))dy$. Also, assume $R_x^{\{3\}}(y)$ satisfy the formula

$$\partial_y^6 R_x^{\{3\}}(y) = -\delta(x-y), \delta \text{ dirac-delta function,} \tag{32}$$

so $\left\langle z(y), R_x^{\{3\}}(y)\right\rangle_{W_2^3} = z(x)$. For conduct of proceedings of expression form of $R_x^{\{3\}}(y)$, we note that the auxiliary formula of Eq. (32) is $\lambda^6 = 0$ and its auxiliary values are $\lambda = 0$ with multiplicity 6. So, let the expression form of $R_x^{\{3\}}(y)$ be as defined in Eq. (8). But on the other hand of Eq. (32), let $R_x^{\{3\}}(y)$ satisfy $\partial_y^i R_x^{\{3\}}(x+0) = \partial_y^i R_x^{\{3\}}(x-0), i = 0,1,...,4$. Thus by integrating $\partial_y^6 R_x^{\{3\}}(y) = -\delta(x-y)$ from $x - \varepsilon$ to $x + \varepsilon$ with respect to y and letting $\varepsilon \to 0$, we have jump degree of $\partial_y^5 R_x^{\{3\}}(y)$ at $y = x$ such that $\partial_y^5 R_x^{\{3\}}(x+0) - \partial_y^5 R_x^{\{3\}}(x-0) = -1$. Therefore, the unknown coefficients $a_i(x)$ and $b_i(x)$, $i = 0,1,...,5$ of Eq. (8) can be obtained as in Remark 7.1. Similarly for part (*b*), by several integration by parts of $\int_0^1 z^{(5)}(y)\partial_y^5 R_x^{\{5\}}(y)dy$, we have

$$\begin{aligned}\left\langle z(y), R_x^{\{5\}}(y)\right\rangle_{W_2^5} &= \sum_{i=0}^{2} z^{(i)}(0)\partial_y^i R_x^{\{5\}}(0) + \sum_{i=0}^{1} z^{(i)}(1)\partial_y^i R_x^{\{5\}}(1) \\ &+ \sum_{i=0}^{4}(-1)^{4-i} z^{(i)}(y)\partial_y^{9-i} R_x^{\{5\}}(y)\,|_{y=0}^{y=1} - \int_0^1 z(y)\partial_y^{10} R_x^{\{5\}}(y)dy.\end{aligned}$$

Hence, if $R_x^{\{5\}}(y) \in W_2^5[0,1]$ satisfy the following

$$R_x^{\{5\}}(0) = \partial_y^2 R_x^{\{5\}}(0) = R_x^{\{5\}}(1) = \partial_y^2 R_x^{\{5\}}(1) = 0$$

$$\partial_y^i R_x^{\{5\}}(0) = \partial_y^i R_x^{\{5\}}(1) = 0, i = 5,6,$$

$$\partial_y^1 R_x^{\{5\}}(0) + \partial_y^8 R_x^{\{5\}}(0) = 0,$$

$$\partial_y^1 R_x^{\{5\}}(1) - \partial_y^8 R_x^{\{5\}}(1) = 0,$$

$$\partial_y^i R_x^{\{5\}}(x+0) = \partial_y^i R_x^{\{5\}}(x-0), i = 0,1,...,8,$$

$$\partial_y^9 R_x^{\{5\}}(x-0) - \partial_y^9 R_x^{\{5\}}(x+0) = 1,$$

then, the unknown coefficients $c_i(x)$ and $d_i(x)$, $i = 0,1,...,9$ of Eq. (9) can be obtained as in Remark 7.2. This completes the proof. □

