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Uniform and Pointwise Estimates for Algebraic Polynomials in Regions with Interior and Exterior Zero Angles

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Abstract. In this study, we give some estimates on the Nikolskii-type inequalities for complex algebraic polynomials in regions with piecewise smooth curves having exterior and interior zero angles.

1. Introduction

Let $\mathbb{C}$ be a complex plane, and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be the bounded Jordan region, with $0 \in G$ and the boundary $L := \partial G$ be a closed Jordan curve, $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = extL$. Let $\wp_n$ denotes the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

Let $0 < p \leq \infty$. For a rectifiable Jordan curve L, we denote

$$\begin{aligned} \|P_n\|_p &:= \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int_L h(z) \left|P_n(z)\right|^p |dz| \right)^{1/p}, \quad 0 < p < \infty; \\ \|P_n\|_\infty &:= \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z \in L} \left|P_n(z)\right|, \qquad p = \infty. \end{aligned}$$

Clearly, $\|\cdot\|_{\mathcal{L}_p}$ is the quasinorm (i.e. a norm for $1 \leq p \leq \infty$ and a $p-$norm for $0 < p < 1$).

Denoted by $w = \Phi(z)$, the univalent conformal mapping of Ω onto $\Delta := \{w : |w| > 1\}$ with normalization $\Phi(\infty) = \infty$, $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$ and $\Psi := \Phi^{-1}$. For $t \geq 1$, we set

$$L_t := \{z : \ |\Phi(z)| = t\}, \quad L_1 \equiv L, \quad G_t := intL_t, \quad \Omega_t := extL_t.$$

Let $\{z_j\}_{j=1}^m$ be a fixed system of distinct points on curve L which is located in the positive direction. For some fixed R_0, $1 < R_0 < \infty$, and $z \in G_{R_0}$, consider a so-called generalized Jacobi weight function $h(z)$ being

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defined as follows

$$h(z) := h_0(z) \prod_{j=1}^{m} \left| z - z_j \right|^{\gamma_j}, \tag{1}$$

where $\gamma_j > -1$, for all $j = 1, 2, ..., m$, and h_0 is uniformly separated from zero in G_{R_0}, i.e. there exists a constant $c_0 := c_0(G_{R_0}) > 0$ such that for all $z \in G_{R_0}$

$$h_0(z) \geq c_0 > 0.$$

In this work, we study the following Nikol'skii-type inequality:

$$\|P_n\|_{\mathcal{L}_q(h,L)} \leq c\mu_n(L, h, p, q) \|P_n\|_{\mathcal{L}_p(h,L)}, \ 0 < p < q \leq \infty, \tag{2}$$

where $c = c(G, p, q) > 0$ is the constant independent of n and P_n, and $\mu_n(L, h, p, q) \to \infty$, $n \to \infty$, depending on the geometrical properties of curve L and weight function h and of p. In particular, it was studied the behavior of the $|P_n(z)|$ on L ($q = \infty$), where the boundary curve L and weight function h having some singularity on the L. First result of (2)-type, in case $h(z) \equiv 1$ and $L = \{z : |z| = 1\}$ for $0 < p < \infty$ was found in [9]. The another results, similar to (2), for the sufficiently smooth curve, was obtained in [20] ($h(z) \equiv 1$), and in [21, Part 4] ($h(z) \neq 1$). The estimation of (2)-type for $0 < p < \infty$ and $h(z) \equiv 1$ when L is a rectifiable Jordan curve was investigated in [12, 13, 18, 21, 22], [15, pp.122-133]. In [8, Theorem 6] obtained identical inequalities for more general curves and for another weighed function. There are more references regarding the inequality of (2)-type, we can find in Milovanovic et al. [14, Sect.5.3].

Further, analogous estimates of (2) for some regions and the weight function $h(z)$ were obtained: in [6] ($p > 1$) and in [16] ($p > 0$, $h \equiv h_0$) for regions bounded by rectifiable quasiconformal curve having some general properties; in [3] ($p > 1$) for piecewise Dini-smooth curve with interior and exterior cusps; in [2] ($p > 1$) for regions bounded by piecewise smooth curve with exterior cusps but without interior cusps; in [4] ($p > 0$) for regions bounded by piecewise rectifiable quasiconformal curve with cusps; in [5] ($p > 0$) for regions bounded by piecewise quasismooth (by Lavrentiev) curve with cusps.

In this work, we investigate similar problem in regions bounded by piecewise smooth curve having interior and exterior zero angles (cusps) for weight function h defined in (1) and for all $p > 0$.

2. Main Results

Let us give some definitions and notations that will be used later in the text.

In what follows, we always assume that $p > 0$ and the constants $c, c_0, c_1, c_2, ...$ are positive and constants $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive (generally, are different in different relations), which depends on G in general and, on parameters inessential for the argument, otherwise, the dependence will be explicitly stated. Also note that, for any $k \geq 0$ and $m > k$, notation $j = \overline{k, m}$ denotes $j = k, k+1, ..., m$.

Let Γ be a rectifiable Jordan curve or arc and $z = z(s)$, $s \in [0, \ell]$ denote the natural parametrization of Γ, where $\ell := mes\, \Gamma$ is the length of Γ.

Definition 2.1. A Jordan curve (arc) Γ is called C_θ-curve (-arc), denote by $\Gamma \in C_\theta$, if Γ has a continuous tangent $\theta(z) = \theta(z(s))$ at every point $z(s)$. We will write $G \in C_\theta$ for a Jordan region G, if its boundary ∂G is a C_θ-curve, that is $\partial G \in C_\theta$.

