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Global Existence Results for Second Order Functional Differential Equations with Delay

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Abstract. Our aim in this work is to study the existence of global mild solutions of a second order functional differential equation with delay. We use the Schauder fixed point theorem combined with the cosine family of linear bounded operators for the existence of solutions.

1. Introduction

In this work we provide sufficient conditions for the existence of global mild solutions for two classes of second order semilinear functional equations with delay. Our investigations will be situated in the Banach space of real continuous and bounded functions on the real half axis $[0, +\infty)$. First, we will consider the following problem

$$y''(t) = Ay(t) + f(t, y_t), \quad \text{a.e. } t \in J := [0, +\infty) \tag{1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \quad y'(0) = \varphi, \tag{2}$$

where $f : J \times C([-r, 0], E) \to E$ is given function, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$, on E, $\phi : [-r, 0] \to E$ is given continuous function, and $(E, |\cdot|)$ is a real Banach space. For any function y defined on $[-r, +\infty)$ and any $t \in J$, we denote by y_t the element of $C([-r, 0], E)$ defined by $y_t(\theta) = y(t + \theta), \theta \in [-r, 0]$. Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.
Later, we consider the following problem

$$y''(t) = Ay(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J := [0, +\infty) \tag{3}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \quad y'(0) = \varphi, \tag{4}$$

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where $f : J \times \mathcal{B} \to E$ is given function, $A : D(A) \subset E \to E$ as in problem (1)-(2), $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$, and $(E, |\cdot|)$ is a real Banach space. For any function y defined on $[-r, +\infty)$ and any $t \in J$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta), \theta \in (-\infty, 0]$. We assume that the histories y_t belongs to some abstract phases $\mathcal{B}$. The main results are based upon Schauder's fixed theorem combined with the family of cosine operators.

The cosine function theory is related to abstract linear second order differential equations in the same manner that the semigroup theory of bounded linear operators is related to first order partial differential equations and its equally appealing devoted their generality and simplicity. For basic concepts and applications of this theory, we refer to the reader to Fattorini [20], Travis and Weeb [41].

In the literature there are many papers study the problems of differential equations using different methods. Among them, the fixed point method combined by semigroup theory in Fréchet space, see for instance Baghli and Benchohra [5–7]. Our purpose in this work is to consider a simultaneous generalization of the classical second order abstract Cauchy problem studied by Travis and Weeb in [40, 41]. Additionally, we observe that the ideas and techniques in this paper permit the reformulation of the problems studied in [8, 12, 21, 35, 36] to the context of partial second order differential equations, see ([40], pp. 557) and the referred papers for details.

Complicated situations in which the delay depends on the unknown functions have been studied in the recent years (see for instance [1, 4, 38, 42, 43] and the references therein). Over the past several years it has become apparent that equations with state-dependent delay arise also in several areas such as classical electrodynamics [16], in population models [10], models of commodity price fluctuations [11, 33], and models of blood cell productions [34]. These equations are frequently called equations with state-dependent delay. The literature devoted differential equations with state-dependent delay is concerned fundamentally with first order functional differential equations for which the state belong to some finite dimensional space, see among another works [2, 9, 13, 15, 17–19]. The problem of the existence of solutions for first and second order partial functional differential with state-dependent delay have treated recently in [3, 25–29, 32, 37, 38]. The literature relative second order differential system with state-dependent delay is very restrict, and related this matter we only cite [39] for ordinary differential system and [24] for abstract partial differential systems.

To the best of our knowledge, the study of the existence of solutions for abstract second order functional differential equations with state-dependent delay on unbounded interval is an untreated topic in the literature and this fact, is the main motivation of the present work.

This paper is organized as follows. In Section 2 we introduce some preliminary results which are used in the following section. Our main result will be presented in Section 3, while in Section 4 we provide two examples to illustrate the abstract theory.