Remark 7.1. The coefficients of $R_x^{\{3\}}(y)$ in $W_2^3[0,1]$ are obtained by

$$\begin{aligned}
a_0(x) &= 0, b_0(x) = \frac{1}{120}x^5, \\
a_1(x) &= \frac{1}{156}\left(-x + 36 - 30x - 10x^2 + 5x^3 - x^4\right), b_1(x) = \frac{1}{312}x\left(72 - 60x - 20x^2 - 3x^3 - 2x^4\right), \\
a_2(x) &= \frac{1}{624}x\left(-120 + 126x - 10x^2 + 5x^3 - x^4\right), b_2(x) = \frac{1}{624}x\left(-120 + 126x + 42x^2 + 5x^3 - x^4\right), \\
a_3(x) &= \frac{1}{1872}x\left(-120 + 126x - 10x^2 + 5x^3 - x^4\right), b_3(x) = \frac{1}{1872}x\left(-120 - 30x - 10x^2 + 5x^3 - x^4\right), \\
a_4(x) &= \frac{1}{3744}x\left(-36 + 30x + 10x^2 - 5x^3 + x^4\right), b_4(x) = \frac{1}{3744}x\left(120 + 30x + 10x^2 - 5x^3 + x^4\right), \\
a_5(x) &= \frac{1}{18720}\left(156 - 120x - 30x^2 - 10x^3 + 5x^4 - x^5\right), b_5(x) = \frac{1}{18720}x\left(-120 - 30x - 10x^2 + 5x^3 - x^4\right).
\end{aligned}$$

Remark 7.2. The coefficients of $R_x^{\{5\}}(y)$ in $W_2^5[0,1]$ are obtained by

$$
\begin{aligned}
c_0(x) &= 0,\ d_0(x) = \frac{1}{362880}x^9,\\
c_1(x) &= \frac{1}{725764}\left(362884x - 725782x^3 + 362903x^4 - 12x^7 + 9x^8 - 2x^9\right),\\
d_1(x) &= \frac{1}{7315701120}x\left(3657870720 - 7315882560x^2 + 3658062240x^3 - 120960x^6 - 90721x^7 - 20160x^8\right),\\
c_2(x) &= 0,\ d_2(x) = \frac{1}{10080}x^7,\\
c_3(x) &= \frac{1}{43894206720}(-43895295360x + 87800025610x^3 - 43910173465x^4 + 10160696x^5 - 5806148x^7\\
&\quad +1088673x^8 - 6x^9),\\
d_3(x) &= \frac{1}{43894206720}x\left(-43895295360 + 87800025610x^2 - 43910173465x^3 - 5806148x^6 + 1088673x^7 - 6x^8\right),\\
c_4(x) &= \frac{1}{87788413440}x(43896746880 - 87820346930x^2 + 43950816165x^3 - 30482088x^4 + 4354668x^6\\
&\quad -1088709x^7 + 14x^8),\\
d_4(x) &= \frac{1}{87788413440}x\left(43896746880 - 87820346930x^2 + 43950816165x^3 + 4354668x^6 - 1088709x^7 + 14x^8\right),\\
c_5(x) &= 0,\ d_5(x) = -\frac{1}{2880}x^4,\\
c_6(x) &= 0,\ d_6(x) = \frac{1}{4320}x^3,\\
c_7(x) &= \frac{1}{21947103360}x\left(-362880 + 2177292x - 2903074x^2 + 1088667x^3 - 12x^6 + 9x^7 - 2x^8\right),\\
d_7(x) &= \frac{1}{21947103360}x\left(-362880 - 2903074x^2 + 1088667x^3 - 12x^6 + 9x^7 - 2x^8\right),\\
c_8(x) &= \frac{1}{29262804480}x\left(-362884 + 725782x^2 - 362903x^3 + 12x^6 - 9x^7 + 2x^8\right),\\
d_8(x) &= \frac{1}{29262804480}x\left(362880 + 725782x^2 - 362903x^3 + 12x^6 - 9x^7 + 2x^8\right),\\
c_9(x) &= \frac{1}{131682620160}\left(362882 - 362880x - 18x^3 + 21x^4 - 12x^7 + 9x^8 - 2x^9\right),\\
d_9(x) &= \frac{1}{131682620160}x\left(-362880 - 18x^2 + 21x^3 - 12x^6 + 9x^7 - 2x^8\right).
\end{aligned}
$$