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An Efficient Analytical-Numerical Technique for Handling Model of Fuzzy Differential Equations of Fractional-Order

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Abstract. This paper adds in our hands a different analytic numeric method to solve a class of fuzzy fractional differential equations (FFDEs) based on the residual power series method (RPSM) under strongly generalized differentiability. The analytic and approximate solutions are provided with the series form according to their parametric form. The new method explained in the current paper has a lot of advantages as follows: First, its nature is global according to the obtainable solutions along with being able to solve numerous problems such as mathematical, physical and engineering ones. Second. It is easily noted that it is precise, needs few efforts to have the required results achieved, alongside being developed for nonlinear problems and cases. As for the third advantage, it can be said that any point in the interval of interest will be possibly picked, in addition, to have the approximate solutions applied. Fourth, the method does not need the variables discretization, also it is not implemented by computational round off errors. At last, the results reached in the current paper show several features concerning the new method such as potentiality, generality and superiority to handle such problems arising in physics and engineering as well.

1. Introduction

Fuzzy theory of fractional differential equations is an important branch of mathematics. It has ample applications due to the fact that many practical problems in industrial engineering, computer science, physics, artificial intelligence, and operations research may be converted to uncertain process problems of fractional order. Recently, it has gained considerable attention of researchers in modeling a lot of phenomena which are inherently vague. However, the theory of fractional calculus which is a generalization of classical calculus, deals with the discussion of the integrals and derivatives of non-integer order, has a long history, dates back to the seventeenth century. It was primarily developed in the past times to be a pure mathematical and theoretical field, is still effectively used in various fields such as rheology, viscoelasticity, electrochemistry and diffusion processes (see for example [2, 11, 17]). Indeed, there are several significant works on the topic of fractional calculus, but the most newly influential work was the monograph of Podlubny [32] and Kilbas et al. [25]. It should be noted that the fuzzy fractional differential equations can be effectively and strongly applied to model a collection of real world physical problems, which really requires a lot of investigations. As a fact, there are many examples about functioning modeling in real life situations such as earthquakes model, the model of fluid dynamic traffic with fractional derivatives, the

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process of measuring of viscoelastic material properties, etc [7]. Accordingly, numerous research papers were conducted to examine and investigate the aforementioned theory and solutions of fuzzy and fractional differential equations (see [3, 4, 21, 30, 34–36]).

A lot of mathematicians and authors have recently presented the idea of solutions for fuzzy differential equations of fractional order. To mention a few, the authors in [6] have deemed the concept of Riemann–Liouville differentiability which depends on Hukuhara differentiability to solve fuzzy fractional differential equations. The concepts concerning the generalized Hukuhara fractional Riemann–Liouville and Caputo differentiability of fuzzy-valued functions have been investigated in [8, 9]. In [28], a modified fractional Euler method was selected and applied to investigate the solution of fuzzy fractional initial value problem under Caputo fuzzy fractional derivatives. The method of the fuzzy Laplace transform is applied to solve the fuzzy fractional differential equations along with the parallel fuzzy initial and boundary value problems under Riemann–Liouville H-differentiability ([38, 39]). Further research papers regarding numerical techniques for differential equations, we refer to [5, 12–15]. A new model with fractional order which include the fuzzy parameter was solved by [7] using a new spectral tau method. In [31], the optimization technique was used to solve the differential equation of fractional order, utilizing the artificial neural network (ANN). Their work which is based on an ANN scheme was validated by different types of examples of fuzzy differential equations (FDEs).

The aim of this work is to extend the application of the residual power series method (RPSM) under the assumption of strongly generalized differentiability to provide numerical approximate solutions for FDEs of the general form

$$x^{\alpha}(t) = f(t, x(t)),\ a \leq t \leq b, \tag{1}$$

subject to the fuzzy initial condition

$$x(0) = x_0, \tag{2}$$

where $f : [a, b] \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is a continuous fuzzy-valued function, $\alpha \in (0, 1]$, in which $\mathbb{R}_{\mathcal{F}}$ denote the set of fuzzy numbers on $\mathbb{R}$.

There are three main approaches in solving fuzzy differential equations with initial conditions. The first approach, postulates that even if the initial value is only fuzzy number, the solution will be fuzzy function, where as a result, the derivatives must be regarded as fuzzy derivatives. To achieve these derivatives, Hukuhara derivative for fuzzy-valued functions has to be used [33]. However, there is a shortcoming related to this approach, where the solution turns to be fuzzier as time passes, therefore, the fuzzy solution acts in a way that is quite dissimilar from the crisp solution. As for the second approach, the fuzzy equation is converted to a crisp one as a family of differential inclusions [25]. The main drawback concerning the use of differential inclusions is that it does not contain a fuzzification of the differential operator, instead, the solution is not essentially a fuzzy interval-valued function. The third approach stems from Zadeh's extension principle, where the related crisp equation is solved and the initial fuzzy values, which are in the solution, are replaced in terms of real constants as well as arithmetic operations regarded as operations on fuzzy numbers in the final solution [32]. Zadeh's extension principle's weak point lies in having the solution in the fuzzy setting re-written, making the techniques of solution less friendly and more constrained with a lot of steps of computation to do. The latest solutions approach, which focuses on searching fuzzy set of real-valued functions, not fuzzy-valued functions examplified with these real-valued functions, fulfills the said restrictions.

The proposed method has the following characteristics. First, the technique yields Taylor expansions of the solutions, as a result, the exact solutions are available when the solutions are polynomials. Second, the solutions along with their derivatives can be applied for each arbitrary point in the given interval. Third, the aforementioned method does not require modifications while converting from the lower to the higher order. Consequently, this method can be easily and directly applied to the given system by selecting an appropriate value for the initial guesses approximations. Fourth, the RPSM requires minor computational requirements with less time and more accuracy. It was developed as an efficient method to determine values of the coefficients of the power series solution for fuzzy differential equations ([10, 19, 20, 26, 27, 29**?**]).

The current paper is mainly organized into six sections including the introduction. As for section two, the primary definitions and basic results relating to fuzzy calculus along with fuzzy fractional calculus will be shown. As for section three, a general review of the FFDEs theory will be presented. Section 4 will present a fractional power series techniuqe as an effective tool to solve the FFDEs of order 0¡1. Numerical solutions for two examples in order to illustrate the proposed, RPSM's abilities are introduced in Section 5. The conclusion of the current paper will rest in section 6.

2. Fuzzy Calculus Theory

A fuzzy set v in a nonempty set S is described by its membership function $v : S \to [0,1]$. So, for each $s \in S$ the degree of membership of an element s in v is defined by $v(s)$. For each $s, t \in \mathbb{R}$ and $0 \le \lambda \le 1$, if $v(\lambda s + (1-\lambda)t) \ge \min\{v(s), v(t)\}$, then we call v is convex on $\mathbb{R}$, and if for $\sigma \in [0,1]$, the set $\{s \in \mathbb{R} | v(s) \ge \sigma\}$ is closed, then we say v is upper semicontinuous. Also, we say that v is normal if $\exists\, s \in \mathbb{R}$ such that $v(s) = 1$. A fuzzy set which is defined as $\{s \in \mathbb{R} | v(s) > 0\}$ is called the support of v.

Definition 2.1. ([23]). Let v be a fuzzy number. One says that v is a fuzzy subset of the real line with it is upper semicontinuous membership function of bounded support, normal, and convex.

Let $[v]^\sigma = \{s \in \mathbb{R} \mid v(s) \ge \sigma\}$ and $[v]^0 = \overline{\{s \in \mathbb{R} \mid v(s) > 0\}}$, $\forall \sigma \in [0,1]$, and the symbol $\overline{\{*\}}$ refer the closure of $\{*\}$. Then, it is easy to establish that v is a fuzzy number if and only if $[v]^\sigma$ is compact convex subset of $\mathbb{R}$ for each $\sigma \in [0,1]$ and $[v]^1 \neq \phi$ [23]. So, if v is a fuzzy number, then $[v]^\sigma = [v_1(\sigma), v_2(\sigma)]$, where $v_1(\sigma) = \min\{s \mid s \in [v]^\sigma\}$ and $v_1(\sigma) = \max\{s \mid s \in [v]^\sigma\}$ for each $\sigma \in [0,1]$. We call the symbol $[v]^\sigma$ the $\sigma-$cut representation or the form of parametric for a fuzzy number v.