2. Preliminaries

In this section we present briefly some notations, definition, and a theorem that are used throughout this work. In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [23] and follow the terminology used in [30]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold :
(i) $y_t \in \mathcal{B}$;
(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
(iii) There exist two functions $L(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with L continuous and bounded, and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq L(t) \sup\{ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Denote
$$l = \sup\{L(t) : t \in J\},$$
and
$$m = \sup\{M(t) : t \in J\}.$$

Remark 2.1. 1. *(ii) is equivalent to $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.*

2. *Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.*
3. *From the equivalence of in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$: We necessarily have that $\phi(0) = \psi(0)$.*

Example 2.2. *(The phase space $(\mathbf{C_r} \times \mathbf{L^p}(\mathbf{g}, \mathbf{E}))$.)*
Let $g : (-\infty, -r) \to \mathbb{R}$ be a positive Lebesgue integrable function and assume that there exists a non-negative and locally bounded function γ on $(-\infty, 0]$ such that $g(\xi + \theta) \leq \gamma(\xi)g(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(g, E)$ consists of all classes of functions $\varphi(-\infty, 0] \to \mathbb{R}$ such that ϕ is continuous on $[-r, 0]$, Lebesgue-measurable and $g\|\phi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(g, E)$ is defined by
$$\|\phi\|_{\mathcal{B}} := \sup\{\|\phi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} g(\theta)\|\phi(\theta)\|^p d\theta\right)^{\frac{1}{p}}$$
Assume that $g(\cdot)$ verifies the condition $(g-5), (g-6)$ and $(g-7)$ in the nomenclature [30]. In this case, $\mathcal{B} = C_r \times L^p(g, E)$ verifies assumptions $(A_1), (A_2), (A_3)$ see ([30] Theorem 1.3.8) for details. Moreover, when $r = 0$ and $p = 2$ we have that $H = 1, M(t) = \gamma(-t)^{\frac{1}{2}}$ and $L(t) = 1 + \left(\int_{-t}^{0} g(\theta)d\theta\right)^{\frac{1}{2}}$ for $t \geq 0$.

By BUC we denote the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E.

Next we mention a few results and notations respect of the cosine function theory which are needed to establish our results. Along of this section, A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on Banach space $(E, \|\cdot\|)$. We denote by $(S(t))_{t\in\mathbb{R}}$ the sine function associated with $(C(t))_{t\in\mathbb{R}}$ which is defined by $S(t)y = \int_0^t C(s)y ds$, for $y \in E$ and $t \in \mathbb{R}$.

The notation $[D(A)]$ stands for the domain of the operator A endowed with the graph norm $\|y\|_A = \|y\| + \|Ay\|, y \in D(A)$. Moreover, in this work, X is the space formed by the vector $y \in E$ for which $C(\cdot)y$ is of class C^1 on $\mathbb{R}$. It was proved by Kisinsky [31] that X endowed with the norm
$$\|y\|_X = \|y\| + \sup_{0\leq t\leq 1} \|AS(t)y\|, y \in X,$$
is a Banach space. The operator valued function
$$G(t) = \begin{pmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{pmatrix}$$
is a strongly continuous group of bounded linear operators on the space $X \times E$ generated by the operator
$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

defined on $D(A)\times X$. It follows that $AS(t): X\to E$ is a bounded linear operator and $AS(T)y \to 0,\ t\longrightarrow 0$, for each $y\in X$. Furthermore, if $y:[0,+\infty)\to E$ is a locally integrable function, then $z(t)=\displaystyle\int_0^t S(t-s)y(s)ds$ defined an $X-$valued continuous function. This is a consequence of the fact that:

$$\int_0^t G(t-s)\begin{pmatrix} 0\\ y(s)\end{pmatrix}ds=\begin{pmatrix}\displaystyle\int_0^t S(t-s)y(s)ds\\ \displaystyle\int_0^t C(t-s)y(s)ds\end{pmatrix}$$

defines an $X\times E-$ valued continuous function. The existence of solutions for the second order abstract Cauchy problem

$$\begin{cases} y''(t)=Ay(t)+h(t), & t\in J:=[0,+\infty),\\ y(0)=y_0, \quad y'(0)=y_1, \end{cases} \tag{5}$$

where $h: J\to E$ is an integrable function has been discussed in [40]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem has been treated in [41].

