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Best Proximity Point Results for Suzuki-Edelstein Proximal Contractions via Auxiliary Functions

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Abstract. In this paper we study the notion of modified Suzuki-Edelstein proximal contraction under some auxiliary functions for non-self mappings and obtain best proximity point theorems in the setting of complete metric spaces. As applications, we derive best proximity point and fixed point results for such contraction mappings in partially ordered metric spaces. Some examples are given to show the validity of our results. Our results extend and unify many existing results in the literature.

1. Introduction

A wide variety of problems arising in different areas of pure and applied mathematics, such as difference and differential equations, discrete and continuous dynamic systems and variational analysis can be modeled as fixed point equations of the form $x = Tx$. Therefore, fixed point theory is quite useful in finding the solutions of above equation and structural optimizations in science and engineering [17, 20, 29]. The Banach contraction principle [15] was the key principle to provide the existence of solution of fixed point equation $x = Tx$.

On contrary, in the case that T is not a self-mapping, it is probable that the equation $Tx = x$ possesses no solution, for a solution of the preceding equation necessitates the equality between an element in the domain and an element in the co-domain of the mapping. In such scenarios, it is worthwhile to determine an approximate solution that is optimal in the sense that the error due to approximation is minimum. That is, if $T : A \to B$ is a non-self-mapping in the framework of a metric space, one desires to compute an approximate solution $x \in A$ such that the error $d(x, Tx)$ is minimum. The value $min_{x\in A} d(x, Tx)$ is basically an ideal optimal approximate solution to the equation $Tx = x$ which is unlikely to have a solution when T is supposed to be a non-self mapping. Considering the fact that $d(x, Tx)$ is at least $d(A, B)$ for all $x \in A$, a solution x to the aforementioned non-linear programming problem becomes an approximate solution with the lowest possible error to the corresponding equation $Tx = x$ if it satisfies the condition that $d(x, Tx) = d(A, B)$. Indeed, such a solution x is known as a best proximity point of the mapping and the results that investigate the existence of best proximity points for non-self mappings are called best proximity

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point theorems. Best proximity point theorems for several types of non-self mappings have been derived in [1–3, 7–13, 27, 28, 31, 35].

In 1962, Edelstein [19] obtained the following well known result:

Theorem 1.1. *Let (X, d) be a compact metric space, and let T be a mapping on X. Assume $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point.*

In 2008, Suzuki [36] introduced a new type of mapping and presented a generalization of the Banach contraction principle in which the completeness can also be characterized by the existence of a fixed point of these mappings.

Theorem 1.2. *Let (X, d) be a complete metric space, and let T be a mapping on X. Define a non-increasing function θ from $[0, 1)$ onto $(\frac{1}{2}, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2} \\ \frac{1-r}{r^2} & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}} \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that $\theta(r)d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

Inspired by Theorem 1.2, Suzuki [36, 37] proved a generalization of Edelstein's fixed point theorem.

Theorem 1.3. *Let (X, d) be a compact metric space, and let T be a mapping on X. Assume that $(\frac{1}{2})d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$. Then T has a unique fixed point.*

Salimi et al. [33] prove the existence and uniqueness of a fixed point results of Suzuki-Edelstein mapping, in a partially ordered complete metric space. In 2005, Eldred et al. [21] gave existence and convergence of best proximity points in the setting of a uniformly convex Banach space. Al-Thagafi et al. [6] studied the existence results of best proximity points for cyclic φ-contraction. In 2011, Sadiq Basha [14] stated some best proximity point theorems for proximal contractions. Jleli et al. [26] prove the existence of best proximity point for generalized $\alpha - \psi$-proximal contraction in the setting of complete metric space. Nawab et al. [22] obtained best proximity point results for modified Suzuki α-ψ-proximal contractions.

2. Preliminaries

Let (X, d) be a metric space, A and B two nonempty subsets of X. Define

$$\begin{aligned} d(A, B) &= \inf\{d(a, b) : a \in A, b \in B\}, \\ A_0 &= \{a \in A : \text{ there exists some } b \in B \text{ such that } d(a, b) = d(A, B)\}, \\ B_0 &= \{b \in B : \text{ there exists some } a \in A \text{ such that } d(a, b) = d(A, B)\}. \end{aligned}$$

Definition 2.1. Let (X, d) be a metric space and A, B two nonempty subsets of X. An element $x \in A$ is said to be a best proximity point of the mapping $T : A \to B$ if $d(x, Tx) = d(A, B)$.

Definition 2.2. ([38]) Let (X, d) be a metric space and $A_0 \neq \phi$, we say that the pair (A, B) has weak P-property if and only if

$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{cases} \quad \text{implies } d(x_1, x_2) \leq d(y_1, y_2)$$

for all $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

In 2012, Samet et al. [34] introduced the concept of α-admissible mappings. This nice concept was generalized and extended in many directions.

Definition 2.3. Let (X, d) be a metric space, T be a self-mapping on X and $\alpha : X \times X \to [0, \infty)$ be a function. The mapping T is an α-admissible if

$$\forall\, x, y \in X,\ \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 2.4. ([32]) Let (X, d) be a metric space, T be a self-mapping on X and $\alpha, \eta : A \times A \to [0, \infty)$ be two functions. The mapping T is an α-admissible with respect to η if

$$\forall\, x, y \in X,\ \alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Note that, if we take $\eta(x, y) \geq 1$, then this definition reduces to the definition of α-admissible mapping.