Theorem 2.1. ([23]). *Suppose that $v_1, v_2 : [0,1] \to \mathbb{R}$ satisfy the following conditions:*

1. *v_1 is a bounded nondecreasing function,*
2. *v_2 is a bounded nonincreasing function,*
3. *$v_1(1) \le v_2(1)$,*
4. *for each $k \in (0,1]$, $\lim_{\sigma\to k^-} v_1(\sigma) = v_1(k)$ and $\lim_{\sigma\to k^-} v_2(\sigma) = v_2(k)$,*
5. *$\lim_{\sigma\to 0^+} v_1(\sigma) = v_1(0)$ and $\lim_{\sigma\to 0^+} v_2(\sigma) = v_2(0)$.*

Then $v : \mathbb{R} \to [0,1]$ defined by $v(s) = \sup\{\sigma \mid v_1(\sigma) \le s \le v_2(\sigma)\}$ is a fuzzy number with parametrization $[v_1(\sigma), v_2(\sigma)]$. Furthermore, if $v_1, v_2 : [0,1] \to \mathbb{R}$ is a fuzzy number with parametrization $[v_1(\sigma), v_2(\sigma)]$, then the functions v_1 and v_2 satisfy the aforementioned conditions.

Definition 2.2. ([41]). The complete metric structure on $\mathbb{R}_\mathcal{F}$ is given by the Hausdorff distance mapping $D_H : \mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F} \to \mathbb{R}^+ \cup \{0\}$ such that

$$D_H(v, w) = \sup_{0\le\sigma\le1} \max\{|v_{1\sigma} - w_{1\sigma}|, |v_{2\sigma} - w_{2\sigma}|\},$$

for arbitrary fuzzy numbers v and w.

Two fuzzy numbers v and w are equal if $[v]^\sigma = [w]^\sigma$ for each $\sigma \in [0,1]$. For arithmetic operations on fuzzy numbers, the following results are well-known and coincide with the theory of interval analysis.

Theorem 2.2. ([40]). *If v and w are two fuzzy numbers and $\lambda \in \mathbb{R} \setminus \{0\}$, then for each $\sigma \in [0,1]$, we have*

1. $[v + w]^\sigma = [v]^\sigma + [w]^\sigma = [v_{1\sigma} + w_{1\sigma}, v_{2\sigma} + w_{2\sigma}]$;
2. $[v - w]^\sigma = [v]^\sigma - [w]^\sigma = [v_{1\sigma} - w_{2\sigma}, v_{2\sigma} - w_{1\sigma}]$;
3. $[\lambda v]^\sigma = \lambda[v]^\sigma = [\min\{\lambda v_{1\sigma}, \lambda v_{2\sigma}\}, \max\{\lambda v_{1\sigma}, \lambda v_{2\sigma}\}]$;
4. $[vw]^\sigma = [\min\{v_{1\sigma}w_{1\sigma}, v_{1\sigma}w_{2\sigma}, v_{2\sigma}w_{1\sigma}, v_{2\sigma}w_{2\sigma}\}, \max\{v_{1\sigma}w_{1\sigma}, v_{1\sigma}w_{2\sigma}, v_{2\sigma}w_{1\sigma}, v_{2\sigma}w_{2\sigma}\}]$.

Definition 2.3. ([22]). Let $v, w \in \mathbb{R}_{\mathcal{F}}$. If there exists an element $\mathcal{P} \in \mathbb{R}_{\mathcal{F}}$ such that $v = w + \mathcal{P}$, then we say that $\mathcal{P}$ is the Hukuhara difference (H- difference) of v and w, denoted by $v \ominus w$.

Remark 2.1. The sign $\ominus$ stands always for Hukuhara difference. Thus, it should be noted that $v \ominus w \neq v + (-1)w$. Normally, $v + (-1)w$ denoted by $v - w$. If the H-difference $v \ominus w$ exists, then $[v \ominus w]^{\sigma} = [v_1(\sigma) - w_1(\sigma), v_2(\sigma) - w_2(\sigma)]$.

Definition 2.4. ([40]). Let y be a fuzzy function on $[a, b]$. Then the σ-cut function on $[a, b]$ is an interval valued function $y_{\sigma} : [a, b] \to \mathbb{R}_{\mathbb{C}}$ defined by $y_{\sigma}(x) = [y(x)]^{\sigma}, \forall \sigma \in [0, 1]$. Hence, $y_{\sigma}(x) = [y_{1\sigma}(x), y_{2\sigma}(x)]$ where $y_{1\sigma}$ and $y_{2\sigma}$ are real valued functions on $[a, b]$ given by $y_{1\sigma}(x) = \min\{y_{\sigma}(x)\}$ and $y_{2\sigma}(x) = \max\{y_{\sigma}(x)\}, \forall \sigma \in [0, 1]$.

As we turn now to present the concept of fuzzy function differentiation, it should be noted that Hukuhara presented the derivative for set valued mappings in late sixties. Later on, Puri and Ralescu extended it for the fuzzy valued mappings in early eighties.

Definition 2.5. ([33]). A mapping $y : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is said to be Hukuhara differentiable, or simply H-differentiable, at $x_* \in [a, b]$ if there is a fuzzy number $y'(x_*)$ such that $\lim_{h\to 0^+} \frac{y(x_*+h)\ominus y(x_*)}{h}$ and $\lim_{h\to 0^+} \frac{y(x_*)\ominus y(x_*-h)}{h}$ exist and are equal to $y'(x_*)$ which is called the H-derivative.

Here, the limit is taken in the metric space $(\mathbb{R}_{\mathcal{F}}, D_H)$ and at the endpoints of $[a, b]$, we consider only one-sided derivatives.

Definition 2.6. ([16]). Let $y : [a, b] \to \mathbb{R}_{\mathcal{F}}$ and for fixed $x_* \in [a, b]$. Then y is called a strongly generalized differentiable at x_*, if there is an element $y'(x_*) \in \mathbb{R}_{\mathcal{F}}$ such that either

i. The H-differences $y(x_* + \xi) \ominus y(x_*)$, $y(x_*) \ominus y(x_* - \xi)$ exist, for each $\xi > 0$ sufficiently tends to 0 and $\lim_{\xi\to 0^+} \dfrac{y(x_* + \xi) \ominus y(x_*)}{\xi} = y'(x_*) = \lim_{\xi\to 0^+} \dfrac{y(x_*) \ominus y(x_* - \xi)}{\xi}$,

ii. The H-differences $y(x_*) \ominus y(x_* + \xi), y(x_* - \xi) \ominus y(x_*)$ exist, for each $\xi > 0$ sufficiently tends to 0 and $\lim_{\xi\to 0^+} \dfrac{y(x_*) \ominus y(x_* + \xi)}{-\xi} = y'(x_*) = \lim_{\xi\to 0^+} \dfrac{y(x_* - \xi) \ominus y(x_*)}{-\xi}$.

As well, it should be noted that the limits within Definition 2.6 is taken in the complete metric space $(\mathbb{R}_{\mathcal{F}}, D_H)$ and at the endpoints of $[a, b]$ by considering only one-sided derivatives.

Definition 2.7. ([24]). Let $y : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is differentiable for any point $x \in (a, b)$. Then y is differentiable on (a, b). Moreover, if y is differentiable in term of the first condition of definition 2.6, where its derivative at x_* is given by $y'(x_*) = D_1^1 y(x_*)$, then we say that y is (1)-differentiable on (a, b). As well, if y is differentiable in term of second condition of definition 2.6, where its derivative at x_* is given by $y'(x_*) = D_2^1 y(x_*)$, then we say that y is (2)-differentiable on (a, b). However, if $D_1^1 y(x_*)$ exists, then $D_2^1 y(x_*)$ does not exists.