Definition 2.3. *The function $y(\cdot)$ given by*

$$y(t)=C(t)y_0+S(t)y_1+\int_0^t S(t-s)h(s)ds,\ t\in J,$$

is called mild solution of (5).

Remark 2.4. *Let $y_0\in X$, then y is continuously differentiable and we have*

$$y'(t)=AS(t)y_0+C(t)y_1+\int_0^t C(t-s)h(s)ds.$$

For additional details about cosine function theory, we refer to the reader to [40, 41].

We need the following definitions in the sequel.

Definition 2.5. *A map $f: J\times C([-r,0],E)\to E$ is said to be Carathéodory if*

(i) $t\to f(t,y)$ is measurable for all $y\in C([-r,0],E)$,

(ii) $y\to f(t,y)$ is continuous for almost each $t\in J$.

Remark 2.6. *In the case of state-dependent delay we substitute $C([-r,0),E)$ by phase space $\mathcal{B}$.*

Theorem 2.7. *(Schauder's fixed point [22])*
Let B be a closed, convex and nonempty subset of a Banach space $\mathcal{E}$. Let $N: B\to B$ be a continuous mapping such that $N(B)$ is a relatively compact subset of $\mathcal{E}$. Then N has at least one fixed point in B, that is, there exists $y\in B$ such that $Ny=y$.

Lemma 2.8. *(Corduneanu [14])*
Let $D\subset BC([0,+\infty),E)$. Then D is relatively compact if the following conditions hold:

(a) D is bounded in BC.

(b) The function belonging to D is almost equicontinuous on $[0,+\infty)$, i.e., equicontinuous on every compact of $[0,+\infty)$.

(c) The set $D(t):=\{y(t): y\in D\}$ is relatively compact on every compact of $[0,+\infty)$.

(d) The function from D is equiconvergent, that is, given $\epsilon>0$, responds $T(\epsilon)>0$ such that $|u(t)-\lim\limits_{t\to+\infty}u(t)|<\epsilon$, for any $t\geq T(\epsilon)$ and $u\in D$.

3. Existing result for the finite delay case

In this section by $BC := BC([-r,+\infty))$ we denote the Banach space of all bounded and continuous functions from $[-r,+\infty)$ into E equipped with the standard norm

$$\|y\|_{BC} = \sup_{t\in[-r,+\infty)} |y(t)|.$$

Now we give our main existence result for problem (1)-(2). Before starting and proving this result, we give the definition of a mild solution.

Definition 3.1. *We say that a continuous function $y : [-r,+\infty) \to E$ is a mild solution of problem (1)-(2) if $y(t) = \phi(t)$, $t \in [-r,0]$, $y(.)$ and $y'(0) = \varphi$,and*

$$y(t) = C(t)\phi(0) + S(t)\varphi + \int_0^t C(t-s)f(s,y_s)ds,\ t \in J.$$

Let

$$M = \sup\{\|C(t)\|_{B(E)} : t \geq 0\},\ M' = \sup\{\|S(t)\|_{B(E)} : t \geq 0\}.$$

Let us introduce the following hypotheses.

(H_1) $C(t)$ is compact for $t > 0$.

(H_2) The function $f : J \times C([-r,0],E) \to E$ is Carathéodory.

(H_3) There exists a continuous function $k : J \to [0,+\infty)$ such that:

$$|f(t,u)| \leq k(t)\|u\|,\ t \in J,\ u \in C([-r,0],E)$$

and

$$k^* := \sup_{t\in J} \int_0^t k(s)ds < \infty.$$

(H_4) For each bounded $B \subset BC$ and $t \in J$ the set :

$$\left\{\int_0^t C(t-s)f(s,y_t)ds : y \in B\right\}$$

is relatively compact in E.