Now, we shall define a new class of regions with piecewise smooth boundary, which have corners, interior and exterior cusps at some finite number boundary points. Let $C_+^2[0, 1]$ denote the class of all functions $f : [0, \varepsilon_0] \to \mathbb{R}$ which are twice differentiable such that $f(0) = 0$ and $f^{(k)}(x) > 0$ for all $0 < x \leq \varepsilon_0$, $k = 0, 1, 2$.

Definition 2.2. It is called that a Jordan region $G \in C_\theta(\lambda_1, ..., \lambda_{m_1}; f_{m_1+1}, ..., f_m)$, $0 < \lambda_i \leq 2$, $i = \overline{1, m_1}$, $f_j \in C_+^2[0, 1]$, $j = \overline{m_1 + 1, m}$, if $L = \partial G$ consists a union of finite number of C_θ-arcs $\{L_j\}_{j=0}^m$, joining at the points $\{z_j\}_{j=0}^m \in L$, such that L is locally smooth at z_0, and the following properties hold:

1. for all $i = \overline{1, m_1}$, the arcs L_{i-1} and L_i meet at the point z_i with the exterior (respect to $\overline{G}$) angle $\lambda_i \pi$, $0 < \lambda_i \le 2$;
2. for all $j = \overline{m_1 + 1, m}$, the arcs L_{j-1} and L_j meet at the point z_j with f_j-type exterior zero angle, that is there exists an ε_j-neighborhood of z_j such that in a local coordinate system, with the origin at z_j, we have

$$\{z = x + iy : \quad |z| < \varepsilon_j,\ c_{1,j} f_j(x) \le y \le c_{2,j} f_j(x),\ x \in [0, \varepsilon_j]\} \subset \overline{\Omega}$$

and

$$\{z = x + iy : \quad |z| < \varepsilon_j,\ |y| \ge \varepsilon_0 x,\ x \in [0, \varepsilon_j]\} \subset \overline{G}$$

for some constants $-\infty < c_{1,j} < c_{2,j} < \infty$ and $\varepsilon_0, \varepsilon_j$.

When $m_1 = m = 0$, this definition yields a Jordan region whose boundary is a C_θ-curve. We write $G \in C_\theta(\lambda_1)$, when $m_1 = m = 1$.

For the simplicity of exposition and in order to avoid cumbersome calculations, without loss of generality, we consider, a Jacobi weight function h given by (1) with $h_0 = 1$ and the region $G \in C_\theta(\lambda_1; f_2)$ with $m_1 = 1$, $m = 2$, $0 < \lambda_1 \le 2$ and the function $f_2(x) = x^{1+\alpha_2}$, $\alpha_2 > 0$, as the function f_2 in the Definition 2.2. We will use the notation $G \in C_\theta(\lambda_1; f_2)$ for this construction. Therefore, $G \in C_\theta(\lambda_1; f_2)$ denote that the region G may have exterior $\lambda_1\pi$, $0 < \lambda_1 \le 2$, (also interior zero) angle at the point z_1 and exterior zero angle at the point z_2 of $f_2(x) = x^{1+\alpha_2}$–touching. Correspondingly, we will use the notation $G \in C_\theta(\lambda_1, \lambda_2)$, if $m_1 = m = 2$, i.e., the region G may have only exterior $\lambda_i\pi$, $0 < \lambda_i \le 2$, (also interior zero) angles at the point z_i, $i = 1, 2$, without exterior zero angles, and notation $G \in C_\theta(f_1, f_2)$, if $m_1 = 0$, $m = 2$, i.e., the region G may have only exterior zero angles of $f_j(x) = x^{1+\alpha_j}$–touching at the point z_j, $j = 1, 2$,without exterior (also interior zero) angles at the point z_j .

Now we can state our new results.

Theorem 2.3. *Let $G \in C_\theta(\lambda_1; f_2)$ for some $0 < \lambda_1 \le 2$ and $f_2(x) = cx^{1+\alpha_2}$, $\alpha_2 > 0$; $h(z)$ be defined as in (1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_j > -1$, $j = 1, 2$, and arbitrary small $\varepsilon > 0$, we have:*

$$\|P_n\|_{C(\overline{G})} \le c_1 \left(n^{\frac{\tilde{\gamma}_1 + 1}{p} \widehat{\lambda}_1} + n^{\frac{\tilde{\gamma}_{2,\alpha_2} + 1 + 2\alpha_2}{p(1+\alpha_2)} + \varepsilon} \right) \|P_n\|_p , \tag{3}$$

where $c_1 = c_1(G, \gamma_j, \lambda_1, \alpha_2, p, \varepsilon) > 0$ is the constant, independent from z and n;

$$\begin{aligned} \tilde{\gamma}_1 &= \max\{\gamma_1, 0\},\ \tilde{\gamma}_{2,\alpha_2} = \max\{\gamma_2, -\alpha_2\}; \\ \widehat{\lambda}_1 &= \begin{cases} \max\{\lambda_1, 1\} + \varepsilon, & \text{if} \quad 0 < \lambda_1 < 2, \\ 2, & \text{if} \quad \lambda_1 = 2. \end{cases} \end{aligned} \tag{4}$$

An analogue of Theorem 2.3 for $p > 1$ and $\lambda_1 \ne 2$ has been given in [2, Th.1.3]. Combining Theorem 2.3 with [2], we get:

