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Fuzzy Soft Metric and Fuzzifying Soft Topology Induced by Fuzzy Soft Metric

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Abstract. Our main goal with this paper is to construct soft topology and fuzzifying soft topology induced by fuzzy soft metric. For this, we present fuzzy soft metric spaces compatible to soft set theory and studied some of its basic properties. Then we investigate soft topological structures induced by fuzzy soft metrics.

1. Introduction

Conventional classical methods are not enough to solve complex problems in economics, engineering, environmental science, physics, computer science, social sciences, health sciences and other fields. Some mathematical theories such as probability theory, fuzzy set theory, intuitionistic fuzzy set, rough set theory, vague set theory and interval mathematics have been developed to define uncertainty.

One of those the theory of fuzzy sets was introduced by Zadeh in 1965 [44]. After C.L. Chang [11] introduced the fuzzy topological space in 1968, concepts in the general topology began to move into fuzzy topological space. Fuzzy topology, as an important research field in fuzzy set theory, has been developed into a quite mature discipline [24, 25, 43], etc. has also been applied to other areas such as fuzzy automata theory [12, 36]. Many authors have tried successfully to generalize the theory of general topology to the fuzzy setting with crisp methods. From a different direction, the fundamental idea of a topology itself being fuzzy, first appeared in 1980 in [20] and in 1991 in [27] in which a topology was a fuzzy subset of a traditional power set of crisp subsets. Besides one of the most important problems in fuzzy topology is finding an appropriate fuzzy metric space concept. This problem has been tackled in different ways by various researchers [16, 19, 23], etc. In particular, George and Veeramani [19] have introduced and studied a notion of fuzzy metric space with the help of continuous t-norms. Until recently only crisp topological structures induced by fuzzy metric have been studied. Lately some authors have studied fuzzification of topologies by using the "fuzziness" of fuzzy metric in [30–32], etc.

In 1999, Molodtsov pointed out in [33] previous concepts (probability theory, fuzzy set theory, intuitionistic fuzzy set and the other) have their own difficulties because of the inadequacy of the parametrization tool in these theory and gave the concept of "soft set theory", a completely new approach to models of uncertainty and indecision. At present, works on the soft set theory are progressing rapidly. Maji et al. [27, 28] presented an application of soft sets in decision making problems. The topological structure of soft

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set theory were first studied by Shabir and Naz [41]. Some authors [3, 35, 39, 45], etc. also studied some of basic properties of soft topological spaces. The algebraic structure of soft set theory has also been studied in more details [1, 18, 22], etc. Recently, Das and Samanta [14, 15] are introduced soft metric and they gave some of the properties of soft metric spaces.

In 2001, Maji et al. [26] combined fuzzy sets and soft sets and introduced the concept of fuzzy soft sets, and they presented an application of fuzzy soft sets in a decision making problem. Ahmad and Kharal [5] presented some more properties of fuzzy soft sets. After then topological structures of fuzzy soft sets studied by many authors [4, 10, 34], etc. Recently, many authors [6–8, 17], etc. studied on fuzzy soft metric space. Beaula et al. gave the notion of fuzzy soft point and introduce fuzzy soft metric space in terms of fuzzy soft point in [6] and Beaula and Priyanga [7] gave fuzzy soft normed linear space. In 2017, Erduran et al [17] defined the soft fuzzy metric space by considering soft point. And then, in [37, 38] application of fuzzy soft sets to the fixed point theory is studied.

In this paper, in order to construct fuzzy soft metric we consider soft elements instead of soft points. The previous constructions of fuzzy soft metric are related with soft points. Therefore we define fuzzy soft metric in terms of soft elements. In this point of view our definition is not comparable with the presented definitions of fuzzy soft metric until now. Then we show the relation between George and Veeramani's fuzzy metric and fuzzy soft metric. We investigate some characteristic properties of fuzzy soft metric and give some related examples. Then we give the definition of restricted soft element in order to obtain soft topological structures. After that we construct soft topology and fuzzifying soft topology induced by strong fuzzy soft metric and study their properties.

2. Preliminaries

2.1. Fuzzy Metric

Definition 2.1. ([40]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous triangular norm(t-norm) if for all $a, b, c, d \in [0,1]$ the following condition hold:

1. $a * b = b * a$
2. $a * 1 = a$
3. $(a * b) * c = a * (b * c)$
4. $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$
5. $*$ operation is continuous.

Definition 2.2. ([19]) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty)set, $*$ is a continuous t-norm and $M : X \times X \times (0, \infty) \to (0,1]$ is a map satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$:
F1 $M(x, y, t) > 0$,
F2 $M(x, y, t) = 1$ if and only if $x = y$,
F3 $M(x, y, t) = M(y, x, t)$,
F4 $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
F5 $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous.
A fuzzy metric space is called a strong fuzzy metric space [21] if it is satisfies the following condition:
(SF) $M(x, y, t) * M(y, z, t) \leq M(x, z, t)$ for all $x, y, z \in X$ and for all t ¿ 0.

Notice that every strong fuzzy metric space is itself a fuzzy metric space.