Definition 2.5. ([26]) Let (X, d) be a metric space and A, B two subsets of X, a non-self mapping $T : A \to B$ is called α-proximal admissible if

$$\begin{cases} \alpha(x_1, x_2) \geq 1, \\ d(u_1, Tx_1) = d(A, B), \quad \text{implies} \quad \alpha(u_1, u_2) \geq 1 \\ d(u_2, Tx_2) = d(A, B). \end{cases}$$

for all $x_1, x_2, u_1, u_2 \in A$ where $\alpha : A \times A \to [0, \infty)$.

Clearly, if $A = B$, T is α-proximal admissible implies that T is α-admissible.

Definition 2.6. ([22]) Let $T : A \to B$ and $\alpha, \eta : A \times A \to [0, \infty)$ be a functions. Then T is called α-proximal admissible with respect to η if

$$\begin{cases} \alpha(x_1, x_2) \geq \eta(x_1, x_2), \\ d(u_1, Tx_1) = d(A, B), \quad \text{implies} \quad \alpha(u_1, u_2) \geq \eta(u_1, u_2) \\ d(u_2, Tx_2) = d(A, B). \end{cases}$$

for all $x_1, x_2, u_1, u_2 \in A$. Note that if we take $\eta(x, y) = 1$ for all $x, y \in A$, then this definition reduced to the definition of α-proximal admissible.

In consistence with [33], we denote Φ_φ the set of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following condition:

$$\varphi(t) \leq \frac{1}{2}t \ \ \forall\ t \geq 0.$$

We denote by Φ the set of nondecreasing functions $\phi : [0, \infty) \to [0, \infty)$ such that

$$\lim_{n\to\infty} \phi^n(t) = 0 \ \ \forall\ t > 0.$$

Lemma 2.7. ([30]) *If $\phi \in \Phi$, then $\phi(t) < t$ for all $t > 0$.*

3. Best Proximity Point Results in Metric Space

We start this section with the following definition:

Definition 3.1. Suppose that A and B are two non-empty subsets of a metric space (X, d). A non-self mapping $T : A \to B$ is said to be modified Suzuki-Edelstein α-proximal contraction if

$$\varphi(d(x, Tx)) - 2d(A, B) \leq \alpha(x, y)d(x, y) \ \Rightarrow\ \alpha(x, y)d(Tx, Ty) \leq \phi(d(x, y)) \tag{1}$$

$\forall\, x, y \in A$, where $\varphi \in \Phi_\varphi$, $\phi \in \Phi$ and $\alpha : A \times A \to [0, \infty]$.

Theorem 3.2. *Suppose that A and B are two non empty closed subsets of a complete metric space (X, d) with A_0 is nonempty and let $T : A \to B$ with $T(A_0) \subseteq B_0$ be continuous modified Suzuki-Edelstein α-proximal admissible mapping with respect to $\eta(x, y) = 2$ and the pair (A, B) satisfies the weak P-property. Moreover, the elements x_0 and x_1 in A_0 with $d(x_1, Tx_0) = d(A, B)$ satisfies $\alpha(x_0, x_1) \geq 2$. Then T has a unique best proximity point.*

Proof. Consider x_0 in A_0, since $T(A_0) \subseteq B_0$, there exists element x_1 in A_0 such that

$$d(x_1, Tx_0) = d(A, B),$$

then by assumption, $\alpha(x_0, x_1) \geq 2$. Since $x_1 \in A_0$ and $T(A_0) \subseteq B_0$ then there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

Since T is $\alpha-$proximal admissible with respect to $\eta(x, y) = 2$, we have that $\alpha(x_0, x_1) \geq 2$.
Continuing in this fashion, we get that

$$d(x_{n+1}, Tx_n) = d(A, B) \text{ satisfies } \alpha(x_n, x_{n+1}) \geq 2 \tag{2}$$

for all $n \in \mathbb{N}$. Now

$$\begin{aligned} \varphi(d(x_{n-1}, Tx_{n-1})) &\leq \frac{1}{2} d(x_{n-1}, Tx_{n-1}) \\ &\leq 2(d(x_{n-1}, Tx_{n-1})) \\ &\leq 2(d(x_{n-1}, x_n) + d(x_n, Tx_{n-1})) \\ &= 2(d(x_{n-1}, x_n) + d(A, B)) \\ &= 2(d(x_{n-1}, x_n)) + 2d(A, B). \end{aligned} \tag{3}$$

By (3), we have

$$\begin{aligned} \varphi(d(x_{n-1}, Tx_{n-1})) - 2d(A, B) &\leq 2d(x_{n-1}, x_n) \\ &\leq \alpha(x_{n-1}, x_n) d(x_{n-1}, x_n). \end{aligned}$$

Then by (1), we have

$$\alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n) \leq \phi(d(x_{n-1}, x_n)).$$

Now

$$d(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n) \leq \phi(d(x_{n-1}, x_n))$$

implies

$$d(Tx_{n-1}, Tx_n) \leq \phi(d(x_{n-1}, x_n)). \tag{4}$$

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, we have

$$d(x_{n_0}, Tx_{n_0}) = d(x_{n_0+1}, Tx_{n_0}) = d(A, B).$$

Then x_{n_0} is the point of best proximity. Therefore, we assume that $x_{n_0} \neq x_{n+1}$, i.e. $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since ϕ is a nondecreasing, by (4) and weak P-property of (A, B), we have

$$d(x_{n+1}, x_n) \leq d(Tx_n, Tx_{n-1}) \leq \phi(d(x_n, x_{n-1})). \tag{5}$$

So

$$\begin{aligned}
d(x_{n+1},x_n) &\leq \phi(d(x_n,x_{n-1}))\\
&\leq \phi(d(Tx_{n-1},Tx_{n-2}))\\
&\leq \phi(\phi(d(x_{n-1},x_{n-2})))\\
&= \phi^2(d(x_{n-1},x_{n-2}))\\
&.\\
&.\\
&.\\
&\leq \phi^n(d(x_1,x_0)).
\end{aligned}$$