Corollary 2.4. *Under the assumptions of Theorem 2.3, we have:*

$$\|P_n\|_{C(\overline{G})} \le c_2 \left(n^{\frac{\tilde{\gamma}_1 + 1}{p} \hat{\lambda}_1} + B_{n,1} \right) \|P_n\|_p ,$$

where $c_2 = c_2(G, \gamma_1, \gamma_2, \lambda_1, \alpha_2, p, \varepsilon) > 0$ is the constant, independent from z and n, and

$$B_{n,1} := \begin{cases} n^{\frac{\tilde{\gamma}_{2,\alpha_2} + 1}{p(1+\alpha_2)} + \frac{2\alpha_2}{p(1+\alpha_2)} + \varepsilon}, & \text{if} \quad 0 < p \le 1 \text{ or } p \ge 2, \\ n^{\frac{\tilde{\gamma}_{2,\alpha_2} + 1}{p(1+\alpha_2)} + \frac{\alpha_2}{(1+\alpha_2)} + \varepsilon}, & \text{if} \quad 1 < p < 2. \end{cases} \tag{5}$$

Now, let's take that the curve L in the both points $z_1,\ z_2 \in L$ have exterior non zero or interior zero angles. In this case we obtain:

Theorem 2.5. *Let $G \in C_\theta(\lambda_1, \lambda_2)$ for some $0 < \lambda_1, \lambda_2 \le 2$; $h(z)$ be defined as in (1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_i > -1$, $i = 1,2$, and arbitrary small $\varepsilon > 0$, we have:*

$$\|P_n\|_{C(\overline{G})} \le c_3 n^{\frac{\tilde{\gamma}+1}{p}\widehat{\lambda}} \|P_n\|_p , \tag{6}$$

where $c_3 = c_3(G, \gamma_i, \lambda_i, p, \varepsilon) > 0$ is the constant, independent from z and n, $\tilde{\gamma} = \max\{\widetilde{\gamma_1}, \widetilde{\gamma_2}\}$, $\widehat{\lambda} = \max\{\widehat{\lambda_1}, \widehat{\lambda_2}\}$, and $\widetilde{\gamma_i}, \widehat{\lambda_i}$, $i = 1,2$, defined as in (4).

Analogously, when the curve L in the both points $z_1, z_2 \in L$ have only exterior zero angles, we have:

Theorem 2.6. *Let $G \in C_\theta(f_1, f_2)$ for some $f_j(x) = cx^{1+\alpha_j}$, $\alpha_j > 0$, $j = 1,2$; $h(z)$ be defined as in (1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_j > -1$, $j = 1,2$, and arbitrary small $\varepsilon > 0$, we have:*

$$\|P_n\|_{C(\overline{G})} \le c_4 B^*_{n,1} \|P_n\|_p ,$$

where $c_4 = c_4(G, \gamma_j, \alpha_j, p, \varepsilon) > 0$ is the constant, independent from z and n;

$$\begin{aligned} B^*_{n,1} &:= \begin{cases} n^{\frac{\tilde{\gamma}+1}{p(1+\alpha_*)} + \frac{2\alpha^*}{p(1+\alpha^*)} + \varepsilon} & \text{if} \quad 0 < p \le 1 \text{ or } p \ge 2. \\ n^{\frac{\tilde{\gamma}+1}{p(1+\alpha_*)} + \frac{\alpha^*}{(1+\alpha^*)} + \varepsilon} & \text{if} \quad 1 < p < 2 \end{cases} ; \\ \tilde{\gamma} &:= \left\{\tilde{\gamma}_{2,\alpha_1}, \tilde{\gamma}_{2,\alpha_2}\right\}, \ \alpha^* := \max\{\alpha_1, \alpha_2\}, \ \alpha_* := \min\{\alpha_1, \alpha_2\}. \end{aligned} \tag{7}$$

Now we will estimate of $|P_n(z)|$ at the critical points $z_j \in L$, $j = 1,2$.

Theorem 2.7. *Let $G \in C_\theta(\lambda_1; f_2)$ for some $0 < \lambda_1 \le 2$ and $f_2(x) = cx^{1+\alpha_2}$, $\alpha_2 > 0$; $h(z)$ be defined as in (1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_j > -1$, $j = 1,2$, and arbitrary small $\varepsilon > 0$, we have:*

$$\left|P_n(z_j)\right| \le c_5 B_{n,2} \|P_n\|_p , \tag{8}$$

where $c_5 = c_5(G, \gamma_1, \gamma_2, \lambda_1, p, \varepsilon) > 0$ is the constant, independent from z and n;

$$B_{n,2} := \begin{cases} n^{\frac{\gamma_1+1}{p}\lambda_1}, & \text{for } j = 1, \\ n^{\frac{\gamma_2+1}{p(1+\alpha_2)} + \frac{2\alpha_2}{p(1+\alpha_2)} + \varepsilon}, \text{ if } 0 < p \le 1 \text{ or } p \ge 2, & \text{for } j = 2, \\ n^{\frac{\gamma_2+1}{p(1+\alpha_2)} + \frac{\alpha_2}{(1+\alpha_2)} + \varepsilon}, \text{ if } 1 < p < 2, & \text{for } j = 2, \end{cases}$$

Combining Corollary 2.4 and Theorem 1.1 of [23], we can obtain estimation for $|P_n(z)|$ in the whole complex plane. For $z \in \mathbb{C}$ and $M \subset \mathbb{C}$, we set: $d(z, M) = dist(z, M) := \inf\{|z - \zeta| : \zeta \in M\}$.