Theorems below assist us to convert the fuzzy fractional differential equations (FFDEs) into a system of ordinary fractional differential equations (OFDEs), ignoring the fuzzy setting approach.

Theorem 2.3. ([24]). *Let $y : [a, b] \to \mathbb{R}_{\mathcal{F}}$, where $[y(x)]^{\sigma} = [y_{1\sigma}(x), y_{2\sigma}(x)]$ for each $\sigma \in [0, 1]$,*

1. *if y is (1)-differentiable, then $y_{1\sigma}$ and $y_{2\sigma}$ are differentiable functions and $\left[D_1^1 y(x)\right]^{\sigma} = \left[y'_{1\sigma}(x), y'_{2\sigma}(x)\right]$,*
2. *if y is (2)-differentiable, then $y_{1\sigma}$ and $y_{2\sigma}$ are differentiable functions and $\left[D_2^1 y(x)\right]^{\sigma} = \left[y'_{2\sigma}(x), y'_{1\sigma}(x)\right]$.*

Let $y : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued function. Then the function y is continuous at $x_* \in [a, b]$ if for every $\epsilon > 0, \exists\, \delta = \delta\,(x_*, \epsilon) > 0$, such that $D_H(y(x), y(x_*)) < \epsilon$ for each $x \in [a, b]$, whenever $|x - x_*| < \delta$. Consequently, we say that y is continuous on $[a, b]$ if y is continuous at each $x_* \in [a, b]$ such that the continuity is one-sided at endpoints of $[a, b]$.

Next, the second fuzzy derivatives definition is given based on the derivative types selection in each differentiation step. For a given fuzzy-valued function $y(x)$, we have two possibilities according to definition 2.7 in order to obtain the derivative of $y(x)$ as follows: $D_1^1 y(x)$ and $D_2^1 y(x)$. Anyhow, for each of these two derivatives, we have again two possibilities of derivatives: $D_1^1\left(D_1^1 y(x)\right), D_2^1\left(D_1^1 y(x)\right)$ and $D_1^1(D_2^1 y(x)), D_2^1\left(D_2^1 y(x)\right)$, respectively.

Definition 2.8. ([24]). Suppose that $y : [a, b] \to \mathbb{R}_{\mathcal{F}}$. One can say $y(x)$ is (n, m)-differentiable on $[a, b]$ if $D_n^1 y(x)$ exist and its (m)-differentiable. The second derivatives of y are denoted by $D_{n,m}^2 y(x)$ for $n, m \in \{1, 2\}$.

Theorem 2.4. ([24]). *Let $D_1^1 y : [a, b] \to \mathbb{R}_{\mathcal{F}}$ and $D_2^1 y : [a, b] \to \mathbb{R}_{\mathcal{F}}$, where $[y(x)]^\sigma = [y_{1\sigma}(x), y_{2\sigma}(x)]$ for each $\sigma \in [0, 1]$:*

1. *$D_1^1 y$ is (1)-differentiable, then $y'_{1\sigma}$ and $y'_{2\sigma}$ are differentiable functions and $\left[D_{1,1}^2 y(x)\right]^\sigma = \left[y''_{1\sigma}(x), y''_{2\sigma}(x)\right]$,*
2. *$D_1^1 y$ is (2)-differentiable, then $y'_{1\sigma}$ and $y'_{2\sigma}$ are differentiable functions and $\left[D_{1,2}^2 y(x)\right]^\sigma = \left[y''_{2\sigma}(x), y''_{1\sigma}(x)\right]$,*
3. *$D_2^1 y$ is (1)-differentiable, then $y'_{1\sigma}$ and $y'_{2\sigma}$ are differentiable functions and $\left[D_{2,1}^2 y(x)\right]^\sigma = \left[y''_{2\sigma}(x), y''_{1\sigma}(x)\right]$,*
4. *$D_2^1 y$ is (2)-differentiable, then $y'_{1\sigma}$ and $y'_{2\sigma}$ are differentiable functions and $\left[D_{2,2}^2 y(x)\right]^\sigma = \left[y''_{1\sigma}(x), y''_{2\sigma}(x)\right]$.*

Definition 2.9. ([37]). Let $y : [a, b] \to \mathbb{R}_{\mathcal{F}}$ and $y \in C^{\mathcal{F}}[a, b] \cap L^{\mathcal{F}}[a, b]$ be a fuzzy set-value function. Then y is said to be Caputo fuzzy H-differentiable at x when $\left({}^C D_{a^+}^\alpha y\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{y'(t)}{(x-t)^\alpha} dt$ exists, where $0 < \alpha \leq 1$.

As well, we say that y is Caputo $[(1) - \alpha]$-differentiable if y is (1)-differentiable, and y is Caputo $[(2) - \alpha]$-differentiable if y is (2)-differentiable.

Theorem 2.5. ([37]). *Let $0 < \alpha \leq 1$ and $y \in AC^{\mathcal{F}}[a, b]$. Then the fuzzy Caputo fractional derivative exists almost everywhere on (a, b) and for all $\sigma \in [0, 1]$, we have*

$$\left[\left({}^C D_{a^+}^\alpha y\right)(x)\right]^\sigma = \left[\frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{y'_{1\sigma}(t)}{(x-t)^\alpha} dt, \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{y'_{2\sigma}(t)}{(x-t)^\alpha} dt\right] = \left[J_{a^+}^{1-\alpha} D y_{1\sigma}(x), J_{a^+}^{1-\alpha} D y_{2\sigma}(x)\right]$$

for (1)-differentiable,

$$\left[\left({}^C D_{a^+}^\alpha y\right)(x)\right]^\sigma = \left[\frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{y'_{2\sigma}(t)}{(x-t)^\alpha} dt, \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{y'_{1\sigma}(t)}{(x-t)^\alpha} dt\right] = \left[J_{a^+}^{1-\alpha} D y_{2\sigma}(x), J_{a^+}^{1-\alpha} D y_{1\sigma}(x)\right]$$

for (2)-differentiable.

3. Fuzzy Fractional Differential Equation

When the world physical phenomena are modeled, the role of the ordinary differential equations (ODEs) can be effectively seen in several fields of discipline such as engineering, economics, physics and applied mathematics. There are experts in such fields who use crisp ODEs it to understand and solve some problems underin their studies. In several situations, information about these related world physical phenomena is prevail and covered with sense of uncertainty. The uncertainty can be strongly appeared in a number of places, namely: the part of experiment, the process of data collection, and measurement in addition to determining the initial values. Therefore, it is necessary to have some mathematical tools in order to understand this uncertainty. Hence, it would be natural to employ fuzzy differential equations (FDEs). Consequently, it is too essential to apply some mathematical tools to comprehend this uncertainty. Hence, it would be natural to employ (FDE).

Let $0 < \alpha \leq 1$. Then the FFDE (1) and (2) is equivalent to one of the following integral equations: $x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{y(s, X(s))\, ds}{(t-s)^{1-\alpha}}$, if $x(t)$ is Caputo $[(1)-\alpha]$-differentiable, and $x(t) = x(0) \ominus \frac{-1}{\Gamma(\alpha)} \int_0^t \frac{y(s, X(s))\, ds}{(t-s)^{1-\alpha}}$, if $x(t)$ is Caputo $[(2) - \alpha]$-differentiable.