Hence

$$d(x_{n+1},x_n) \leq \phi^n d(x_0,x_1).$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} d(x_{n+1},x_n) = 0.$$

Now for a fixed ϵ, there exists $N \in \mathbb{N}$ such that

$$d(x_{n+1},x_n) < \epsilon - \phi(\epsilon)$$

for all $n \geq N$. As ϕ is nondecreasing. we get

$$\phi(d(x_{n+1},x_n)) \leq \phi(\epsilon - \phi(\epsilon)) \leq \phi(\epsilon)$$

for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
d(x_n,x_{n+2}) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2})\\
&< \epsilon - \phi(\epsilon) + \phi(d(x_n,x_{n+1}))\\
&\leq \epsilon - \phi(\epsilon) + \phi(\epsilon) = \epsilon.
\end{aligned}$$

Continuing this process, we get

$$d(x_n,x_{n+k}) < \epsilon$$

for all $n \geq N$ and $k \in \mathbb{N}$. Consequently,

$$\lim_{m,n\to\infty} d(x_m,x_n) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in A. Since X is a complete and A is closed, therefore there exists $z \in A$ such that $x_n \to z$ and by the continuity of T we derive that $Tx_n \to Tz$ as $n \to \infty$. Thus (2) gives

$$d(A,B) = \lim_{n\to\infty} d(x_{n+1},Tx_n) = d(z,Tz).$$

Now we have to show that T has a unique best proximity point. On contrary suppose that $y,z \in A_0$ are two best proximity points of T with $y \neq z$, that is

$$d(y,Ty) = d(z,Tz) = d(A,B).$$

By weak P-property, we get

$$d(y,z) \leq d(Ty,Tz). \tag{6}$$

Now,

$$d(y,Ty) = 2d(A,B) - d(A,B). \tag{7}$$

Since

$$\varphi(d(y, Ty)) \leq \frac{1}{2} d(y, Ty). \tag{8}$$

Eq. (7) gives

$$\begin{aligned} \varphi(d(y, Ty)) &\leq \frac{1}{2} d(y, Ty) \\ &= \frac{1}{2}(2d(A, B) - d(A, B)) \\ &= \frac{1}{2} d(A, B) \\ &\leq 2d(A, B). \end{aligned} \tag{9}$$

So (9) gives

$$\varphi(d(y, Ty)) - 2d(A, B) \leq 0 \leq \alpha(y, z) d(y, z).$$

Thus by (1), we have

$$\alpha(y, z) d(Ty, Tz) \leq \phi(d(y, z)). \tag{10}$$

Since $\alpha(y, z) \geq 2$, from (10), we have

$$d(Ty, Tz) \leq \phi(d(y, z)). \tag{11}$$

By (6), we get

$$d(y, z) \leq \phi(d(y, z)) < d(y, z), \tag{12}$$

which is a contradiction. Hance $y = z$. This completes the proof. □

Example 3.3. Let $X = [0, \infty) \times [0, \infty)$ with metric d defined as $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$. Suppose $A = \{1\} \times [0, \infty)$ and $B = \{0\} \times [0, \infty)$. Then $d(A, B) = d((1, 0), (0, 0)) = 1$ and $A_0 = A, B_0 = B$. Define $T : A \to B$ by

$$T(1, x) = \begin{cases} (0, \frac{x}{3}) & if\ x, \in\ [0, 1] \\ (0, x - \frac{2}{3}) & if x > 1, \end{cases}$$

$\alpha : A \times A \to [0, \infty)$ by

$$\alpha((x, y), (s, t)) = \begin{cases} 2 & if\ (x, y), (s, t) \in\ [0, 1] \times [0, 1] \\ 0 & otherwise, \end{cases}$$

and $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{999}{1000} t$ for all $t \geq 0$. Clearly, $T(A_0) \subseteq B_0$. Now, let $(1, x_1), (1, x_2) \in A$ and $(0, u_1), (0, u_2) \in B$ such that

$$\begin{cases} d((1, x_1), (0, u_1)) = d(A, B) = 1, \\ d((1, x_2), (0, u_2)) = d(A, B) = 1. \end{cases}$$

So

$$d((1, x_1), (1, x_2)) \leq d((0, u_1), (0, u_2)),$$

that is, the pair (A, B) has weak P-property. Suppose

$$\begin{cases} \alpha((1, x_1), (1, x_2)) \geq 1, \\ d((1, u_1), T(1, x_1)) = d(A, B) = 1, \\ d((1, u_2), T(1, x_2)) = d(A, B) = 1, \end{cases}$$

then

$$\begin{cases}(1,x_1),(1,x_2)\in[0,1],\\ d((1,u_1),T(1,x_1))=1,\\ d((1,u_2),T(1,x_2))=1.\end{cases}$$