Corollary 2.8. *Under the assumptions of Theorem 2.3, we have:*

$$|P_n(z)| \le c_6 \|P_n\|_p \begin{cases} \left(n^{\frac{\tilde{\gamma}_1+1}{p}\hat{\lambda}_1} + B_{n,1}\right), & z \in \overline{G}, \\ \frac{|\Phi(z)|^{n+1}}{d^{\frac{2}{p}}(z,L)} B_{n,3}, & z \in \Omega, \end{cases}$$

where $c_6 = c_6(G, \gamma_j, \lambda_1, \alpha_2, p, \varepsilon) > 0$ is the constant, independent from z and n;

$$B_{n,3} := \begin{cases} n^{\frac{\gamma_1-1}{p}\hat{\lambda}_1} + n^{\frac{\gamma_2-1}{p(1+\alpha_2)} + \varepsilon}, & \text{if} \quad \gamma_1, \gamma_2 > 1, \\ (\ln n)^{1/p}, & \text{if} \quad \gamma_1 = \gamma_2 = 1, \\ 1, & \text{if} \quad -1 < \gamma_1, \gamma_2 < 1, \end{cases} ; \tag{9}$$

$B_{n,1}$ and $\widehat{\lambda}_1$ defines as in (5) and(4), respectively.

Analogously to Corollary 2.8, combining Theorems 2.5, 2.6 with correspondingly results from [23, Cor. 1.3 and Cor.1.4], we also can be obtain estimation for $|P_n(z)|$ in the whole complex plane, respectively.

The sharpness of the inequalities given above, for some special cases, has been seen from the following:

Remark 2.9. Let B be the unit disc and $L = \partial B$. For any $n \in \mathbb{N}$ there exists a polynomial $P_n^* \in \wp_n$ and constants $c_7, c_8 > 0$, such that the inequalities

$$\begin{aligned} \left\|P_n^*\right\|_{C(\overline{B})} &\geq c_7 n^{1/p} \left\|P_n^*\right\|_{\mathcal{L}_p(1,L)}, \ p > 1; \\ \left\|P_n^*\right\|_{C(\overline{B})} &\geq c_8 n^{(\gamma_1+1)/p} \left\|P_n^*\right\|_{\mathcal{L}_p(h,L)}, \qquad p > \gamma_1 + 1, \end{aligned}$$

hold, where $h(z) = |z-1|^{\gamma_1}$, $\gamma_1 > 0$.

3. Some Auxiliary Results

Recall that, as noted above throughout this work, $c, c_0, c_1, c_2, ...$ are positive constants (generally, different in different relations), which depend on G in general. Further, for the nonnegative functions $a > 0$ and $b > 0$, we shall use the notations "$a \preceq b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b), respectively. Let $B := \{w : |w| < 1\}$. Let φ be the univalent conformal mapping of G onto B such that $\varphi(0) = 0$ and $\varphi'(0) > 0$. $\psi := \varphi^{-1}$. For $0 < r < 1$, we take $L_r := \left\{z : \left|\varphi(z)\right| = r\right\}$.

Following to [19] (see also [7, 10]), a Jordan curve (or arc) L is called $K-$quasiconformal ($K \geq 1$), if there is a $K-$quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a circle (or line segment). On the other hand, it can be given some geometric criteria of quasiconformality of the curves. We give one of them. Let z_1, z_2 be an arbitrary points on L and $L(z_1, z_2)$ denotes the subarc of L of shorter diameter with end points z_1 and z_2. Then, the curve L is a quasi-circle if and only if the quantity

$$\frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|}$$

is bounded for all $z_1, z_2 \in L$ and $z \in L(z_1, z_2)$ ([11], [17]).

Lemma 3.1. ([1]) *Let L be a $K-$quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \leq d(z_1, L_{r_0})\}$ for some $0 < r_0 < 1$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*

a) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. So the statements $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$ also are equivalent;

b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, then

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2},$$

If we take $z_3 \in L_{R_0}$ for $R_0 = \frac{1}{r_0}$, From Lemma 3.1, we have the following.

Corollary 3.2. *Under the assumptions of Lemma 3.1, we have*

$$|w_1 - w_2|^{K^2} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{K^{-2}}$$

Corollary 3.3. *If $L \in C_\theta$, then we have*

$$|w_1 - w_2|^{1+\varepsilon} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{1-\varepsilon}.$$

for all $\varepsilon > 0$.

For $0 < \delta_j < 1/2$, we put $\Omega(z_j,\delta_j) := \Omega \cap \left\{z : \left|z - z_j\right| \le \delta_j\right\}$, where $z_1 = -1$ and $z_2 = 1$, $j = 1,2$; $\delta := \min\{\delta_1,\delta_2\}$, $\Omega(\delta) := \bigcup\limits_{j=1}^{2} \Omega(z_j,\delta)$, $\widehat{\Omega} := \Omega \backslash \Omega(\delta)$. Additionally, let $\Delta_j := \Phi(\Omega(z_j,\delta))$, $\Delta(\delta) := \bigcup\limits_{j=1}^{m} \Phi(\Omega(z_j,\delta))$.

Lemma 3.4. ([24]). *Let $G \in C_\theta(\lambda_j)$, $0 < \lambda_j < 2$, $j = 1,2$. Then for all $\varepsilon > 0$, we have*

i. $|w - w_j|^{\lambda_j+\varepsilon} \le |\Psi(w) - \Psi(w_j)| \le |w - w_j|^{\lambda_j-\varepsilon}$, *and* $|w - w_j|^{\lambda_j-1+\varepsilon} \le |\Psi'(w)| \le |w - w_j|^{\lambda_j-1-\varepsilon}$, *for any* $w \in \Delta_j$.

ii. $(|w| - 1)^{1+\varepsilon} \le d(\Psi(w),L)| \le (|w| - 1)^{1-\varepsilon}$, *and* $(|w| - 1)^{\varepsilon} \le |\Psi'(w)| \le (|w| - 1)^{-\varepsilon}$, *for any* $w \in \overline{\Delta} \backslash \Delta_j$.