Theorem 3.1. ([37]). *The FFDE (1) and (2) is equivalent to the system of ordinary fractional differential equations (OFDEs): if $x(t)$ is Caputo $[(1)-\alpha]$-differentiable, then*

$$\begin{aligned} &\left({}^{C}D^{\alpha}_{t_0^+}x_{1\sigma}\right)(t) = f_{1\sigma}\left(t, x_{1\sigma}(t), x_{2\sigma}(t)\right) \\ &\left({}^{C}D^{\alpha}_{t_0^+}x_{2\sigma}\right)(t) = f_{2\sigma}\left(t, x_{1\sigma}(t), x_{2\sigma}(t)\right) \\ &x_{1\sigma}(0) = x_{01\sigma} \\ &x_{2\sigma}(0) = x_{02\sigma}. \end{aligned} \tag{3}$$

if $x(t)$ is Caputo $[(2)-\alpha]$-differentiable, then FFDE (1) and (2) is equivalent to the following system of OFDEs:

$$\begin{aligned} &\left({}^{C}D^{\alpha}_{t_0^+}x_{1\sigma}\right)(t) = f_{2\sigma}\left(t, x_{1\sigma}(t), x_{2\sigma}(t)\right) \\ &\left({}^{C}D^{\alpha}_{t_0^+}x_{2\sigma}\right)(t) = f_{1\sigma}\left(t, x_{1\sigma}(t), x_{2\sigma}(t)\right) \\ &x_{1\sigma}(0) = x_{01\sigma} \\ &x_{2\sigma}(0) = x_{02\sigma}. \end{aligned} \tag{4}$$

Algorithm 3.1. To find the solutions of FFDE (1) and (2), we discuss the following two cases:
Case (1): If $x(t)$ is Caputo $[(1)-\alpha]$-differentiable, we convert FFDE (1) and (2) to the system (3) and follow the steps:

Step 1: Solve the system (3).
Step 2: Ensure that $[x_{1\sigma}(t), x_{2\sigma}(t)]$ and $[x'_{1\sigma}(t), x'_{2\sigma}(t)]$ are valid level sets for $\sigma \in [0,1]$.
Step 3: Use Zadeh's principle to construct the solution $x(t)$ whose σ-cut representation is $[x_{1\sigma}(t), x_{2\sigma}(t)]$.

Case (2): If $x(t)$ is Caputo $[(2)-\alpha]$-differentiable, we convert FFDE (1) and (2) to the system (4) and follow the steps:

Step 1: Solve the system (4).
Step 2: Ensure that $[x_{1\sigma}(t), x_{2\sigma}(t)]$ and $[x'_{1\sigma}(t), x'_{2\sigma}(t)]$ are valid level sets for $\sigma \in [0,1]$.
Step 3: Use Zadeh's principle to construct the solution $x(t)$ whose σ-cut representation is $[x_{1\sigma}(t), x_{2\sigma}(t)]$.

4. Fractional Power Series Technique

In this section, we utilize the RPSM for constructing and obtaining FFDE (3) and (4)'s solutions through substituting the expansion of its fractional power series (FPS) among its truncated residual functions. In view of that, the resultant equation helps us to derive a recursion formula for the coefficients computation, where the FPS expansions coefficients can be computed recursively through recurrent fractional differentiating of the truncated residual function. It is necessary to begin with the following theory and definition that are required in the current paper.

Definition 4.1. ([20]). A power series expansion of the form

$$\sum_{m=0}^{\infty} c_m\left(t-t_0\right)^{m\alpha} = c_0 + c_1\left(t-t_0\right)^{\alpha} + c_2\left(t-t_0\right)^{2\alpha} + \cdots, \tag{5}$$

is called fractional PS about $t = t_0$, where t is a variable and c_m's are constants called the coefficients of the series.

Theorem 4.1. ([20]). *Suppose that f has a fractional PS representation at $t = t_0$ of the form*

$$f(t) = \sum_{m=0}^{\infty} c_m\left(t-t_0\right)^{m\alpha}. \tag{6}$$

If $D^{m\alpha} f(t)$ are continuous on $(t_0, t_0 + R)$, $m = 0, 1, 2, \ldots$, then coefficients c_m of Eq. (6) are given by the formula

$$c_m = \frac{D^{m\alpha} f(t) \mid_{t=t_0}}{\Gamma(m\alpha + 1)},\ m = 0, 1, 2, \ldots \tag{7}$$

where $D^{m\alpha} = D^{\alpha} \cdot D^{\alpha} \cdots D^{\alpha}$ ($m-$times).

Next, we employ RPSM to construct the analytic and approximate series solutions for FFDE (1) and (2) with respect to Caputo $[(1) - \alpha]$-differentiable. However, the same technique can be employed to construct solutions under Caputo $[(2) - \alpha]$-differentiable. Throughout this paper, we assume that, $x_{1\sigma}, x_{2\sigma}, f_{1\sigma}$ and $f_{2\sigma}$ are analytic functions on the given interval for each $\sigma \in [0, 1]$.

It is worth noting that the RPSM consists in expressing the solution of system of OFDEs (3) as an FPS expansion about the initial point $t = 0$. To make our goal come true, we assume that these solutions take the form $x_{1\sigma}(t) = \sum_{n=0}^{\infty} x_{n,1\sigma}(t)$ and $x_{2\sigma}(t) = \sum_{n=0}^{\infty} x_{n,2\sigma}(t)$, where $x_{n,1\sigma}(t)$ and $x_{n,2\sigma}(t)$ are two terms of approximation and are represented as:

$$\begin{aligned} x_{n,1\sigma}(t) &= c_n \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)}, \\ x_{n,2\sigma}(t) &= d_n \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)} \end{aligned} \tag{8}$$

We notice that, if $n = 0$, then $x_{0,1\sigma}(t)$ and $x_{0,2\sigma}(t)$ satisfy the initial condition system of OFDEs (3), in which, $x_{0,1\sigma}(0)$ and $x_{0,2\sigma}(0)$ are the initial guesses approximations of $x_{1\sigma}$ and $x_{2\sigma}$, respectively, so from Eq. (8), we obtain $x_{1\sigma}(0) = x_{0,1\sigma} = c_0$ and $x_{2\sigma}(0) = x_{0,2\sigma} = d_0$.

On the other aspect as well, if we choose $x_{0,1\sigma}$ and $x_{0,2\sigma}$ as an initial guesses approximations of $x_{1\sigma}(t)$ and $x_{2\sigma}(t)$, respectively, then we can calculate $x_{n,1\sigma}(t)$ and $x_{n,2\sigma}(t)$, for $n = 0, 1, 2, \ldots$, and approximate the solutions $x_{1\sigma}(t)$ and $x_{2\sigma}(t)$ for system of OFDEs (3) by the kth-truncated series

$$\begin{aligned} x_{k,1\sigma}(t) &= c_0 + \sum_{n=1}^{k} c_n \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)}, \\ x_{k,2\sigma}(t) &= d_0 + \sum_{n=1}^{k} d_n \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)}. \end{aligned} \tag{9}$$

Before applying the RPS algorithm for finding the value of coefficients c_n and d_n, $n = 1, 2, 3, \ldots, k$, in the series expansion, we should define the residual functions $Res_{1\sigma}(t)$ and $Res_{2\sigma}(t)$ for Eq. (3) as follows:

$$\begin{aligned} Res_{1\sigma}(t) &= \left({}^{C}D^{\alpha}_{t_0^+} x_{1\sigma}\right)(t) - f_{1\sigma}\left(t, x_{1\sigma}(t), x_{2\sigma}(t)\right), \\ Res_{2\sigma}(t) &= \left({}^{C}D^{\alpha}_{t_0^+} x_{2\sigma}\right)(t) - f_{2\sigma}\left(t, x_{1\sigma}(t), x_{2\sigma}(t)\right). \end{aligned} \tag{10}$$

As well, for $k = 1, 2, 3, \ldots$. The kth-residual function $Res_{k,1\sigma}$ and $Res_{k,2\sigma}$ as the following of the style form:

$$\begin{aligned} Res_{k,1\sigma}(t) &= \left({}^{C}D^{\alpha}_{t_0^+} x_{k,1\sigma}\right)(t) - f_{1\sigma}\left(t, x_{k,1\sigma}(t), x_{k,2\sigma}(t)\right), \\ Res_{k,2\sigma}(t) &= \left({}^{C}D^{\alpha}_{t_0^+} x_{k,2\sigma}\right)(t) - f_{2\sigma}\left(t, x_{k,1\sigma}(t), x_{k,2\sigma}(t)\right). \end{aligned} \tag{11}$$