Definition 2.3. Let $(X, M, *)$ be a fuzzy metric space. The set

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$$

is called open ball with centre $x \in X$ and radius $r \in (0, 1)$.

2.2. Soft Set and Soft Topology

Throughout this study, X refers to an initial universe set and E is the set of all parameters. Let $P(X)$ denotes the power set of X and $U \subseteq E$.

Definition 2.4. ([28]) A pair (F, U) is called a soft set over the universe X, where F is a mapping given by $F : U \to P(X)$ and $U \subseteq E$.

According to [29], any soft set (F, U) can be extended to a soft set of type (F, E), where $F(e) \neq \emptyset$, for all $e \in U$ and $F(e) = \emptyset$, for all $e \in E - U$. Thus, a more general definition can be given as follows.

Definition 2.5. ([29]) A soft set F_U over the universe X is a mapping from the parameter set E to the power set $P(X)$ of X, i.e., $F_U : E \to P(X)$, where$F_U(e) \neq \emptyset$, if $e \in U \subseteq E$ and $F_U(e) = \emptyset$ if $e \notin U$.

The subscript A in the notation F_U indicates where the image of F_U is non-empty.
A soft set can also be defined by the set of ordered pairs [9]

$$F_U = \{(e, F_U(e)) : e \in E,\ F_U(e) \in P(X)\}.$$

From now on, $S(X, E)$ denotes the family of all soft sets over X.

Definition 2.6. ([28]) The soft set $F_\emptyset \in S(X, E)$ is called null soft set, if $F_\emptyset(e) = \emptyset$, for all $e \in E$. denoted by Φ.

Definition 2.7. ([28]) Let $F_E \in S(X, E)$ be a soft set . If $F_E(e) = X$ for all $e \in E$ then F_E is called absolute soft set, denoted by $\widetilde{E}$.

Definition 2.8. ([28]) Let a soft set $F_U \in S(X, E)$. If $F_A(e) = X$ for all $e \in U$ then F_U is called U-absolute soft set, denoted by $\widetilde{U}$.

Definition 2.9. ([28]) Let $F_U, G_V \in S(X, E)$ be two soft sets. We say that F_U is a soft subset of G_V if $F_U(e) \subset G_V(e)$, for each $e \in E$, denoted by $F_U \sqsubset G_V$.

Definition 2.10. ([28]) The union of two soft sets $F_U, G_V \in S(X, E)$ is a soft set H_W defined by $H_W(e) = F_U(e) \cup G_V(e)$ for all $e \in W$, where $W = U \cup V$. We denote by $(H, W) = (F, U) \sqcup (G, V)$.

Definition 2.11. ([28]) The intersection of two soft sets $F_U, G_V \in S(X, E)$ is a soft set H_W defined by $H_W(e) = F_U(e) \cap G_V(e)$ for all $e \in E$, where $W = U \cap V$.

Definition 2.12. ([41]) A soft topology τ is a family of soft sets over X satisfying the following properties:
(1) $\Phi, \widetilde{E} \in \tau$,
(2) If $F_U, G_V \in \tau$, then $F_U \sqcap G_V \in \tau$,
(3) If $(F_U)_\lambda \in \tau$, for each $\lambda \in \Delta$ then $\bigsqcup_{\lambda \in \Delta} (F_U)_\lambda \in \tau$.

Definition 2.13. ([3]) An enriched soft topology τ is a family of soft sets over X satisfying the following properties:
(1) $\Phi, \widetilde{U} \in \tau$ for all $U \subseteq E$,
(2) If $F_U, G_V \in \tau$, then $F_U \sqcap G_V \in \tau$,
(3) If $(F_U)_\lambda \in \tau$, for each $\lambda \in \Delta$ then $\bigsqcup_{\lambda \in \Delta} (F_U)_\lambda \in \tau$.

Definition 2.14. ([14]) A soft set $F_E \in S(X, E)$ is said to be a soft point, if there exist $x \in X$ and $e \in E$ such that $F(e) = \{x\}$ and $F\left(e'\right) = \emptyset$ some $x \in X$ and for all $e' \in E \setminus \{e\}$. It is denoted by $\widetilde{x}_e$.

Definition 2.15. ([13]) Let X be a non-empty set and E be a non-empty parameter set. Then a function $\widetilde{x} : E \to P(X)$ is said to be a soft element of X, denoted by $\widetilde{x}$, where $\widetilde{x}(e)$ is a singleton for each $e \in E$.

A soft element $\widetilde{x}$ of X is said to belong to a soft set $F_E \in S(X, E)$, denoted by $\widetilde{x} \tilde{\in} F_E$, if $\widetilde{x}(e) \in F(e)$, for all $e \in E$.

Definition 2.16. Let $\widetilde{x}$ be a soft element of X and $U \subseteq E$ be parameter set. Then $\widetilde{x}_U$ is said to be U-restricted soft element of $\widetilde{x}$, where

$$\widetilde{x}_U(e) = \begin{cases} \tilde{x}(e), e \in U \\ \emptyset, \text{otherwise} \end{cases}$$

.