Lemma 3.5. ([2]) *Let L be a rectifiable Jordan curve, $P_n \in \wp_n$, $n \in \mathbb{N}$, $p > 0$ and $R > 1$. Then, the following inequality holds:*

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \le R^{n+\frac{1+\gamma}{p}} \|P_n\|_p\,,$$

where $\gamma = \max\{\gamma_j : \ j = \overline{1,m}\}$.

4. Proofs of the Main Results

Before giving proofs of the main theorems, let us give the geometric notations used in the proofs to prevent the flow of presentation of the proof.

Without loss of generality, we assume that $z_1 = -1$, $z_2 = 1$ and $(-1,1) \subset G$. We will use the notations given in the following: $L^+ := \{z \in L : \ \text{Im}z \ge 0\}$ and $L^- := \{z \in L : \ \text{Im}z < 0\}$, so that $L = L^+ \cup L^-$; $w_j = \Phi(z_j) =: e^{i\varphi_j}$, $0 \le \varphi_j < 2\pi$, $j = 1,2$; $w^+ := e^{i\theta_+}$ and $w^- := -w^+$, where $\theta_+ = \varphi_2 + \frac{|\varphi_1-\varphi_2|}{2}$; $z^\pm := \Psi(w^\pm)$; $l_j^\pm := l_j^\pm(z_i,z^\pm)$ is the sub-arc with endpoints $z^\pm$ and z_j, $j = 1,2$. $L^1 := L^1(z^+,z_1,z^-)$ denote the arc connecting the points z^+ and z^- passing through the point z_1; $L^2 := L^2(z^-,z_2,z^+)$ denote the arc, connecting the points z^- and z^+ passing through the point z_2.

Similar notations for L_R are in the following: $L_R^+ := \{z \in L_R : \ \text{Im}z \ge 0\}$ and $L_R^- := \{z \in L_R : \ \text{Im}z < 0\}$, so that $L_R = L_R^+ \cup L_R^-$; $w_R^+ := Re^{i\theta_+}$ and $w_R^- := -w_R^+$; $z_R^\pm := \Psi(w_R^\pm)$. Let $z_{j,R} \in L_R$ such that $d(z_j,L_R) = |z_j - z_{j,R}|$, $j = 1,2$; $\zeta^\pm \in L^\pm$ such that $d(z_{2,R},L^\pm) = |z_{2,R} - \zeta^\pm|$. $z_j^\pm \in L^j \cap L^\pm$ such that $c_j d(z_j,L_R) = |z_j - z_j^\pm|$, $j = 1,2$; $z_{j,R}^\pm \in L_R^\pm$ such that $c_j d(z_j,L_R) = |z_j - z_{j,R}^\pm|$, $j = 1,2$.

Finally, let us give the following notations: Let $E_{1,R}^{i,\pm} := \{\zeta \in L_R^\pm : \ |\zeta - z_i| < c_i d(z_i,L_R)\}$, and let $E_{2,R}^{i,\pm}$ be the sub-arc of $L_R^\pm$ with endpoints $z_{j,R}^\pm$ and $z_R^\pm$, $j = 1,2$; $E_1^{i,\pm} := \{\zeta \in L^i : \ |\zeta - z_i| < c_i d(z_i,L_R)\}$, and let $E_2^{i,\pm}$ be the sub-arc of $L^\pm$ with endpoints $z_i^\pm$ and $z^\pm$, $i = 1,2$. Let $F_{j,R}^{i,\pm} := \Phi(E_{j,R}^{i,\pm})$, $i = 1,2$, $j = 1,2$.

Let $z_0 \in L$ be a point far from the points z_1 and z_2. Without loss of generality, we assume that $z_0 = z^+$ (or $z_0 = z^-$) to ensure simplicity in calculations.

4.1. Proof of Theorem 2.3

Proof. Let $G \in C_\theta(\lambda_1,\alpha_2)$ and $R = 1 + \frac{1}{n}$, $n \in \mathbb{N}$. Let $w = \varphi_R(z)$ denote the univalent conformal mapping of G_R onto the unit disk $B = \{z : \ |z| < 1\}$ normalized by $\varphi_R(0) = 0$, $\varphi_R'(0) > 0$, and let $\left\{\xi_j\right\}$, $1 \le j \le m \le n$, the zeros of $P_n(z)$, lying on G_R (if such zeros exist). Let

$$b_{m,R}(z) = \prod_{j=1}^{m} \frac{\varphi_R(z) - \varphi_R(\xi_j)}{1 - \overline{\varphi_R(\xi_j)}\varphi_R(z)}$$

denote a Blaschke function with respect to the zeros $\left\{\xi_j\right\}_{j=1}^{m}$. For $p > 0$, let us set:

$$H_{n,p}(z) := \left[\frac{P_n(z)}{b_{m,R}(z)}\right]^{p/2}, \qquad z \in G_R.$$

The function $H_{n,p}$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. We take an arbitrary continuous branch of $H_{n,p}$ and we maintain the same designation for this branch. Its Cauchy integral representation for the region G_R is the following:

$$H_{n,p}(z) = \frac{1}{2\pi i} \int\limits_{L_R} H_{n,p}(\zeta) \frac{d\zeta}{\zeta - z}, \qquad z \in G_R.$$