Consequently, we note that $Res_{n\sigma}(t) = 0$ and $\lim_{k\to 0} Res_{k,n\sigma}(t) = Res_{n\sigma} \equiv 0$ for $n = 1, 2$ and each $t \geq 0$. In fact, these lead to $D_t^{m\alpha} Res_{n\sigma}(t) = 0$, since the fractional derivative of a constant function in the Caputo's sense is zero. Also, the fractional derivatives $D_t^{m\alpha} Res_{n\sigma}(t)$ and $D_t^{m\alpha} Res_{k,n\sigma}(t)$ are equivalent at $t = 0$ for each

$m = 0, 1, 2, \ldots, k$; that is, $D_t^{m\alpha} Res_{n\sigma}(0) = D_t^{m\alpha} Res_{k,n\sigma}(0) = 0$. For finding the coefficient values' c_n and d_n of Eq. (9), for $n = 1, 2, 3, \ldots, k$, we solve the following system:

$$\begin{aligned} &D_t^{(k-1)\alpha} Res_{k,1\sigma}(0) = 0, \\ &D_t^{(k-1)\alpha} Res_{k,2\sigma}(0) = 0. \end{aligned} \tag{12}$$

In other words, our claim is to determine the unknown coefficients c_n and d_n of Eq. (9). So, in order to find those coefficients, we need substituting Eq. (9) into the kth-residual functions, $Res_{k,1\sigma}$ and $Res_{k,2\sigma}$ of Eq. (11). Following, we compute the fractional derivative formula $D_t^{(k-1)\alpha}$ on both sides of $Res_{k,1\sigma}, Res_{k,2\sigma}$ and then we solve the obtained algebraic systems $D_t^{(k-1)\alpha} Res_{k,1\sigma}(0) = 0$ and $D_t^{(k-1)\alpha} Res_{k,2\sigma}(0) = 0$ for $0 < \alpha \leq 1,\ t \geq 0$.

To obtain the coefficient, c_n and d_n for $n = 1, 2, 3, \ldots, k$, we do the following steps:

Firstly, to find c_1 and d_1, we consider $k = 1$, in Eq. (9), then we substitute $x_{1,1\sigma}(t) = c_0 + c_1 \frac{t^\alpha}{\Gamma(1+\alpha)}$ and $x_{1,2\sigma}(t) = d_0 + d_1 \frac{t^\alpha}{\Gamma(1+\alpha)}$, into $Res_{1,1\sigma}(t)$ and $Res_{1,2\sigma}(t)$ of Eq. (11), that is,

$$\begin{aligned} Res_{1,1\sigma}(t) &= \left({}^C D_{t_0^+}^{\alpha} x_{1,1\sigma}\right)(t) - f_{1\sigma}\left(t, x_{1,1\sigma}(t), x_{1,2\sigma}(t)\right), \\ &= c_1 - f_{1\sigma}\left(t, c_0 + c_1 \frac{t^\alpha}{\Gamma(1+\alpha)}, d_0 + d_1 \frac{t^\alpha}{\Gamma(1+\alpha)}\right). \\ Res_{1,2\sigma}(t) &= \left({}^C D_{t_0^+}^{\alpha} x_{1,2\sigma}\right)(t) - f_{2\sigma}\left(t, x_{1,1\sigma}(t), x_{1,2\sigma}(t)\right), \\ &= d_1 - f_{2\sigma}\left(t, c_0 + c_1 \frac{t^\alpha}{\Gamma(1+\alpha)}, d_0 + d_1 \frac{t^\alpha}{\Gamma(1+\alpha)}\right). \end{aligned} \tag{13}$$

Depending on the result of Eq. (12), for $k = 1$, the substitution of $t = 0$ through Eq. (13) will yields

$$\begin{aligned} c_1 &= f_{1\sigma}\left(0, c_0, d_0\right). \\ d_1 &= f_{2\sigma}\left(0, c_0, d_0\right). \end{aligned} \tag{14}$$

Thus, the first RPS approximate solution for system of OFDEs (3) can be written as:

$$\begin{aligned} x_{1,1\sigma}(t) &= c_0 + f_{1\sigma}\left(0, c_0, d_0\right) \frac{t^\alpha}{\Gamma(1+\alpha)}, \\ x_{1,2\sigma}(t) &= d_0 + f_{2\sigma}\left(0, c_0, d_0\right) \frac{t^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \tag{15}$$

Secondly, to determine c_2 and d_2, we set $k = 2$, in Eq. (9), then we substitute $x_{2,1\sigma}(t) = c_0 + c_1 \frac{t^\alpha}{\Gamma(1+\alpha)} + c_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$ and $x_{2,2\sigma}(t) = d_0 + d_1 \frac{t^\alpha}{\Gamma(1+\alpha)} + d_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$ into $Res_{2,1\sigma}(t)$ and $Res_{2,2\sigma}(t)$ of Eq. (11), as follows:

$$\begin{aligned} Res_{2,1\sigma}(t) &= \left({}^C D_{t_0^+}^{\alpha} x_{2,1\sigma}\right)(t) - f_{1\sigma}\left(t, x_{2,1\sigma}(t), x_{2,2\sigma}(t)\right), \\ &= c_1 + c_2 \frac{t^\alpha}{\Gamma(1+\alpha)} - f_{1\sigma}\left(t, c_0 + c_1 \frac{t^\alpha}{\Gamma(1+\alpha)} + c_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, d_0 + d_1 \frac{t^\alpha}{\Gamma(1+\alpha)} + d_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right), \\ Res_{2,2\sigma}(t) &= \left({}^C D_{t_0^+}^{\alpha} x_{2,2\sigma}\right)(t) - f_{2\sigma}\left(t, x_{2,1\sigma}(t), x_{2,2\sigma}(t)\right), \\ &= d_1 + d_2 \frac{t^\alpha}{\Gamma(1+\alpha)} - f_{2\sigma}\left(t, c_0 + c_1 \frac{t^\alpha}{\Gamma(1+\alpha)} + c_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, d_0 + d_1 \frac{t^\alpha}{\Gamma(1+\alpha)} + d_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right). \end{aligned} \tag{16}$$

By applying the operator D_t^{α} on both sides of Eq. (16) gives the $\alpha - th$ fractional derivative of $Res_{2,1\sigma}(t)$ and $Res_{2,2\sigma}(t)$ and then we solve the obtained algebraic equations $D_t^{\alpha}Res_{2,1\sigma}(0) = 0$ & $D_t^{\alpha}Res_{2,2\sigma}(0) = 0$, to conclude

$$\begin{aligned} c_2 &= f_{1\sigma}\left(0, c_1\Gamma(1+\alpha), d_1\Gamma(1+\alpha)\right), \\ d_2 &= f_{2\sigma}\left(0, c_1\Gamma(1+\alpha), d_1\Gamma(1+\alpha)\right). \end{aligned} \tag{17}$$

Therefore, the second RPS approximate solution for the system of OFDEs (3) can be written as

$$\begin{aligned} x_{2,1\sigma}(t) &= c_0 + f_{1\sigma}\left(0, c_0, d_0\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + f_{1\sigma}\left(0, c_1\Gamma(1+\alpha), d_1\Gamma(1+\alpha)\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \\ x_{2,2\sigma}(t) &= d_0 + f_{2\sigma}\left(0, c_0, d_0\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + f_{2\sigma}\left(0, c_1\Gamma(1+\alpha), d_1\Gamma(1+\alpha)\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \end{aligned} \tag{18}$$