Definition 2.17. ([13]) Let $\mathbb{R}$ be the set of real numbers, $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and U be a set of parameters. Then a mapping $F : U \to B(\mathbb{R})$ is called a soft real set.

If, in particular, F_A is a soft element on $\mathbb{R}$ then it will be called a soft real number. The set of soft real numbers is denoted by $\tilde{\mathbb{R}}$.

We use notations $\tilde{r}, \tilde{s}, \tilde{t}$ to denote soft real numbers whereas $\bar{r}, \bar{s}, \bar{t}$ will denote a particular type of soft real numbers such that $\bar{r}(e) = r$ for all $e \in U$ etc. For example $\bar{0}$ is the soft real number where $\bar{0}(e) = 0$, for all $e \in U$.

Definition 2.18. ([14]) For two soft real numbers $\tilde{r}, \tilde{s}$ we define

1. $\tilde{r} \tilde{\leq} \tilde{s}$ if $\tilde{r}(e) \leq \tilde{s}(e)$, for all $e \in U$,
2. $\tilde{r} \tilde{\geq} \tilde{s}$ if $\tilde{r}(e) \geq \tilde{s}(e)$, for all $e \in U$,
3. $\tilde{r} \tilde{<} \tilde{s}$ if $\tilde{r}(e) < \tilde{s}(e)$, for all $e \in U$,
4. $\tilde{r} \tilde{>} \tilde{s}$ if $\tilde{r}(e) > \tilde{s}(e)$, for all $e \in U$.

A soft real number $\tilde{r}$ is said to be positive if $\tilde{r} \tilde{>} \bar{0}$.

Definition 2.19. ([42]) Let $\tilde{r}, \tilde{s} \in \tilde{\mathbb{R}}$ be two soft real numbers. The operation $+$ is $\tilde{r} \tilde{+} \tilde{s} = \left\{ \left(e, \tilde{r}(e) + \tilde{s}(e)\right) : forall\ e \in E \right\}$.

Definition 2.20. ([17]) Let $\bar{I}$ be the set of all soft real numbers taking value at $[0, 1]$ and $*$ be a continuous t-norm. Then $\tilde{*}$ is called a continuous soft t-norm defined by $\left(\tilde{a} \tilde{*} \tilde{b}\right)(e) = \tilde{a}(e) * \tilde{b}(e)$, for each $e \in E$ and $\tilde{a}, \tilde{b} \in \bar{I}$.

Definition 2.21. ([26]) f_A is called a fuzzy soft set over X, where $f : A \to I^X$ is a function ($I = [0, 1]$). That is, for each $a \in A$, $f(a) = f_a : X \to I$ is a fuzzy set on X.

The collection of all fuzzy soft set denoted by $FS(X, E)$.

Definition 2.22. ([2]) A fuzzy soft set f_E on X is called a λ-absolute fuzzy soft set and denoted by $\tilde{E}$, if $f_e = \bar{\lambda}$, for each $e \in E$.

3. Main Result

Throughout this paper $\tilde{*}$ is a continuous soft t-norm, $\tilde{\mathbb{R}}$ is the set of all positive soft real numbers and $\tilde{I}$ be the set of all soft real numbers taking value at $(0, 1]$. F_U be a soft set over X. The collection of all soft elements of F_U is denoted by $SE(F_U)$ and $S(X, E)$ denotes the family of all soft sets over X.

3.1. Fuzzy Soft Metric Spaces

In [7] and [17] there are two definitions of fuzzy soft metric. As well as having some characteristic differences, they are comparable and both associate two soft point with a value. However in this paper we give another definition of fuzzy soft metric by considering soft elements instead of soft points. The purpose of considering soft elements over soft points is the construction of soft elements is compatible with soft set theory. That is, a soft set may be thought of a parameterized family of subset of the universal set. Correlatively, a soft element may also be thought of a parameterized family of points.

Definition 3.1. Let $\mathbf{M} : SE\left(\tilde{E}\right) \times SE\left(\tilde{E}\right) \times \tilde{\mathbb{R}} \to \tilde{I}$ be a map defined by $\mathbf{M}(\tilde{x}, \tilde{y}, \tilde{t})(e) \triangleq \mathbf{M}_e\left(\tilde{x}(e), \tilde{y}(e), \tilde{t}\,(e)\right)$, where $\mathbf{M}_e : X \times X \times (0, \infty) \to (0, 1]$ is a map for each $e \in E$. Then M is said to be a fuzzy soft metric if the following conditions are satisfied for all $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} SE\left(\tilde{E}\right)$ and $\tilde{t}, \tilde{s} \tilde{\in} \tilde{\mathbb{R}}$:

FSM1 $\mathbf{M}(\tilde{x}, \tilde{y}, \tilde{t}) \tilde{>} \bar{0}$,

FSM2 $\mathbf{M}(\tilde{x}, \tilde{y}, \tilde{t}) = \bar{1}$ for all $\tilde{t} > 0$ if and only if $\tilde{x} = \tilde{y}$,