Then, $(x_1,x_2)\in[0,1]\times[0,1]$. We also have $u_1=\frac{x_1}{3}$ *and* $u_2=\frac{x_2}{3}$, that is $(1,u_1=\frac{x_1}{3})\in[0,1]\times[0,1]$ and $(1,u_2=\frac{x_2}{3})\in[0,1]\times[0,1]$, i.e. $x_1,x_2\in[0,1]$. So $\alpha((1,u_1),(1,u_2))\geq 2$. That is, T is a α-proximal admissible mapping with respect to $\eta(x,y)=2$. If $(1,x),(1,y)\in[0,1]\times[0,1]$, then $\alpha((1,x),(1,y))=2$

$$\begin{aligned}\alpha((1,x),(1,y))d(T(1,x),T(1,y)) &\leq \phi(d((1,x),(1,y)))\\ 2d(T(1,x),T(1,y)) &\leq \phi(d((1,x),(1,y)))\\ 2d((0,\frac{x}{3})(0,\frac{y}{3})) &\leq \phi(|1-1|+|x-y|)\\ 2(|0-0|+|\frac{x}{3}-\frac{y}{3}|) &\leq \phi(|x-y|),\\ 2\frac{1}{3}|x-y| &\leq \frac{999}{1000}|x-y|,\\ 0.66|x-y| &\leq 0.99|x-y|.\end{aligned}$$

Otherwise, $\alpha((1,x),(1,y))=0$. That is

$$\varphi(d((1,x),T(1,x))-2d(A,B)\leq\alpha((1,x),(1,y)d((1,x),(1,y)=0$$

implies

$$0=\alpha((1,x),(1,y))d(T(1,x),T(1,y))\leq\phi(d((1,x),(1,y)).$$

Hance,

$$\varphi(d((1,x),T(1,x)))-2d(A,B)\leq\alpha((1,x),(1,y)d((1,x),(1,y))$$

implies

$$\alpha((1,x),(1,y))d(T(1,x),T(1,y))\leq\phi(d((1,x),(1,y))).$$

Thus all the conditions of Theorem 3.2 are satisfied. Hence there exist a unique best proximity point $z=(1,0)$ of T.

Best proximity point can also be obtained if we replace the condition of continuity of T in Theorem 3.2 by the following property:

$\mathcal{H}$: If $\{x_n\}$ is a sequence in A such that $\alpha(x_n,x_{n+1})\geq 2$ and $x_n\to z\in A$ as $n\to\infty$ then $\alpha(x_n,z)\geq 2$ $\forall$ $n\in N$.

Theorem 3.4. *Suppose that A and B are two non-empty closed subsets of a complete metric space (X,d) with A_0 is nonempty and let $T:A\to B$ with $T(A_0)\subseteq B_0$ be modified Suzuki-Edelstein α-proximal admissible mapping with respect to $\eta(x,y)\geq 2$, the pair (A,B) satisfies the weak P-property and the elements x_0 and x_1 in A_0 with $d(x_1,Tx_0)=d(A,B)$ satisfies $\alpha(x_0,x_1)\geq 2$. Moreover property $\mathcal{H}$ holds. Then T has a unique best proximity point.*

Proof. Following the proof of Theorem 3.2, we have a Cauchy sequence $x_n\to z$ as $n\to\infty$. By property $\mathcal{H}$, we have $\alpha(x_n,z)\geq 2$ for all $n\in\mathbb{N}$. From (5)

$$\begin{aligned}d(x_{n+2},x_{n+1}) &\leq \phi(d(x_{n+1},x_n))\\ &< d(x_{n+1},x_n)\end{aligned}$$

implies

$$d(x_{n+2},x_{n+1})<d(x_{n+1},x_n) \tag{13}$$

for all $n \in \mathbb{N}$. Also

$$\begin{aligned}\frac{1}{2}d(x_n, x_{n+1}) &\geq \varphi(d(x_n, x_{n+1})) \\ &> \alpha(x_n, z)d(x_n, z) \\ &\geq d(x_n, z),\end{aligned}$$

that is

$$\frac{1}{2}d(x_n, x_{n+1}) > d(x_n, z). \tag{14}$$

Similarly,

$$\frac{1}{2}d(x_{n+1}, x_{n+2}) > d(x_{n+1}, z) \tag{15}$$

for some $n \in \mathbb{N}$. Now

$$\begin{aligned}d(x_n, x_{n+1}) &\leq d(x_n, z) + d(x_{n+1}, z) \\ &< \frac{1}{2}d(x_n, x_{n+1}) + \frac{1}{2}d(x_{n+1}, x_n) \\ &< \frac{1}{2}d(x_n, x_{n+1}) + \frac{1}{2}d(x_n, x_{n+1}) \\ &= d(x_n, x_{n+1}).\end{aligned}$$

A contradiction. Thus for all $n \in \mathbb{N}$, either

$$\varphi(x_n, x_{n+1}) \leq \alpha(x_n, z)d(x_n, z)$$

or

$$\varphi(x_{n+1}, x_{n+2}) \leq \alpha(x_{n+1}, z)d(x_{n+1}, z)$$

holds. Hence by (1), we have

$$\begin{aligned}d(Tx_n, Tz) &\leq \alpha(x_n, z)d(Tx_n, Tz) \\ &\leq \phi(d(x_n, z))\end{aligned} \tag{16}$$

or

$$\begin{aligned}d(Tx_{n+1}, Tz) &\leq \alpha(x_{n+1}, z)d(Tx_{n+}, Tz) \\ &\leq \phi(d(x_{n+1}, z)).\end{aligned} \tag{17}$$