Thirdly, to derive the coefficients c_3 and d_3, we consider $k = 3$, in Eq. (9), then we substitute $x_{3,1\sigma}(t) = c_0 + c_1\frac{t^{\alpha}}{\Gamma(1+\alpha)} + c_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + c_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$ and $x_{3,2\sigma}(t) = d_0 + d_1\frac{t^{\alpha}}{\Gamma(1+\alpha)} + d_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + d_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$, into the $Res_{3,1\sigma}$ and $Res_{3,2\sigma}$ of Eq. (11), then we have

$$\begin{aligned} Res_{3,1\sigma}(t) &= \left({}^{C}D_{t_0^+}^{\alpha}x_{3,1\sigma}\right)(t) - f_{1\sigma}\left(t, x_{3,1\sigma}(t), x_{3,2\sigma}(t)\right) \\ &= c_1 + c_2\frac{t^{\alpha}}{\Gamma(1+\alpha)} + c_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - f_{1\sigma}\Big(t, c_0 + c_1\frac{t^{\alpha}}{\Gamma(1+\alpha)} + c_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + c_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \\ &\quad d_0 + d_1\frac{t^{\alpha}}{\Gamma(1+\alpha)} + d_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + d_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\Big) \\ Res_{3,2\sigma}(t) &= \left({}^{C}D_{t_0^+}^{\alpha}x_{3,2\sigma}\right)(t) - f_{2\sigma}\left(t, x_{3,1\sigma}(t), x_{3,2\sigma}(t)\right) \\ &= d_1 + d_2\frac{t^{\alpha}}{\Gamma(1+\alpha)} + d_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - f_{2\sigma}\Big(t, c_0 + c_1\frac{t^{\alpha}}{\Gamma(1+\alpha)} + c_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + c_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \\ &\quad d_0 + d_1\frac{t^{\alpha}}{\Gamma(1+\alpha)} + d_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + d_3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\Big). \end{aligned} \tag{19}$$

According to Eq. (12), we solve $D_t^{2\alpha}Res_{3,1\sigma}(t)\mid_{t=0}$ and $D_t^{2\alpha}Res_{3,2\sigma}(t)\mid_{t=0}$, to get the values of c_3 and d_3 as the following:

$$\begin{aligned} c_3 &= f_{1\sigma}\left(0, c_2\Gamma(1+2\alpha), d_2\Gamma(1+2\alpha)\right). \\ d_3 &= f_{2\sigma}\left(0, c_2\Gamma(1+2\alpha), d_2\Gamma(1+2\alpha)\right). \end{aligned} \tag{20}$$

As a matter of fact, from Eq (20) and based on the previous results of Eqs (14) and (17) and the initial gusses approximations, the third RPS approximate solution for system (3) can be summarized in the following expansion:

$$\begin{aligned} x_{3,1\sigma}(t) &= c_0 + f_{1\sigma}\left(0, c_0, d_0\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + f_{1\sigma}\left(0, c_1\Gamma(1+\alpha), d_1\Gamma(1+\alpha)\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &\quad + f_{1\sigma}\left(0, c_2\Gamma(1+2\alpha), d_2\Gamma(1+2\alpha)\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ x_{3,2\sigma}(t) &= d_0 + f_{2\sigma}\left(0, c_0, d_0\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + f_{2\sigma}\left(0, c_1\Gamma(1+\alpha), d_1\Gamma(1+\alpha)\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &\quad + f_{2\sigma}\left(0, c_2\Gamma(1+2\alpha), d_2\Gamma(1+2\alpha)\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}. \end{aligned} \tag{21}$$

By the same argument, the process can be repeated till the arbitrary order coefficients of the FPS solution for the system of OFDE (3) are obtained. So, we have the following recurrence relation for $n \geq 1$.

$$\begin{aligned} c_n &= f_{1\sigma}\left(0, c_{n-1}\Gamma(1+(n-1)\alpha), d_{n-1}\Gamma(1+(n-1)\alpha)\right). \\ d_n &= f_{2\sigma}\left(0, c_{n-1}\Gamma(1+(n-1)\alpha), d_{n-1}\Gamma(1+(n-1)\alpha)\right). \end{aligned} \tag{22}$$

Moreover, the *kth*-RPS approximate solution for system (3) can be written in the following expansion:

$$\begin{aligned} x_{k,1\sigma}(t) &= c_0 + \sum_{n=1}^{k} \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} f_{1\sigma}\left(0, c_{n-1}\Gamma(1+(n-1)\alpha), d_{n-1}\Gamma(1+(n-1)\alpha)\right), \\ x_{k,2\sigma}(t) &= d_0 + \sum_{n=1}^{k} \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} f_{2\sigma}\left(0, c_{n-1}\Gamma(1+(n-1)\alpha), d_{n-1}\Gamma(1+(n-1)\alpha)\right). \end{aligned} \tag{23}$$

It is worth noting that the RPSM is a numerical technique depends on the formula of generalized Taylor series which constructs an analytical solution in the convergent series form. Thus, we can reach a good approximation with the exact solution through the only use of limited numbers of terms. In view of that, the overall errors can be lessened through adding more terms for the approximations of the RPS.

5. Numerical Results

To demonstrate the features of the current new method, namely: efficiency, properties, behavior along with its applicability, two clear examples including linear and nonlinear problems are numerically presented.

Example 5.1. Consider the FFDE:

$$\begin{aligned} &\left({}^{C}D_{0^+}^{\beta} x\right)(t) = -x(t) + sin(t),\ 0 < \beta \leq 1,\ t \in [0,1] \\ &\text{subject to} \\ &\qquad [x(0)]^r = \left[\frac{24}{25} + \frac{1}{25}r, \frac{101}{100} - \frac{1}{100}r\right]. \end{aligned} \tag{24}$$

Depending on the type of differentiability, (24) can be converted to one of the following systems:
Case (1): Under Caputo $[(1)-\beta]$-differentiability, the system of OFDEs corresponding to Caputo $[(1)-\beta]$-differentiable is

$$\begin{aligned} {}^{C}D_{0^+}^{\beta} x_{1r}(t) &= -x_{1r}(t) + sin(t) \\ {}^{C}D_{0^+}^{\beta} x_{2r}(t) &= -x_{2r}(t) + sin(t) \end{aligned} \tag{25}$$

subject to

$$\begin{aligned} x_{1r}(0) &= \frac{24}{25} + \frac{1}{25}r \\ x_{2r}(0) &= \frac{101}{100} - \frac{1}{100}r \end{aligned}$$

If $\beta = 1$, then the exact solution of (25) is:

$$\begin{aligned} x_{1r}(t) &= \frac{1}{2}(sint - cost) + \frac{1}{2}e^{-t} + \left(\frac{24}{25} + \frac{1}{25}r\right)\cosh t - \left(\frac{101}{100} - \frac{1}{100}r\right)\sinh t, \\ x_{2r}(t) &= \frac{1}{2}(sint - cost) + \frac{1}{2}e^{-t} + \left(\frac{101}{100} - \frac{1}{100}r\right)\cosh t - \left(\frac{24}{25} + \frac{1}{25}r\right)\sinh t, \end{aligned} \tag{26}$$

Next some graphical results and tabulated data are presented.

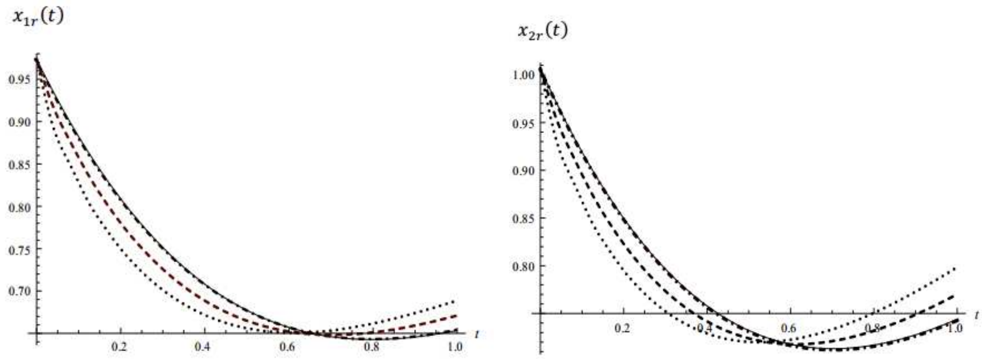

Figure 1: Approximate solutions $x_{1r}(t)$ and $x_{2r}(t)$ at $r = \frac{1}{3}$ for different values of β for Example 1, case1. - - - Exact ($\beta = 1$), Approximate ($\beta = 1$), Approximate ($\beta = 0.9$), . . . Approximate ($\beta = 0.8$).