Since $\left|b_{m,R}(\zeta)\right| \equiv 1$ for all $\zeta \in L_R$ and $\left|b_{m,R}(z)\right| < 1$ for all $z \in G_R$, then we have:

$$|P_n(z)|^{\frac{p}{2}} \le |b_{m,R}(z)|^{\frac{p}{2}} \int\limits_{L_R} \left| \frac{P_n(\zeta)}{b_{m,R}(\zeta)} \right|^{\frac{p}{2}} \frac{|d\zeta|}{|\zeta - z|} \le \int\limits_{L_R} |P_n(\zeta)|^{\frac{p}{2}} \frac{|d\zeta|}{|\zeta - z|},$$

for all $z \in G_R$. Multiplying the numerator and the denominator of the last integrand by $h^{1/2}(\zeta)$, and then applying the Cauchy-Schwarz Inequality, we obtain:

$$|P_n(z)|^{\frac{p}{2}} \le \left(\int\limits_{L_R} h(\zeta) |P_n(\zeta)|^p |d\zeta| \right)^{1/2} \left(\int\limits_{L_R} \frac{|d\zeta|}{h(\zeta) |\zeta - z|^2} \right)^{1/2}.$$

By Lemma 3.5, we have:

$$|P_n(z)| \le \|P_n\|_p \left(\int\limits_{L_R} \frac{|d\zeta|}{h(\zeta) |\zeta - z|^2} \right)^{1/p} := \|P_n\|_p \, J_n^{1/p}, \tag{10}$$

for all $z \in G_R$, where

$$J_n := J_n(L_R) := \int\limits_{L_R} \frac{|d\zeta|}{h(\zeta) |\zeta - z|^2}. \tag{11}$$

Now, we will estimate the integral J_n. For this purpose, we will use the notations and definitions given in the beginning of this section. Under this notations, we can write (11) in the following form:

$$J_n \asymp \sum_{i,j=1}^{2} \int\limits_{E_{j,R}^{i,\pm}} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i} |\zeta - z|^2} := \sum_{i,j=1}^{2} J(E_{j,R}^{i,\pm}). \tag{12}$$

Now, we estimate the integrals $J(E_{j,R}^{i,\pm})$ in (12). First, let us fix a point $z' \in L$, such that $|P_n(z')| = \|P_n\|_{C(\overline{G})}$. So that $z' \in L^1$ or $z' \in L^2$. We will examine both cases. Let $d_{i,R} := d(z_i, L_R)$, $i = 1, 2$.

Case 1. Let $z' \in L^1$. If $\gamma_1 \ge 0$, we have:

$$J(E_{1,R}^{1,+}) + J(E_{1,R}^{1,-}) = \int\limits_{E_{1,R}^{1,+} \cup E_{1,R}^{1,-}} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1} |\zeta - z'|^2} \preceq \frac{1}{d_{1,R}^2} \int\limits_{d_{1,R}}^{c_1 d_{1,R}} \frac{ds}{s^{\gamma_1}} \preceq \frac{1}{d_{1,R}^{1+\gamma_1}}, \tag{13}$$

if $-1 < \gamma_1 < 0$, then we have:

$$J(E_{1,R}^{1,+}) + J(E_{1,R}^{1,-}) \preceq \frac{1}{d_{1,R}^{2+\gamma_1}} \int\limits_{d_{1,R}}^{c_1 d_{1,R}} ds \preceq \frac{1}{d_{1,R}^{1+\gamma_1}}. \tag{14}$$

If $z' \in E_1^{1,\pm}$, then we get:

$$J(E_{2,R}^{1,+}) + J(E_{2,R}^{1,-}) \preceq \int\limits_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|d\zeta|}{(\min\{|\zeta - z_1|, |\zeta - z'|\})^{2+\gamma_1}} \preceq \frac{1}{d_{1,R}^{1+\gamma_1}}, \tag{15}$$

for $\gamma_1 \geq 0$, and taking into account the inequality $(x+y)^r \leq 2^r(x^r + y^r)$, $x, y, r > 0$, we have:

$$\begin{aligned} J(E_{2,R}^{1,+}) + J(E_{2,R}^{1,-}) &= \int\limits_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|\zeta - z_1|^{-\gamma_1}\,|d\zeta|}{|\zeta - z'|^2} \\ &\preceq \int\limits_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|\zeta - z'|^{-\gamma_1} + |z' - z_1|^{-\gamma_1}}{|\zeta - z'|^2}|d\zeta| \\ &\preceq \int\limits_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|d\zeta|}{|\zeta - z'|^{2+\gamma_1}} + (c_1 d_{1,R})^{-\gamma_1} \int\limits_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|d\zeta|}{|\zeta - z'|^2} \\ &\preceq \frac{1}{d_{1,R}^{1+\gamma_1}}. \end{aligned} \tag{16}$$

for $-1 < \gamma_1 < 0$. If $z' \in E_2^{1,\pm}$, then we get:

$$J(E_{2,R}^{1,+}) + J(E_{2,R}^{1,-}) = \int\limits_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1}\,|\zeta - z'|^2} \preceq \frac{1}{d_{1,R}^{1+\gamma_1}}, \tag{17}$$

for $\gamma_1 \geq 0$, and we have

$$J(E_{2,R}^{1,+}) + J(E_{2,R}^{1,-}) = \int\limits_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|\zeta - z_1|^{-\gamma_1}\,|d\zeta|}{|\zeta - z'|^2} \preceq \frac{1}{d_{1,R}}, \tag{18}$$