FSM3 $\mathbf{M}(\tilde{x}, \tilde{y}, \tilde{t}) = \mathbf{M}(\tilde{y}, \tilde{x}, \tilde{t})$,

FSM4 $\mathbf{M}(\tilde{x}, \tilde{z}, \tilde{t} + \tilde{s}) \tilde{\geq} \mathbf{M}(\tilde{x}, \tilde{y}, \tilde{t}) \tilde{*} \mathbf{M}(\tilde{y}, \tilde{z}, \tilde{s})$,

FSM5 $\mathbf{M}(\tilde{x}, \tilde{y}, \bullet) : \tilde{\mathbb{R}} \to \tilde{I}$ is continuous , where $\tilde{\mathbb{R}}$ and $\tilde{I}$ has product topology on it.

Then the triple $\left(SE\left(\tilde{X}\right), \mathbf{M}, \tilde{*}\right)$ is said to be a fuzzy soft metric space.

If the fuzzy soft metric space satisfy the condition

SFSM $\mathbf{M}(\tilde{x}, \tilde{z}, \tilde{t}) \tilde{\geq} \mathbf{M}\left(\tilde{x}, \tilde{y}, \tilde{t}\right) \tilde{*} \mathbf{M}\left(\tilde{y}, \tilde{z}, \tilde{s}\right)$ for all $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} SE\left(\tilde{E}\right)$ and for all $\tilde{t}, \tilde{s} \tilde{\in} \tilde{\mathbb{R}}$, then the triple $\left(SE\left(\tilde{E}\right), \mathbf{M}, \tilde{*}\right)$ is called a strong fuzzy soft metric space.

Example 3.2. Let X be an arbitrary nonempty set, $E = (0, 1)$ be a parameter set and d_e be a metric on X for each $e \in E$. Define $\left(\tilde{a} \tilde{*} \tilde{b}\right)(e) = \tilde{a}\,(e)\,.\tilde{b}\,(e)$ and $\mathbf{M}(\tilde{x}, \tilde{y}, \tilde{t})\,(e) = \frac{\tilde{t}(e)}{\tilde{t}(e)+d_e(\tilde{x}(e),\tilde{y}(e))}$, where $t\,(e) > 0$ for each $e \in E$.Then $\mathbf{M}$ is a fuzzy soft metric.

Remark 3.3. A fuzzy soft metric $\mathbf{M} : SE\left(\tilde{E}\right) \times SE\left(\tilde{E}\right) \times \tilde{\mathbb{R}} \to \tilde{I}$ can be represented by a mapping from E to $I^{X \times X \times (0,\infty)}$ and the conditions SM1-SM5 can be written as follows for all $e \in E$, $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} SE\left(\tilde{X}\right)$ and $\tilde{t}, \tilde{s} \tilde{\in} \tilde{\mathbb{R}}$,

FSM1 $\mathbf{M}_e(\tilde{x}\,(e)\,, \tilde{y}\,(e)\,, \tilde{t}\,(e)) > 0$,

FSM2 $\mathbf{M}_e(\tilde{x}\,(e)\,, \tilde{y}\,(e)\,, \tilde{t}\,(e)) = 1$if and only if $\tilde{x} = \tilde{y}$,

FSM3 $\mathbf{M}_e(\tilde{x}\,(e)\,, \tilde{y}\,(e)\,, \tilde{t}\,(e)) = \mathbf{M}_e(\tilde{y}\,(e)\,, \tilde{x}\,(e)\,, \tilde{t}\,(e))$,

FSM4 $\mathbf{M}_e(\tilde{x}\,(e)\,, \tilde{z}\,(e)\,, \tilde{t}\,(e) + \tilde{s}\,(e)) \geq \mathbf{M}_e(\tilde{x}\,(e)\,, \tilde{y}\,(e)\,, \tilde{t}\,(e)) * \mathbf{M}_e(\tilde{y}\,(e)\,, \tilde{z}\,(e)\,, \tilde{s}\,(e))$,

FSM5 $\mathbf{M}_e(\tilde{x}\,(e)\,, \tilde{y}\,(e)\,, \bullet) : (0, \infty) \to (0, 1]$ is continuous.

It is clear that $\mathbf{M}_e$ is a fuzzy metric on X, for each $e \in E$ by the sense of George and Veeramani [19].

If $\mathbf{M}$ is strong fuzzy soft metric then $\mathbf{M}_e$ is strong fuzzy metric for each $e \in E$.

This remark shows that the concept of fuzzy soft metric is compatible with the soft set theory.

Example 3.4. Let U be a nonempty finite subset of parameters and $\left(SE\left(\tilde{E}\right), \mathbf{M}, \tilde{*}\right)$ be a fuzzy soft metric space. The mapping $D : SE\left(\tilde{E}\right) \times SE\left(\tilde{E}\right) \times \tilde{\mathbb{R}} \to \tilde{I}$

$$D\left(\tilde{x}, \tilde{y}, t\right) = \min_{e} \mathbf{M}_e(\tilde{x}\,(e)\,, \tilde{y}\,(e)\,, \bar{t}\,(e))$$

is a fuzzy metric on $SE\left(\widetilde{E}\right)$.