Table 1: The absolute error of approximating $x_{1r}(t)$ for Eq. (25).

t	$r = 0$	$r = 0.25$	$r = 0.5$	$r = 0.75$	$r = 1$
0.2	0.786561×10^{-10}	0.37662×10^{-10}	0.417873×10^{-10}	0.356916×10^{-10}	0.418666×10^{-10}
0.4	0.919491×10^{-9}	0.135547×10^{-9}	0.743873×10^{-9}	0.778036×10^{-9}	0.659107×10^{-9}
0.6	0.742514×10^{-8}	0.462194×10^{-8}	0.809546×10^{-8}	0.072125×10^{-8}	0.431608×10^{-8}
0.8	0.081362×10^{-7}	0.484576×10^{-7}	0.483455×10^{-7}	0.066379×10^{-7}	0.156841×10^{-7}
1	0.293303×10^{-6}	0.988609×10^{-6}	0.076106×10^{-6}	0.381373×10^{-6}	0.830071×10^{-6}

Case (2): Under Caputo $[(2)-\beta]$-differentiability, the system of OFDEs corresponding to Caputo $[(2)-\beta]$-differentiable is

$$\begin{aligned} {}^{C}D^{\beta}_{0^+} x_{1r}(t) &= -x_{2r}(t) + sin(t) \\ {}^{C}D^{\beta}_{0^+} x_{2r}(t) &= -x_{1r}(t) + sin(t) \end{aligned} \tag{27}$$

subject to

$$\begin{aligned} x_{1r}(0) &= \frac{24}{25} + \frac{1}{25} r \\ x_{2r}(0) &= \frac{101}{100} - \frac{1}{100} r \end{aligned}$$

If $\beta = 1$, then the exact solution of (27) is:

$$\begin{aligned} x_{1r}(t) &= \frac{1}{2}(sint - cost) + \left(\frac{1}{2} + \frac{24}{25} + \frac{1}{25} r\right) e^{-t}, \\ x_{2r}(t) &= \frac{1}{2}(sint - cost) + \left(\frac{1}{2} + \frac{101}{100} - \frac{1}{100} r\right) e^{-t}. \end{aligned} \tag{28}$$

Next some graphical results and tabulated data are present

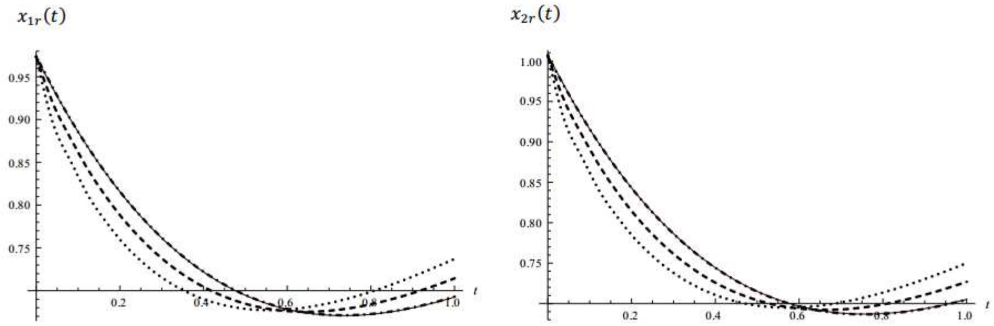

Figure 2: Approximate solutions $x_{1r}(t)$ and $x_{2r}(t)$ at $r = \frac{1}{3}$ for different values of β for Example 1, case 2. - - - Exact ($\beta = 1$), Approximate ($\beta = 1$), Approximate ($\beta = 0.9$), . . . Approximate ($\beta = 0.8$).

Table 2: The absolute error of approximating $x_{1r}(t)$ for Eq. (27).

t	$r = 0$	$r = 0.25$	$r = 0.5$	$r = 0.75$	$r = 1$
0.2	0.987369×10^{-10}	0.584262×10^{-10}	0.899036×10^{-10}	0.841897×10^{-10}	0.159156×10^{-10}
0.4	0.160604×10^{-9}	0.23812×10^{-9}	0.683479×10^{-9}	0.526032×10^{-9}	0.689592×10^{-9}
0.6	0.035859×10^{-8}	0.016719×10^{-8}	0.956339×10^{-8}	0.937085×10^{-8}	0.767948×10^{-8}
0.8	0.865089×10^{-7}	0.361176×10^{-7}	0.027726×10^{-7}	0.803166×10^{-7}	0.909273×10^{-7}
1	0.544426×10^{-6}	0.352355×10^{-6}	0.713772×10^{-6}	0.502104×10^{-6}	0.395093×10^{-6}

Next numerical results of RPS method with the reproducing kernel Hilbert space method (RKHSM) are presented for Example 1 as given in Tables 3 and 4:

Table 3: Numerical results case1 for Example 1 at $\beta = 1$.

	t	$r = 0$	$r = 0.25$	$r = 0.5$	$r = 0.75$	$r = 1$
	0.4	1.1463×10^{-5}	9.8552×10^{-6}	8.2467×10^{-6}	1.5596×10^{-6}	6.6383×10^{-6}
RKHSM	0.8	3.0507×10^{-5}	2.8105×10^{-5}	2.5702×10^{-5}	1.2702×10^{-5}	2.3300×10^{-5}
RPSM	0.4	0.9194×10^{-9}	0.1355×10^{-9}	0.7438×10^{-9}	0.77803×10^{-9}	0.6591×10^{-9}
	0.8	0.0813×10^{-7}	0.4845×10^{-7}	0.4834×10^{-7}	0.0663×10^{-7}	0.1568×10^{-7}

Table 4: Numerical results case1 for Example 1 at $\beta = 1$.

	t	$r = 0$	$r = 0.25$	$r = 0.5$	$r = 0.75$	$r = 1$
	0.4	8.3323×10^{-6}	7.7676×10^{-6}	7.2030×10^{-6}	8.5869×10^{-6}	6.63832×10^{-6}
RKHSM	0.8	2.4029×10^{-5}	2.3786×10^{-5}	2.3543×10^{-5}	5.9829×10^{-5}	2.3300×10^{-5}
RPSM	0.4	0.1606×10^{-9}	0.23812×10^{-9}	0.6834×10^{-9}	0.5260×10^{-9}	0.6895×10^{-9}
	0.8	0.8650×10^{-7}	0.361176×10^{-7}	0.0277×10^{-7}	0.8031×10^{-7}	0.9092×10^{-7}

Example 5.2. Consider the following FFD equation:

$$\begin{aligned} &\left({}^{C}D^{\beta}_{0^+}x\right)(t) = 2tx(t) + tu,\ 0 < \beta \le 1,\ t \in [0,1] \\ &\text{subject to} \\ &\qquad x(0) = u \in \mathbb{R}_{\mathcal{F}},\ u(s) = \max\{0, 1 - |s|\}, \textit{where} s \in \mathbb{R}. \end{aligned} \tag{29}$$

The r-cut representation of $x(0)$ is $[x(0)]^r = [r - 1, 1 - r]$. Hence, the FFDE (29) can be converted to the following system of OFDEs:

Case (1): Under Caputo $[(1)-\beta]$-differentiability, the system of OFDEs corresponding to Caputo $[(1)-\beta]$-differentiable is

$$\begin{aligned} {}^{C}D^{\beta}_{0^+}x_{1r}(t) &= 2tx_{1r}(t) + t(r-1) \\ {}^{C}D^{\beta}_{0^+}x_{2r}(t) &= 2tx_{2r}(t) + t(1-r) \end{aligned} \tag{30}$$

subject to

$$\begin{aligned} x_{1r}(0) &= r-1, \\ x_{2r}(0) &= 1-r, \end{aligned}$$

If $\beta = 1$, then the exact solution of (30) is:

$$\begin{aligned} x_{1r}(t) &= \frac{1}{2}\left(3e^{t^2}-1\right)(r-1), \\ x_{2r}(t) &= \frac{1}{2}\left(3e^{t^2}-1\right)(1-r). \end{aligned} \tag{31}$$

Next some graphical results and tabulated data are presented.