Taking limit as $n \to \infty$ in (16) and (17), we get

$$Tx_n \to Tz \text{ or } Tx_{n+1} \to Tz.$$

Consequently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k} \to Tz$ as $x_{n_k} \to z$. Therefore,

$$d(A, B) = \lim_{k\to\infty} d(x_{n_{k+1}}, Tx_{n_k}) = d(z, Tz).$$

Uniqueness follows from Theorem 3.2. □

Theorem 3.5. *Suppose that A and B are two non empty closed subsets of a complete metric space (X, d) with A_0 is nonempty, a function $\delta : [0, 1) \to (0, \frac{1}{2}]$ and for all $x, y \in A$ a mapping $T : A \to B$ be such that*

$$\delta(r)(\varphi(d(x, Tx)) - 2d(A, B)) \leq d(x, y) \;\Rightarrow\; \delta(r)d(Tx, Ty) \leq \phi(d(x, y)) \tag{18}$$

where $r \in [0, 1)$, $\varphi \in \Phi_\varphi$ and $\phi \in \Phi$. Moreover, $T(A_0) \subseteq B_0$ and the pair (A, B) satisfies the weak P-property. Then T has a unique best proximity point.

Proof. For a fixed $r \in [0,1)$, define $\alpha_r(x,y) = \frac{1}{\delta(r)}$ for all $x,y \in A$. Since $\frac{1}{\delta(r)} \geq 2$ for all $r \in [0,1)$, then $\alpha_r(x,y) \geq 2$ for all $x,y \in A$. Now since $\alpha_r(x,y)$ is constant and $\alpha_r(x,y) \geq 2$ for all $x,y \in A$, so T is an α_r-proximal admissible mapping with respect to $\eta(x,y) = 2$ and property $\mathcal{H}$ holds. Now if

$$\varphi(d(x,Tx)) - 2d(A,B) \leq \alpha_r(x,y)d(x,y)$$

then

$$\delta(r)(\varphi(d(x,Tx)) - 2d(A,B)) \leq d(x,y),$$

so by (18) we have

$$\delta(r)d(Tx,Ty) \leq \phi(d(x,y)).$$

Hence all the conditions of Theorem 3.4 holds and T has a unique best proximity point. □

If we take $\phi(t) = kt$, $k \in [0,1)$, in Theorem 3.5 we have the following:

Corollary 3.6. *Suppose that A and B are two non empty closed subsets of a complete metric space (X,d) with $A_0 \neq \phi$, a function $\delta : [0,1) \to (0,\frac{1}{2}]$ and for all $x,y \in A$ a mapping $T : A \to B$ be such that*

$$\delta(r)(\varphi(d(x,Tx)) - 2d(A,B)) \leq d(x,y) \;\Rightarrow\; \delta(r)d(Tx,Ty) \leq kd(x,y), \tag{19}$$

where $r \in [0,1)$, $\varphi \in \Phi_\varphi$. Moreover $T(A_0) \subseteq B_0$ and the pair (A,B) satisfies the weak P-property. Then T has a unique best proximity point.

Corollary 3.7. *Suppose that A and B are two non empty closed subsets of a complete metric space (X,d) with $A_0 \neq \phi$, a non-increasing function $\theta : [0,1) \to (\frac{1}{2},1]$ defined by*

$$\theta(r) = \begin{cases} 1 & if\ 0 \leq (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & if\ (\sqrt{5}-1)/2 < r < 2^{-1/2}, \\ (1+r)^{-1} & if\ 2^{-1/2} \leq r < 1, \end{cases}$$

and for all $x,y \in A$ a mapping $T : A \to B$ be such that

$$\frac{1}{2}\theta(r)(\varphi(d(x,Tx)) - 2d(A,B)) \leq d(x,y) \;\Rightarrow\; \frac{1}{2}\theta(r)d(Tx,Ty) \leq kd(x,y) \tag{20}$$

where $r \in [0,1)$, $\varphi \in \Phi_\varphi$. Moreover $T(A_0) \subseteq B_0$ and the pair (A,B) satisfies the weak P-property. Then T has a unique best proximity point.

Proof. Take $\delta(r) = \frac{1}{2}\theta(r)$ in Corollary 3.6, we get the required result. □

Corollary 3.8. *Suppose that A and B are two non empty closed subsets of a complete metric space (X,d) with $A_0 \neq \phi$, a non-increasing function $\beta : [0,1) \to (\frac{1}{2},1]$ defined by*

$$\beta(r) = \frac{1}{2(1+r)}$$

and for all $x,y \in A$ a mapping $T : A \to B$ be such that

$$\beta(r)(\varphi(d(x,Tx)) - 2d(A,B)) \leq d(x,y) \;\Rightarrow\; \beta(r)d(Tx,Ty) \leq kd(x,y) \tag{21}$$

where $r \in [0,1)$, $\varphi \in \Phi_\varphi$. Moreover $T(A_0) \subseteq B_0$ and the pair (A,B) satisfies the weak P-property. Then T has a unique best proximity point.