Definition 3.5. Let $\left(SE\left(\widetilde{E}\right), \mathbf{M}, \widetilde{*}\right)$ be a fuzzy soft metric space and $\widetilde{r}\,\widetilde{\in}\,(0,\widetilde{1})$ and $\widetilde{t} \in \widetilde{\mathbb{R}}$. Then

$$B\left(\widetilde{x}, \widetilde{r}, \widetilde{t}\right) = \left\{\widetilde{y}\,\widetilde{\in}\, SE\left(\widetilde{E}\right) : \mathbf{M}\left(\widetilde{x}, \widetilde{r}, \widetilde{t}\right) \widetilde{>} \bar{1} - \widetilde{r}\right\}$$

is called open ball with center $\widetilde{x}$ and radius $\widetilde{r}$.

Example 3.6. Let $X = \mathbb{R}$, $E = \{1, 2, 3\}$ be a parameter set, the map $\mathbf{M}_e\left(\widetilde{x}, \widetilde{y}, \widetilde{t}\right) = \frac{\widetilde{t}(e)}{\widetilde{t}(e)+d_e\left(\widetilde{x}(e),\widetilde{y}(e)\right)}$ be fuzzy metric where

$$\begin{aligned} d_{e_1}(x, y) &= \left|x - y\right| \\ d_{e_2}(x, y) &= \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases} \\ d_{e_3}(x, y) &= \frac{\left|x - y\right|}{1 - \left|x - y\right|} \end{aligned}$$

Then open ball $B\left(\tilde{x}, \bar{\frac{1}{3}}, \bar{1}\right)$ with center $\tilde{x} = \{(1, \{3\}), (2, \{5\}), (3, \{2\})\}$ as follows;

$$\begin{aligned} B\left(\tilde{x}, \bar{\frac{1}{3}}, \bar{1}\right) &= \left\{\tilde{y}\,\widetilde{\in}\, SE\left(\tilde{X}\right) : \mathbf{M}\left(\tilde{x}, \tilde{y}, \bar{1}\right) \widetilde{>} \bar{1} - \bar{\frac{1}{3}}\right\} \\ &= \left\{\left(1, \left(\frac{5}{2}, \frac{7}{2}\right)\right), (2, \{5\}), (3, (1, 3))\right\}. \end{aligned}$$

Definition 3.7. Let $\left(SE\left(\widetilde{E}\right), \mathbf{M}, \widetilde{*}\right)$ be a fuzzy soft metric space. Then the soft set F_U is said to be open if and only if for each $\tilde{x}\widetilde{\in} SE(F_U)$ there exists $\widetilde{r}\,\widetilde{\in}\,(0,\widetilde{1})$ and $\widetilde{t}\,\widetilde{\in}\,\widetilde{\mathbb{R}}$ such that $B\left(\tilde{x}, \widetilde{r}, \widetilde{t}\right) \sqsubset SE(F_U)$.

Theorem 3.8. *Every open ball in fuzzy soft metric space is an open set.*

Proof. Let $(SE\left(\tilde{X}\right), \mathbf{M}, \widetilde{*})$ be a fuzzy soft metric space. Consider an open ball $B(\tilde{x}, \widetilde{r}, \widetilde{t})$ and $\tilde{y}\widetilde{\in} B(\tilde{x}, \widetilde{r}, \widetilde{t})$. This means that $\tilde{y}(e)\,\widetilde{\in} B(\tilde{x}(e), \widetilde{r}(e), \widetilde{t}(e))$ for each $e \in E$. Then $M_e\left(\tilde{x}(e), \tilde{y}(e), \widetilde{t}(e)\right) > 1 - \widetilde{r}(e)$ for all $e \in E$. We can find a t_0^e such that $M_e\left(\tilde{x}(e), \tilde{y}(e), t_0^e\right) > 1 - \widetilde{r}(e)$ for all $e \in E$.

Let $\widetilde{t}_0(e) = t_0^e$ and $\widetilde{r}_0(e) \overset{\Delta}{=} r_0^e = M_e(\tilde{x}(e), \tilde{y}(e), \widetilde{t}_0(e)) > 1 - \widetilde{r}(e)$. Since $r_0^e > 1 - \widetilde{r}(e)$, we can find a $\widetilde{s}(e) \overset{\Delta}{=} s_e \in (0, 1)$ such that $r_0^e > 1 - s_e > 1 - \widetilde{r}(e)$.

Now for a given r_0^e and s_e such that $r_0^e > 1 - s_e$, we can find $\widetilde{r}_1(e) \overset{\Delta}{=} r_1^e \in (0, 1)$, such that $r_0^e * r_1^e \geq 1 - s_e$.

Now consider the ball $B(\tilde{y}(e), 1 - r_1^e, \widetilde{t}(e) - t_0^e)$ for each $e \in E$. We claim $B(\tilde{y}(e), 1 - r_1^e, \widetilde{t}(e) - t_0^e) \subset B(\tilde{x}(e), \widetilde{r}(e), \widetilde{t}(e))$.