for $-1 < \gamma_1 < 0$. From the inequalities (13)-(18), we conclude that

$$\sum_{j=1}^{2}(J(E_{j,R}^{1,+}) + J(E_{j,R}^{1,-}) \preceq \frac{1}{d_{1,R}^{1+\tilde{\gamma}_1}}, \tag{19}$$

where $\tilde{\gamma}_1 := \max\{\gamma_1, 0\}$. By the relation (25) and Lemma 2.2 in [2], we write the inequality (19) in the following form:

$$\sum_{j=1}^{2}(J(E_{j,R}^{1,+}) + J(E_{j,R}^{1,-}) \preceq n^{(1+\tilde{\gamma}_1)\hat{\lambda}_1}, \qquad \text{for all } \varepsilon > 0, \tag{20}$$

where

$$\hat{\lambda}_1 = \begin{cases} \max\{\lambda_1, 1\} + \varepsilon, & \text{if} \quad 0 < \lambda_1 < 2, \\ 2, & if \quad \lambda_1 = 2. \end{cases}$$

.

Case 2. Let $z' \in L^2$ and $w' = \Phi(z')$. By changing the variable $\tau = \Phi(\zeta)$ in the integral (12), we have:

$$J_n \asymp \sum_{i,j=1}^{2} \int\limits_{F_{j,R}^{i,\pm}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau) - \Psi(w_i)|^{\gamma_i}\,|\Psi(\tau) - \Psi(w)|^2} := \sum_{i,j=1}^{2} J(F_{j,R}^{i,\pm}). \tag{21}$$

Now, we estimate the integrals $J(F^{i,\pm}_{j,R})$ in (21). First, we assume that $z' \in E^{2,\pm}_1$. Then, by the relation (25) and Lemma 2.2 in [2], we have:

$$\begin{aligned} J(F^{2,+}_{1,R}) + J(F^{2,-}_{1,R}) &= \int\limits_{F^{2,+}_{1,R}\cup F^{2,-}_{1,R}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}\,|\Psi(\tau)-\Psi(w')|^2} \\ &\asymp \int\limits_{F^{2,+}_{1,R}\cup F^{2,-}_{1,R}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}\,|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)} \\ &\preceq \int\limits_{F^{2,+}_{1,R}\cup F^{2,-}_{1,R}} \frac{|d\tau|}{d^{\gamma_2-1}_{2,R}\,|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)}. \end{aligned}$$

We estimate only the integral over $F^{2,+}_{1,R}$, since the other estimate is similar. By using the inequality

$$|\Psi(\tau)-\Psi(w')| \geq \min\{d_{2,R}, d(\Psi(\tau),L^+)\} \succcurlyeq \min\{d_{2,R}, d^{1+\alpha_2}_{2,R}\} \succcurlyeq d^{1+\alpha_2}_{2,R},$$

we have

$$\begin{aligned} \int\limits_{F^{2,+}_{1,R}} \frac{|d\tau|}{d^{\gamma_2-1}_{2,R}\,|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)} &\preceq \int\limits_{F^{2,+}_{1,R}} \frac{|d\tau|}{d^{\gamma_2-1+2(1+\alpha_2)}_{2,R}(|\tau|-1)} \\ &\preceq \int\limits_{F^{2,+}_{1,R}} \frac{|d\tau|}{|z^+_{2,R}-z^+_2|^{\frac{\gamma_2-1}{1+\alpha_2}+2}(|\tau|-1)} \\ &\preceq \int\limits_{F^{2,+}_{1,R}} \frac{|d\tau|}{(|\tau|-1)^{\frac{\gamma_2-1}{1+\alpha_2}+3+\varepsilon}} \\ &\preceq n^{\frac{\gamma_2-1}{1+\alpha_2}+2+\varepsilon}, \end{aligned}$$

for arbitrary $\varepsilon > 0$. So that, we get

$$J(F^{2,+}_{1,R}) + J(F^{2,-}_{1,R}) \preceq n^{\frac{\gamma_2-1}{1+\alpha_2}+2+\varepsilon}, \qquad \text{for all } \varepsilon > 0. \tag{22}$$

The same conclusion in (22) can be drawn for the case $z' \in E^{2,\pm}_2$, by similar arguments.

Now, assume that $z' \in E^{2,\pm}_1$. If $\gamma_2 > 0$, then

$$\begin{aligned} J(F^{2,+}_{2,R}) + J(F^{2,-}_{2,R}) &= \int\limits_{F^{2,+}_{2,R}\cup F^{2,-}_{2,R}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}\,|\Psi(\tau)-\Psi(w')|^2} \\ &\preceq \int\limits_{F^{2,+}_{2,R}} \frac{|d\tau|}{d^{\gamma_2}_{2,R} d(\Psi(\tau),L)^2(|\tau|-1)^{\varepsilon}} \\ &\preceq \int\limits_{F^{2,+}_{2,R}} \frac{|d\tau|}{|z_{2,R}-z^+_2|^{\frac{\gamma_2}{1+\alpha_2}}\,(|\tau|-1)^{2+\varepsilon}} \\ &\preceq \int\limits_{F^{2,+}_{2,R}} \frac{|d\tau|}{(|\tau|-1)^{\frac{\gamma_2}{1+\alpha_2}+2+\varepsilon}} \preceq n^{\frac{\gamma_2}{1+\alpha_2}+1+\varepsilon}, \text{ for all } \varepsilon > 0, \end{aligned}$$