Proof. Take $\delta(r) = \frac{1}{2(1+r)}$ in Corollary 3.6, we get the required result. □

Corollary 3.9. *Suppose that A and B are two non empty closed subsets of a complete metric space* (X, d) *with* $A_0 \neq \phi$ *and for all* $x, y \in A$ *a mapping* $T : A \to B$ *be such that*

$$\frac{1}{2}(\varphi(d(x, Tx)) - 2d(A, B)) \leq d(x, y) \;\Rightarrow\; \frac{1}{2}rd(Tx, Ty) \leq kd(x, y) \tag{22}$$

where $r \in [0, 1)$, $\varphi \in \Phi_{\varphi}$. *Moreover* $T(A_0) \subseteq B_0$ *and the pair* (A, B) *satisfies the weak P-property. Then T has a unique best proximity point.*

Proof. Take $\delta(r) = \frac{1}{2}$ in Corollary 3.6, we get the required result. □

4. Best Proximtiy Point Results in Partially Ordered Metric Spaces

Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [[5, 23]] and references therein). The existence of best proximity and fixed point results in partially ordered metric spaces has been considered recently by many authors [4, 16, 18, 24, 25]. The aim of this section is to deduce some best proximity and fixed point results in the context of partially ordered metric spaces. Moreover, we obtain certain recent fixed point results as corollaries in partially ordered metric spaces.

Definition 4.1. ([23]) A mapping $T : A \to B$ is said to be proximally order-preserving if and only if it satisfies the condition

$$\begin{cases} x_1 \preceq x_2, \\ d(u_1, Tx_1) = d(A, B), & \Rightarrow u_1 \preceq u_2, \\ d(u_2, Tx_2) = d(A, B). \end{cases}$$

for all $x_1, x_2, u_1, u_2 \in A$.

Clearly, if $B = A$, then the proximally order-preserving map $T : A \to A$ reduces to a nondecreasing map.

Theorem 4.2. *Let A and B be two non-empty closed subsets of a partially ordered complete metric space* $(X, d, \preceq)$ *with* A_0 *is nonempty, let* $T : A \to B$ *with* $T(A_0) \subseteq B_0$ *is continuous proximally order preserving map, the pair* (A, B) *satisfies the weak P-property and*

$$\frac{1}{2}\varphi(d(x, Tx)) - d(A, B) \leq d(x, y) \Rightarrow \frac{1}{2}d(Tx, Ty) \leq \phi(d(x, y)) \tag{23}$$

hold, where $\varphi \in \Phi_{\varphi}$ *and* $\phi \in \Phi$. *Moreover the elements* x_0 *and* x_1 *in* A_0 *with* $d(x_1, Tx_0) = d(A, B)$ *satisfies* $x_0 \preceq x_1$. *Then T has a unique best proximity point.*

Proof. Define $\alpha : A \times A \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 2, & if x \preceq y, \\ 0, & otherwise. \end{cases}$$

Now we prove that T is an α-proximal admissible mapping with respect to $\eta(x, y) = 2$. For this, assume

$$\begin{cases} \alpha(x, y) \geq 2, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B), \end{cases}$$

so

$$\begin{cases} x \preceq y, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B). \end{cases}$$

Now, since T is proximally order-preserving, $u \preceq v$. Thus, $\alpha(u, v) \geq 2$. Furthermore, by assumption that for the comparable elements x_0 and x_1 in A_0 with $d(x_1, Tx_0) = d(A, B)$ satisfies $\alpha(x_0, x_1) \geq 2$. Finally, suppose that

$$\varphi(d(x, Tx)) - 2d(A, B) \leq \alpha(x, y)d(x, y).$$

Then for all comparable $x, y \in A$ we have $\alpha(x, y) \geq 2$ and hence by (23), we have

$$\alpha(x, y)d(Tx, Ty) \leq \phi(d(x, y)).$$

That is, T is a modified Suzuki-Edelstein α-proximal contraction. Thus all conditions of Theorem 3.2 hold and T has a unique best proximity point. □

$\mathcal{H}'$**:** If $\{x_n\}$ is a non-decreasing sequence in A such that $x_n \to x \in A$ as $n \to \infty$ then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Theorem 4.3. *Let A and B be two non-empty closed subsets of a partially ordered complete metric space $(X, d, \preceq)$ with $A_0 \neq \phi$ let $T : A \to B$ with $T(A_0) \subseteq B_0$ is continuous proximally order preserving map, the pair (A, B) satisfies the weak P-property and*

$$\frac{1}{2}\varphi(d(x, Tx)) - d(A, B) \leq d(x, y) \Rightarrow \frac{1}{2}d(Tx, Ty) \leq \phi(d(x, y)) \tag{24}$$

hold. Moreover the elements x_0 and x_1 in A_0 with $d(x_1, Tx_0) = d(A, B)$ satisfies $x_0 \preceq x_1$ along with property $\mathcal{H}'$. Then T has a unique best proximity point.

Proof. Following the definition of $\alpha : A \times A \to [0, \infty)$ as in the proof of Theorem 4.2, one can easily observe that T is an α-proximal admissible mapping with respect to $\eta(x, y) = 2$ and is modified Suzuki-Edelstein α-proximal contraction. Suppose that $\alpha(x_n, x_{n+1}) \geq 2$ such that $x_n \to x$ as $n \to \infty$, then $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. Hence by property $\mathcal{H}'$, we conclude $x_n \preceq x$ and so $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Thus all the conditions of Theorem 3.4 are satisfied and T has a unique best proximity point. □

If we take $\phi(t) = kt$ in Theorem 4.2, we obtain the following result:

Corollary 4.4. *Let A and B be two non-empty closed subsets of a partially ordered complete metric space $(X, d, \preceq)$ with A_0 is nonempty let $T : A \to B$ with $T(A_0) \subseteq B_0$ is continuous proximally order preserving map and the pair (A, B) satisfies the weak P-property and*

$$\frac{1}{2}\varphi(d(x, Tx)) - d(A, B) \leq d(x, y) \Rightarrow \frac{1}{2}d(Tx, Ty) \leq kd(x, y) \tag{25}$$

hold, where $0 \leq k < 1$ and $\varphi \in \Phi_\varphi$. Moreover the elements x_0 and x_1 in A_0 with $d(x_1, Tx_0) = d(A, B)$ satisfies $x_0 \preceq x_1$. Then T has a unique best proximity point.