Now $\tilde{z}(e) \in B(\tilde{y}(e), 1 - r_1^e, \widetilde{t}(e) - t_0^e)$, implies that $M_e(\tilde{y}(e), \tilde{z}(e), \widetilde{t}(e) - t_0^e) > r_1^e$. Therefore

$$\begin{aligned} M_e(\tilde{x}(e), \tilde{z}(e), \widetilde{t}(e)) &\geq M_e(\tilde{x}(e), \tilde{y}(e), t_0^e) * M_e(\tilde{y}(e), \tilde{z}(e), \widetilde{t}(e) - t_0^e) \\ &\geq r_0^e * r_1^e \\ &\geq 1 - s_e \\ &> 1 - \widetilde{r}(e). \end{aligned}$$

Therefore $\tilde{z}(e) \in B(\tilde{x}(e), \tilde{r}(e), \tilde{t}(e))$ and hence $B(\tilde{y}(e), 1 - r_1^e, \tilde{t}(e) - t_0^e) \subseteq B(\tilde{x}(e), \tilde{r}(e), \tilde{t}(e))$ for each $e \in E$. □

Remark 3.9. Unfortunately the collection $\left\{B\left(\tilde{x}, \tilde{r}, \tilde{t}\right) : \tilde{x} \tilde{\in} SE(F_U),\ \tilde{r} \tilde{\in} (\tilde{0}, 1),\ \tilde{t} \tilde{\in} \tilde{\mathbb{R}}\right\}$ may not induce a soft topology over X. Indeed, the non-null soft set $B\left(\tilde{x}, \tilde{r}_1, \tilde{t}\right) \sqcap B\left(\tilde{y}, \tilde{r}_2, \tilde{s}\right)$ may not have any soft element. In order to obtain soft topology we define following soft set by using restricted soft element ,

$$B\left(\tilde{x}_U, \tilde{r}, \tilde{t}\right) = \begin{cases} B\left(\tilde{x}(e), \tilde{r}(e), \tilde{t}(e)\right), e \in U \\ \emptyset, \text{ otherwise} \end{cases}$$

Corollary 3.10. *The open ball* $B\left(\tilde{x}_U, \tilde{r}, \tilde{t}\right)$ *of* $\tilde{x}_U$ *is an open set.*

The family

$$B = \left\{B\left(\tilde{x}_U, \tilde{r}, \tilde{t}\right) : \tilde{x}_U \text{ is } U - \text{restricted soft element of } \tilde{x}, \tilde{r} \in (\tilde{0}, 1),\ \tilde{t} \tilde{\in} \tilde{\mathbb{R}}\right\}$$

is a base for soft topology. Indeed, let $B\left(\tilde{x}_U, \tilde{r}_1, \tilde{t}\right) \sqcap B\left(\tilde{y}_V, \tilde{r}_2, \tilde{s}\right) \neq \Phi$. Then there exists $W \subseteq U \cap V$ and a soft element $\tilde{z}$ such that $\tilde{z}_w \tilde{\in} B\left(\tilde{x}_U, \tilde{r}_1, \tilde{t}\right) \sqcap B\left(\tilde{y}_V, \tilde{r}_2, \tilde{s}\right)$. By the Theorem 3.8, we have $\tilde{p}$ and $\tilde{q}$ such that $B\left(\tilde{z}_w, \tilde{p}, \tilde{k}\right) \subseteq B\left(\tilde{x}_U, \tilde{r}_1, \tilde{t}\right)$ and $B\left(\tilde{z}_w, \tilde{q}, \tilde{l}\right) \subseteq B\left(\tilde{y}_V, \tilde{r}_2, \tilde{s}\right)$. If we define $\tilde{r}_0(e) = \min\left\{\tilde{p}(e), \tilde{q}(e)\right\}$ and $\tilde{t}_0(e) = \min\left\{\tilde{k}(e), \tilde{l}(e)\right\}$, for each $e \in E$,then $B\left(\tilde{z}_w, \tilde{r}_0, \tilde{t}_0\right) \sqsubseteq B\left(\tilde{x}_U, \tilde{r}_1, \tilde{t}\right) \sqcap B\left(\tilde{y}_V, \tilde{r}_2, \tilde{s}\right)$.

From Corollary 3.10 and $\tilde{U} = \bigsqcup_{\tilde{x}_U \in \tilde{U}} B\left(\tilde{x}_U, \tilde{r}, \tilde{t}\right)$ we have the following theorem.

Theorem 3.11. *The collection*

$$\tau_{\mathbf{M}} = \left\{F_U \tilde{\in} S(X, E) | \forall \tilde{x}_U \tilde{\in} F_U\ \ \exists \tilde{t} \tilde{\in} \tilde{\mathbb{R}} \text{ and } \exists \tilde{r} \tilde{\in} (\tilde{0}, 1) : B(\tilde{x}_U, \tilde{r}, \tilde{t}) \sqsubseteq SE(F_U)\right\}$$

is an enriched soft topology.