Table 5: The absolute error of approximating $x_{1r}(t)$ for Eq. (30).

t	$r=0$	$r=0.25$	$r=0.5$	$r=0.75$	$r=1$
0.2	0.256841×10^{-10}	0.37662×10^{-10}	0.417873×10^{-10}	0.356916×10^{-10}	0.418666×10^{-10}
0.4	0.4289513×10^{-9}	0.135547×10^{-9}	0.743873×10^{-9}	0.778036×10^{-9}	0.659107×10^{-9}
0.6	0.742514×10^{-8}	0.462194×10^{-8}	0.875546×10^{-8}	0.2072125×10^{-8}	0.431608×10^{-8}
0.8	0.081362×10^{-7}	0.564576×10^{-7}	0.483455×10^{-7}	0.066379×10^{-7}	0.156841×10^{-7}
1	0.2293303×10^{-6}	0.988609×10^{-6}	0.076106×10^{-6}	0.781373×10^{-6}	0.52071×10^{-6}

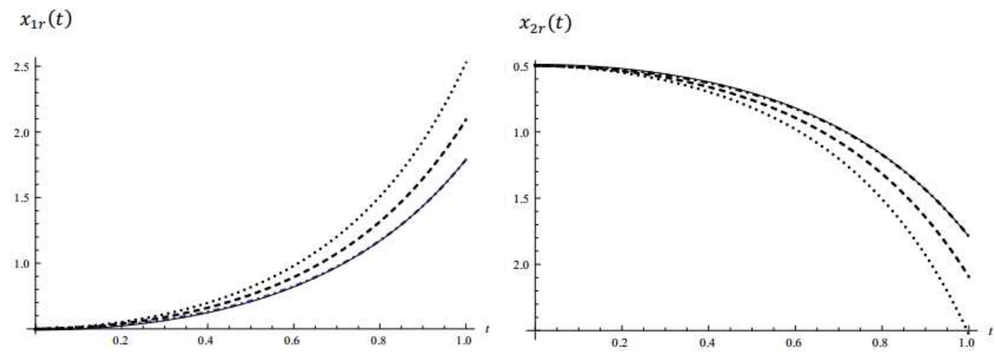

Figure 3: Approximate solutions $x_{1r}(t)$ and $x_{2r}(t)$ at $r = 0.25$ for different values of β for Example 2, case 1. - - - Exact ($\beta = 1$), Approximate ($\beta = 1$), Approximate ($\beta = 0.9$), . . . Approximate ($\beta = 0.8$).

Case (2): Under Caputo $[(2)-\beta]$-differentiability, the system of OFDEs corresponding to Caputo $[(2)-\beta]$-differentiable is

$$\begin{aligned} {}^{C}D^{\beta}_{0^+}x_{1r}(t) &= 2tx_{2r}(t) + t(1-r) \\ {}^{C}D^{\beta}_{0^+}x_{2r}(t) &= 2tx_{1r}(t) + t(r-1) \end{aligned} \tag{32}$$

subject to

$$x_{1r}(0) = r-1,\ x_{2r}(0) = 1-r,$$

If $\beta = 1$, then the exact solution of (32) is:

$$\begin{aligned} x_{1r}(t) &= \frac{1}{2}(r-1)\left(3e^{-t^2}-1\right), \\ x_{2r}(t) &= \frac{1}{2}(1-r)\left(3e^{-t^2}-1\right). \end{aligned} \tag{33}$$

Next some graphical results and tabulated data are presented.

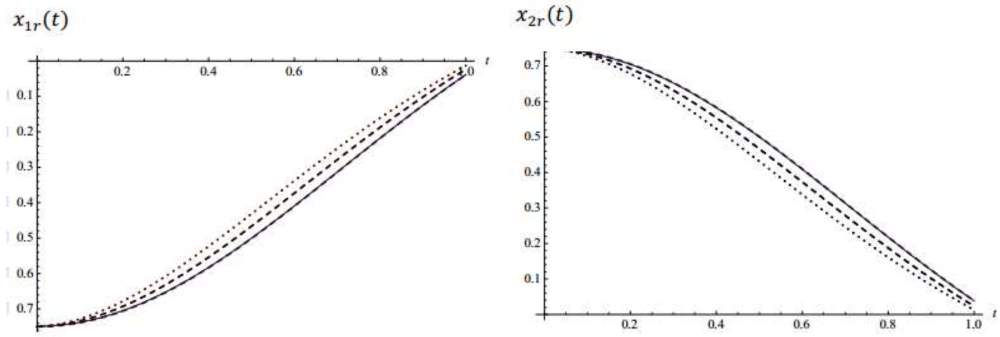

Figure 4: Approximate solutions $x_{1r}(t)$ and $x_{2r}(t)$ at $r = \frac{1}{3}$ for different values of β for Example 1, case 2. - - - Exact ($\beta = 1$), Approximate ($\beta = 1$), Approximate ($\beta = 0.9$), . . . Approximate ($\beta = 0.8$).

Table 6: The absolute error of approximating $x_{1r}(t)$ for Eq. (32).

t	$r = 0$	$r = 0.25$	$r = 0.5$	$r = 0.75$	$r = 1$
0.2	0.786561×10^{-10}	0.037662×10^{-10}	0.417873×10^{-10}	0.356916×10^{-10}	0.418666×10^{-10}
0.4	0.919491×10^{-9}	0.135547×10^{-9}	0.743873×10^{-9}	0.778036×10^{-9}	0.659107×10^{-9}
0.6	0.742514×10^{-8}	0.462194×10^{-8}	0.809546×10^{-8}	0.072125×10^{-8}	0.431608×10^{-8}
0.8	0.081362×10^{-7}	0.484576×10^{-7}	0.483455×10^{-7}	0.066379×10^{-7}	0.156841×10^{-7}
1	0.293303×10^{-6}	0.988609×10^{-6}	0.076106×10^{-6}	0.381373×10^{-6}	0.830071×10^{-6}

Next numerical results of RPS with the reproducing kernel Hilbert space method (RKHSM) are presented for Example 2 as given in Tables 7 and 8:

Table 7: Numerical results case 1 for Example 2 at $\beta = 1$.

	t	$r = 0$	$r = 0.25$	$r = 0.5$	$r = 0.75$	$r = 1$
	0.4	1.5195×10^{-4}	1.1396×10^{-4}	7.5976×10^{-5}	3.7988×10^{-5}	0
RKHSM	0.8	1.9332×10^{-5}	1.4499×10^{-5}	9.6663×10^{-6}	4.8331×10^{-6}	0
RPSM	0.4	0.0813×10^{-7}	0.5645×10^{-7}	0.4834×10^{-7}	0.0663×10^{-7}	0.1568×10^{-7}
	0.8	0.0813×10^{-7}	0.4845×10^{-7}	0.4834×10^{-7}	0.0663×10^{-7}	0.1568×10^{-7}

Table 8: Numerical results case 2 for Example 2 at $\beta = 1$.

	t	$r = 0$	$r = 0.25$	$r = 0.5$	$r = 0.75$	$r = 1$
	0.4	1.1155×10^{-4}	8.3662×10^{-5}	5.5775×10^{-5}	2.7887×10^{-5}	0
RKHSM	0.8	5.7274×10^{-5}	4.2956×10^{-5}	2.8637×10^{-5}	1.4318×10^{-5}	0
RPSM	0.4	0.9194×10^{-9}	0.1355×10^{-9}	0.7438×10^{-9}	0.7780×10^{-9}	0.6591×10^{-9}
	0.8	0.0813×10^{-7}	0.4845×10^{-7}	0.4834×10^{-7}	0.0663×10^{-7}	0.1568×10^{-7}

6. Concluding remarks

In this paper, we introduce a novel numerical method in order to solve FFDEs by using the RPSM. Besides, we assert the idea that the RPSM is able to have the solutions of FFDEs approximated under strongly generalized differentiability. The number-one reason behind the use of the RPS method lies in its applicability in function approximation. The obtained results show that the RPS approach is very totally operable, effective and appropriate to handling such model.

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