5. Applications

As an application of our results, we deduce new fixed point results for Suzuki-Edelstien contraction in the frame work of metric and partially ordered metric spaces.

If we take $A = B = X$ in Theorem 3.2 and 3.4, we obtain the following fixed point results:

Theorem 5.1. *Suppose that (X, d) is a complete metric space and let $T : X \to X$ be continuous α-admissible mapping with respect to $\eta(x, y) = 2$ such that*

$$\varphi(d(x, Tx)) \leq \alpha(x, y)d(x, y) \;\Rightarrow\; \alpha(x, y)d(Tx, Ty) \leq \phi(d(x, y))$$

for all $x, y \in X$, where $\varphi \in \Phi_\varphi$ and $\phi \in \Phi$. Moreover, there exists element $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 2$. Then T has a unique fixed point.

Theorem 5.2. *Suppose that (X, d) is a complete metric space and let $T : X \to X$ be continuous α-admissible mapping with respect to $\eta(x, y) = 2$ such that*

$$\varphi(d(x, Tx)) \leq \alpha(x, y)d(x, y) \;\Rightarrow\; \alpha(x, y)d(Tx, Ty) \leq \phi(d(x, y))$$

for all $x, y \in X$, where $\varphi \in \Phi_\varphi$ and $\phi \in \Phi$. Moreover, there exists element $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 2$ along with property $\mathcal{H}$. Then T has a unique fixed point.

If $\phi(t) = kt$ in Theorem 5.1 and 5.2, where $0 \leq k < 1$, we get the following result:

Theorem 5.3. *Suppose that (X, d) is a complete metric space and let $T : X \to X$ be α-admissible mapping with respect to $\eta(x, y) = 2$ such that*

$$\varphi(d(x, Tx)) \leq \alpha(x, y)d(x, y) \;\Rightarrow\; \alpha(x, y)d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$, where $\varphi \in \Phi_\varphi$. Moreover, there exists element $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 2$ and either T is continuous or property $\mathcal{H}$ holds. Then T has a unique fixed point.

If we take $A = B = X$ in Theorem 4.2 and 4.3, we obtain the following fixed point results of [33] in complete partially ordered metric spaces:

Theorem 5.4. *Let $(X, d, \preceq)$ be partially ordered complete metric space, $T : X \to X$ be continuous non-decreasing satisfying*

$$\frac{1}{2}\varphi(d(x, Tx)) - d(A, B) \leq d(x, y) \Rightarrow \frac{1}{2}d(Tx, Ty) \leq \phi(d(x, y)) \tag{26}$$

for all comparable $x, y \in X$ with $x \preceq y$ where $\varphi \in \Phi_\varphi$ and $\phi \in \Phi$. Moreover, there exist an elements $x_0 \in X$ such that $x_0 \preceq Tx_0$. Then T has a unique fixed point.

Theorem 5.5. *Let $(X, d, \preceq)$ be partially ordered complete metric space, $T : X \to X$ be non-decreasing satisfying*

$$\frac{1}{2}\varphi(d(x, Tx)) - d(A, B) \leq d(x, y) \Rightarrow \frac{1}{2}d(Tx, Ty) \leq \phi(d(x, y)) \tag{27}$$

for all comparable $x, y \in X$ with $x \preceq y$ where $\varphi \in \Phi_\varphi$ and $\phi \in \Phi$. Moreover, there exist an elements $x_0 \in X$ such that $x_0 \preceq Tx_0$ along with property $\mathcal{H}'$. Then T has a unique fixed point.

If $\phi(t) = kt$ in Theorem 5.4 and 5.5, where $0 \leq k < 1$, we get the following result:

Theorem 5.6. *Let $(X, d, \preceq)$ be partially ordered complete metric space, $T : X \to X$ be non-decreasing satisfying*

$$\frac{1}{2}\varphi(d(x, Tx)) - d(A, B) \leq d(x, y) \Rightarrow \frac{1}{2}d(Tx, Ty) \leq kd(x, y)$$

for all comparable $x, y \in X$ with $x \preceq y$ where $\varphi \in \Phi_\varphi$. Moreover, there exist an elements $x_0 \in X$ such that $x_0 \preceq Tx_0$ and either T is continuous or property $\mathcal{H}'$ holds. Then T has a unique fixed point.