3.2. Fuzzifying Soft Topology

Definition 3.12. ([4]) A mapping $\tau : E \to [0, 1]^{FS(X,E)}$is called an enriched fuzzy soft topology on X if it satisfies the following conditions for each $e \in E$:

(1) $\tau_e\left(\tilde{E}^{\lambda}\right) = 1$ for each $\lambda \in [0, 1]$,

(2) $\tau_e(f_U \sqcap g_B) \geq \tau_e(f_U) \bigwedge \tau_e(g_B)$, for each $f_U, g_B \in FS(X, E)$,

(3) $\tau_e\left(\bigsqcup_{i \in \Delta} (f_U)_i\right) \geq \bigwedge_{i \in \Delta} \tau_e\left((f_U)_i\right)$, for $(f_U)_i \in FS(X, E), i \in \Delta$.

Definition 3.13. ([10]) A mapping $\tau : E \to [0, 1]^{S(X,E)}$is called an enriched fuzzifying soft topology on X if it satisfies the following conditions for each $e \in E$:

(1) $\tau_e(\Phi) = 1$ and $\tau_e\left(\tilde{U}\right) = 1$ for each nonempty $U \subset E$,

(2) $\tau_e(F_U \sqcap G_B) \geq \tau_e(F_U) \bigwedge \tau_e(G_B)$, for each $F_U, G_B \in S(X, E)$,

(3) $\tau_e\left(\bigsqcup_{i \in \Delta} (F_U)_i\right) \geq \bigwedge_{i \in \Delta} \tau_e((F_U)_i)$, for $(F_U)_i \in S(X, E), i \in \Delta$

In order to construct a fuzzifying soft topology, let consider the mapping $\varphi : \tilde{\mathbb{R}} \to (0,\tilde{1})$, defined by $\varphi\left(\tilde{t}\right)(e) = \frac{\tilde{t}(e)}{\tilde{t}(e)+1}$,for each $e \in E$. Obviously the inverse of φ is the mapping $\psi : (0,\tilde{1}) \to \tilde{\mathbb{R}}$, where $\psi\left(\tilde{\alpha}\right)(e) = \frac{\tilde{\alpha}(e)}{1-\tilde{\alpha}(e)}$for each $e \in E$.

Let $\left(SE\left(\tilde{E}\right), \mathbf{M}, \tilde{*}\right)$ be a strong fuzzy soft metric space. For fixed $\tilde{\alpha} \in (0,\tilde{1})$ consider the family

$$B_{\tilde{\alpha}} = \left\{B(\tilde{x}_U, \tilde{r}, \tilde{t}) \,\middle|\, \tilde{x}\tilde{\in}SE\left(\tilde{E}\right),\ \tilde{r}\tilde{\in}(0,\tilde{1})\right\},\ \text{where } t = \psi(\tilde{\alpha}).$$

Since $\mathbf{M}^{\tilde{t}}\left(\tilde{x}, \tilde{y}\right) = \mathbf{M}\left(\tilde{x}, \tilde{y}, \tilde{t}\right)$ is a fuzzy soft metric for fixed $\tilde{t} \tilde{\in} \tilde{\mathbb{R}}$, $B_{\tilde{\alpha}}$ is a base of an enriched soft topology. We can characterize this soft topology as the following way

$$F_U \in T_{\tilde{\alpha}} \Leftrightarrow \exists \tilde{r} \in (0,\tilde{1}) : B(\tilde{x}_V, \tilde{r}, \tilde{t}) \subseteq F_U \text{ where } \tilde{t} = \psi(\tilde{\alpha}).$$

Theorem 3.14. *Let $\tilde{\alpha} \tilde{\in} (0,\tilde{1})$ and let F_U be a soft set. Then $F_U \in T_{\tilde{\alpha}}$ if and only if for each $\tilde{x}_V \tilde{\in} F_U$ there exists $\tilde{r}\tilde{\in}(0,\tilde{1})$ such that $B(\tilde{x}_V, \tilde{r}, \tilde{t}) \subseteq F_U$,where $\tilde{t} \in \psi(\tilde{\alpha})$.*