and if $-1 < \gamma_2 \leq 0$, then we get:

$$J(F_{2,R}^{2,+}) + J(F_{2,R}^{2,-}) \leq \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{d(\Psi(\tau), L)^2(|\tau|-1)^{\varepsilon}} \leq \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{(|\tau|-1)^{2+\varepsilon}} \leq n^{1+\varepsilon},$$

for arbitrary $\varepsilon > 0$. In the case $z' \in E_2^{2,\pm}$, if $\gamma_2 > 0$, then we get:

$$\begin{aligned} J(F_{2,R}^{2,+}) + J(F_{2,R}^{2,-}) &\leq \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{d_{2,R}^{\gamma_2}(|\tau|-1)^{2+\varepsilon}} + \int\limits_{F_{2,R}^{2,-}} \frac{|d\tau|}{d_{2,R}^{\gamma_2+2}(|\tau|-1)^{\varepsilon}} \\ &\leq \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{|z_{2,R} - z_2^+|^{\frac{\gamma_2}{1+\alpha_2}}(|\tau|-1)^{2+\varepsilon}} + \int\limits_{F_{2,R}^{2,-}} \frac{|d\tau|}{|z_{2,R} - z_2^+|^{\frac{\gamma_2+2}{1+\alpha_2}}(|\tau|-1)^{\varepsilon}} \\ &\leq \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{(|\tau|-1)^{\frac{\gamma_2}{1+\alpha_2}+2+\varepsilon}} + \int\limits_{F_{2,R}^{2,-}} \frac{|d\tau|}{(|\tau|-1)^{\frac{\gamma_2+2}{1+\alpha_2}+\varepsilon}} \\ &\leq n^{\frac{\gamma_2}{1+\alpha_2}+1+\varepsilon} + n^{\frac{\gamma_2+2}{1+\alpha_2}-1+\varepsilon}, \text{ for all } \varepsilon > 0, \end{aligned}$$

and if $-1 < \gamma_2 \leq 0$, then we have:

$$\begin{aligned} J(F_{2,R}^{2,+}) + J(F_{2,R}^{2,-}) &\leq \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{(|\tau|-1)^{2+\varepsilon}} + \int\limits_{F_{2,R}^{2,-}} \frac{|d\tau|}{d_{2,R}^{2}(|\tau|-1)^{\varepsilon}} \\ &\leq n^{1+\varepsilon} + \int\limits_{F_{2,R}^{2,-}} \frac{|d\tau|}{(|\tau|-1)^{\frac{2}{1+\alpha_2}+\varepsilon}} \\ &\leq n^{1+\varepsilon} + n^{\frac{2}{1+\alpha_2}-1+\varepsilon}, \qquad \varepsilon > 0. \end{aligned}$$

Consequently, if $z' \in L^2$, we obtain:

$$J(F_{2,R}^{2,+}) + J(F_{2,R}^{2,-}) \leq n^{\frac{\tilde{\gamma}_2}{1+\alpha_2}+1+\varepsilon}, \qquad \varepsilon > 0, \tag{23}$$

where $\tilde{\gamma}_2 = \max\{\gamma_2, 0\}$. If we put the obtained results (22) and (23) in (21), we get the following inequality for the case $z' \in L^2$:

$$J_n \leq n^{\frac{\tilde{\gamma}_{2,\alpha_2}-1}{1+\alpha_2}+2+\varepsilon}, \qquad \text{for all } \varepsilon > 0, \tag{24}$$

where $\tilde{\gamma}_{2,\alpha_2} = \max\{\gamma_2, -\alpha_2\}$.

For the general case, that is $z' \in L$, from (20) and (24), we have:

$$J_n \leq n^{(1+\tilde{\gamma}_1)\tilde{\lambda}} + n^{\frac{\tilde{\gamma}_{2,\alpha_2}-1}{1+\alpha_2}+2+\varepsilon}, \qquad \text{for all } \varepsilon > 0. \tag{25}$$

Finally, if we put the estimation (25) in (10), we obtain the desired result. □

4.2. Proof of Theorems 2.5 and 2.6.

Proof. The proof of Theorems 2.5 and 2.6 we can get from (25) firstly for the region $G \in C_\theta(\lambda_1, \lambda_2)$, $0 < \lambda_j \leq 2$, and after for the region $G \in C_\theta(f_1, f_2)$ with $f_j(x) = x^{1+\alpha_j}$, $j = 1, 2$. □

4.3. Proof of Theorem 2.7.

Proof. The proof of Theorem 2.7 we get from the scheme of the proof of Theorem 2.3, if we take everywhere in $z = z_1$. □

4.4. *Proof of Remark 2.9*

Proof. Let us consider the polynomial $P_n^*(z) = 1 + z + z^2 + ... + z^{n-1}$. It is easy to see that $\left\|P_n^*\right\|_{C(\overline{B})} = n$. Thus, the desired results are seen from the following relations which have been obtained in [20]:

$$\left\|P_n^*\right\|_{\mathcal{L}_p(1,L)} \asymp \begin{cases} n^{1-\frac{1}{p}}, & \text{if} \quad p > 1, \\ \ln n, & \text{if} \quad p = 1, \\ 1, & \text{if} \quad 0 < p < 1, \end{cases}$$

and

$$\left\|P_n^*\right\|_{\mathcal{L}_p(h,L)} \asymp n^{1-\frac{(\gamma_1+1)}{p}}, \qquad \text{if} \quad p > \gamma_1 + 1.$$

□

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