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References

[1] M. Abbas, A. Hussain, P. Kumam, A coincidence best proximity point problem in G-metric spaces, Abst. Appl. Anal. 2015 (2015), Article ID 243753, 12 pages.
[2] A. Abkar, M. Gabeleh, Generalized cyclic contractions in partially ordered metric spaces, Optim. Lett. 6 (2012) 1819–1830.
[3] A. Abkar, M. Gabeleh, Global optimal solutions of noncyclic mappings in metric spaces, J. Optim. Theory Appl. 153 (2012) 298–305.
[4] A. Abkar, M. Gabeleh, Best proximity points for cyclic mappings in ordered metric spaces, J. Optim. Theory Appl. 150 (2011) 188–193.
[5] R.P. Agarwal, N. Hussain, M.A. Taoudi, Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations, Abstr. Appl. Anal. 2012 (2012), Article ID 245872.
[6] M.A. Al-Thagafi, N. Shahzad, Convergence and existence results for best proximity points, Nonlinear Anal. 70 (2009) 3665-3671.
[7] S.S. Basha, Best proximity point theorems generalizing the contraction principle, Nonlinear Anal. 74 (2011) 5844–5850.
[8] S.S. Basha, Extensions of Banachs contraction principle, Numer. Funct. Anal. Optim. 31 (2010) 569–576.
[9] S.S. Basha, Best proximity points: global optimal approximate solution, J. Glob. Optim. 49 (2011) 15–21.
[10] S.S. Basha, Best proximity point theorems generalizing the contraction principle, Nonlinear Anal. 74 (2011) 5844–5850.
[11] S.S. Basha, N. Shahzad, R. Jeyaraj, Best proximity points: approximation and optimization, Optim. Lett. 7 (2011) 145–155.
[12] S.S. Basha, Common best proximity points: global minimization of multi-objective functions, J. Glob. Optim. 54 (2012) 367–373.
[13] S.S. Basha, Veeramani, P.: Best proximity pair theorems for multifunctions with open fibres, J. Approx. Theory 103 (2000) 119-129.
[14] S.S. Basha, Best proximity point theorems, J. Approx. Theory 163 (2011), 1772-1781.
[15] S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, Fundam. Math. 3 (1922) 133–181.
[16] S.S. Basha, Best proximity point theorems on partially ordered sets, Optim. Lett., 7(5) (2012) 1035-1043.
[17] H.H. Bauschke, R.S. Burachik, P.L. Combettes, V. Elser, D.R. Luke, H. Wolkowicz, Fixed point algorithms for inverse problems in science and engineering, Optimization and its Applications (1st edition), Springer, New York, NY, USA, 2011.
[18] L. Ćirić, M. Abbas, R. Saadati, N. Hussain, Common fixed points of almost generalized contractive mappings in ordered metric spaces, Appl. Math. Comput. 217 (2011) 5784–5789.
[19] M. Edelstein, On fixed and periodic points under contractive mappings, J. London Math. Soc. 37 (1962) 740-79.
[20] H. El-Dessouky, S. Bingulac, A fixed point iterative algorithm for solving equations modeling the multi-stage flash desalination process, Computer Methods Appl. Mech. Engin. 141 (1997) 95-115.
[21] A.A. Eldred, P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl. 323 (2006) 1001-1006.
[22] N. Hussain, M.A. Kutbi, P. Salimi, Best proximity point results for modified α-ψ-proximal rational contractions, Abstr. Appl. Anal. 2013 (2013), Article ID 927457.
[23] N. Hussain, A.R. Khan, R.P. Agarwal, Krasnoselskii and Ky Fan type fixed point theorems in ordered Banach spaces, J. Nonlinear Convex Anal. 11 (2010) 475–489.
[24] N. Hussain, S. Al-Mezel, P. Salimi, Fixed points for ψ-graphic contractions with application to integral equations, Abstr. Appl. Anal. 2013 (2013), Article ID 575869.
[25] N. Hussain, A. Latif, P. Salimi, Best proximity point results for modified Suzuki α-ψ-proximal contractions, Fixed Point Theory Appl. 2014:10 (2014).
[26] M. Jleli, B. Samet, Best proximity points for α-ψ-proximal contractive type mappings and applications, Bull. Sci. Math. 137 (2013) 977-995.
[27] P. Kumam, P. Salimi, C. Vetro, Best proximity point results for modified α-proximal C-contraction mappings, Fixed Point Theory Appl. 2014:99 (2014).
[28] A. Latif, M. Abbas, A. Hussain, Coincidence best proximity point of F_g-weak contractive mappings in partially ordered metric spaces, J. Nonlinear Sci. Appl. 9 (2016) 2448–2457.
[29] R. Levy, Fixed point theory and structural optimization, Engineering Optim. 17 (1991).
[30] J. Matkowski, Fixed point theorems for mappings with a contractive iterate at a point, Proc. Amer. Math. Soc. 62 (1977) 344-348.
[31] C. Mongkolkeha, Y.J. Cho, P. Kumam, Best proximity points for Geraghtys proximal contraction mappings. Fixed Point Theory Appl. 2013:180 (2013).
[32] P. Salimi, A. Latif, N. Hussain, Modified α-ψ-contractive mappings with applications, Fixed Point Theory Appl., 2013 (2013), Article ID 151.
[33] P. Salimi, E. Karapinar, Suzuki-Edelstein Type Contractions via Auxiliary Functions, Math. Problems Engin. 2013 (2013), Article ID 648528, 8 pages.
[34] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anal. 75 (2012) 2154–2165.
[35] N. Shahzad, S.S. Basha, R. Jeyaraj, Common best proximity points: global optimal solutions, J. Optim. Theory Appl. 148 (2011) 69–78.
[36] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc. 136 (2008) 1861-1869.
[37] T. Suzuki, A new type of fixed point theorem in metric spaces, Nonlinear Analysis Theory: Methods Appl. 71 (2009) 5313-5317.
[38] J. Zhang, Y. Su, Q. Cheng, A note on A best proximity point theorem for Geraghty-contractions, Fixed Point Theory Appl. 2013:99 (2013).