Proof. Let $F_U \in T_{\tilde{\alpha}}$ for fixed $\tilde{\alpha} \in (0,\tilde{1})$ and $\tilde{x}_V \in F_U$. Then there exists $\tilde{y}_V \tilde{\in} F_U$ and $\tilde{\varepsilon} \in (0,\tilde{1})$such that $\tilde{x}_V \in B\left(\tilde{y}_V, \tilde{\varepsilon}, \tilde{t}\right) \subseteq F_U$ where $\tilde{t} \in \psi(\tilde{\alpha})$. One can find $0 < \delta_e < \varepsilon(e)$ for each $e \in E$ such that $\mathbf{M}_e\left(\tilde{x}(e), \tilde{y}(e), \tilde{t}(e)\right) > 1 - \delta_e$, for each $e \in V$. Since $*$ is continuous for $e \in E$ and $1 - \delta_e > 1 - \tilde{\varepsilon}(e)$ there exists $r_e \in (0,1)$ such that $(1 - r_e) * (1 - \delta_e) > 1 - \varepsilon(e)$ for each $e \in E$. Hence $B(\tilde{x}_V, \tilde{r}, \tilde{t}) \subseteq B\left(\tilde{y}_V, \tilde{\varepsilon}, \tilde{t}\right)$. Indeed, if $\tilde{z}_V \in B(\tilde{x}_V, \tilde{r}, \tilde{t})$ then $\mathbf{M}_e\left(\tilde{x}(e), \tilde{z}(e), \tilde{t}(e)\right) > 1 - \tilde{r}(e)$ for each $e \in V$. Since M is strong, it follows that $\mathbf{M}_e\left(\tilde{y}(e), \tilde{z}(e), \tilde{t}(e)\right) \geq \mathbf{M}_e\left(\tilde{y}(e), \tilde{x}(e), \tilde{t}(e)\right) * \mathbf{M}_e\left(\tilde{x}(e), \tilde{z}(e), \tilde{t}(e)\right) > (1 - \delta_e) * (1 - r_e) > 1 - \varepsilon(e)$ for each $e \in V$. Thus $\tilde{z}_V \in B\left(\tilde{y}_V, \tilde{\varepsilon}, \tilde{t}\right)$.

Conversely, let assume that for each $\tilde{x}_V \in F_U$ there exists $\tilde{r}_x\tilde{\in}(0,\tilde{1})$ such that $B(\tilde{x}_V, \tilde{r}_x, \tilde{t}) \subseteq F_U$. Then $F_U = \bigsqcup_{\tilde{x}_V \in F_U} B(\tilde{x}_V, \tilde{r}_x, \tilde{t})$. It follows that $F_U \in T_\alpha$. □

Corollary 3.15. *If $F_U \in T_{\tilde{\alpha}}$, then $F_U \in T_{\tilde{\beta}}$ whenever $\tilde{\beta}\tilde{\leq}\tilde{\alpha}$.*

Corollary 3.16. *$\left\{T_{\tilde{\alpha}} : \tilde{\alpha} \in (0,\tilde{1})\right\}$ is a decreasing family of soft topologies.*

Theorem 3.17. *Let $\tau^M : E \to I^{SE(X)}$ be a fuzzy soft set defined by $\tau^M_e(F_U) = \bigvee\{\tilde{\alpha}(e) : F_U \in T_{\tilde{\alpha}}\}$. Then τ^M is a fuzzifying soft topology over $\tilde{E}$.*

Proof. 1) $\tau^M(\Phi) = \tau^M\left(\tilde{E}\right) = 1$, since $\Phi, \tilde{A} \in T_{\tilde{\alpha}}$ for every $\tilde{\alpha} \in (0,\tilde{1})$.

2) Let $\tau(F_U) = \tilde{\alpha}$ and $\tau(G_V) = \tilde{\beta}$ and assume that $\tilde{\alpha} \leq \tilde{\beta}$. If $\tilde{\alpha} = \bar{0}$ then the statement is obvious. Otherwise $\tilde{\alpha}(e) = \sup\left\{\lambda(e) : \bar{0} < \tilde{\lambda} < \tilde{\alpha}\right\}$ and $F_U, G_V \in T_{\tilde{\lambda}}$ for every $\tilde{\lambda} < \tilde{\alpha}$. Hence $F_U \sqcap G_V \in T_{\tilde{\lambda}}$, whenever $\tilde{\lambda} < \tilde{\alpha}$ and therefore $\tau^M(F_U \sqcap G_V) \geq \tilde{\alpha} = \tau^M(F_U) \bigwedge \tau^M(G_V)$.

3) Let consider a family $\{(F_U)_i : i \in I\} \subset S(X,E)$ and $\bigwedge_i \tau^M((F_U)_i) = \tilde{\alpha}$.

If $\tilde{\alpha} = \bar{0}$ then the statement is obvious. Otherwise $\tilde{\alpha}(e) = \sup\left\{\lambda(e) : \bar{0} < \tilde{\lambda} < \tilde{\alpha}\right\}$ and hence $\tau^M((F_U)_i) \tilde{\geq} \tilde{\lambda}$, for every $i \in I$ and every $\tilde{\lambda} < \tilde{\alpha}$. Therefore for every $\tilde{\lambda} < \tilde{\alpha}$, $\{(F_U)_i : i \in I\} \subseteq T_{\tilde{\lambda}}$. This means that $\bigsqcup_{i \in I}(F_U)_i \in T_{\tilde{\lambda}}$ for every $\tilde{\lambda} < \tilde{\alpha}$ and hence $\tau^M(\bigcup(F_U)_i) \geq \alpha = \bigwedge_i \tau^M((F_U)_i)$. □

4. Conclusion

In this paper we have given the definition of fuzzy soft metric space. Then we have defined the collection of $B\left(\tilde{x}_U, \tilde{r}, \tilde{t}\right)$ and by using this collection we have obtained a soft topology. Then we have constructed a fuzzifying soft topology induced by fuzzy soft metric. And we have investigated its properties.

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