Remark 3.2. *Condition (H_4) is satisfied if for each $t \in J$, the function $u \mapsto f(t,u)$ maps bounded sets into relatively compact sets.*

Theorem 3.3. *Assume that $(H_1) - (H_4)$ hold. If $K^*M < 1$, then the problem (1)-(2) has at least one mild solution.*

Proof. We transform the problem (1)-(2) into a fixed point problem. Consider the operator: $N : BC \to BC$ define by:

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ C(t)\,\phi(0) + S(t)\varphi + \displaystyle\int_0^t C(t-s)\,f(s,y_s)\,ds, & \text{if } t \in J. \end{cases}$$

The operator N maps BC into BC; indeed the map $N(y)$ is continuous on $[-r, +\infty)$ for any $y \in BC$, and for each $t \in J$, we have

$$\begin{aligned} |N(y)(t)| &\leq M\|\phi\| + M'\|\varphi\| + M\int_0^t |f(s, y_s)|ds \\ &\leq M\|\phi\| + M'\|\varphi\| + M\int_0^t k(s)\|y_s\|ds \\ &\leq M\|\phi\| + M'\|\varphi\| + M\|y\|_{BC}k^* := c. \end{aligned}$$

Hence, $N(y) \in BC$. Let

$$C = M\|\phi\| + M'\|\varphi\|.$$

Moreover, let $r > 0$ be such that $r \geq \frac{C}{1-Mk^*}$, and B_r be the closed ball in BC centered at the origin and of radius r. Let $y \in B_r$ and $t \in [0, +\infty)$. Then,

$$|N(y)(t)| \leq \quad C + Mk^*r.$$

Thus,

$$\|N(y)\|_{BC} \leq r,$$

which means that the operator N transforms the ball B_r into itself.

Now we prove that $N : B_r \to B_r$ satisfies the assumptions of Schauder's fixed theorem. The proof will be given in several steps.

Step 1: N is continuous in B_r.
Let $\{y_n\}$ be a sequence such that $y_n \to y$ in B_r. We have

$$|N(y_n)(t) - N(y)(t)| \quad \leq M\int_0^t |f(s, y_{s_n}) - f(s, y_s)|ds.$$

Then by (H_2) we have $f(s, y_{s_n}) \to f(s, y_s)$, as $n \to \infty$, for a.e. $s \in J$, and by the Lebesgue dominated convergence theorem we have

$$\|N(y_n) - N(y)\|_{BC} \to 0, \text{ as } n \to \infty.$$

Thus, N is continuous.

Step 2 : $N(B_r) \subset B_r$ this is clear.

Step 3: $N(B_r)$ is equicontinuous on every compact interval $[0, b]$ of $[0, +\infty)$ for $b > 0$. Let $\tau_1, \tau_2 \in [0, b]$

with $\tau_2 > \tau_1$, we have

$$
\begin{aligned}
& |N(y)(\tau_2) - N(y)(\tau_1)| \\
\leq\ & \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}\|\phi\| + \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}\|\varphi\| \\
+\ & \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}|f(s, y_s)|ds \\
+\ & \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)}|f(s, y_s)|ds \\
\leq\ & \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}\|\phi\| + \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}\|\varphi\| \\
+\ & \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}|f(s, y_s)|ds \\
+\ & \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)}|f(s, y_s)|ds \\
\leq\ & \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}\|\phi\| + \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}\|\varphi\| \\
+\ & r\int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}k(s)ds \\
+\ & r\int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)}k(s)ds.
\end{aligned}
$$

When $\tau_2 \to \tau_2$, the right-hand side of the above inequality tends to zero, since $C(t), S(t)$ are a strongly continuous operator and the compactness of $C(t), S(t)$ for $t > 0$, implies the continuity in the uniform operator topology (see [40, 41]). This proves the equicontinuity of the set $N(B_r)$.

Step 4:$N(B_r)$ is relatively compact on every compact interval of $[0, \infty)$ by (H_5).

Step 5: $N(B_r)$ is equiconvergent.
Let $y \in B_r$ and $t \in J$. We have

$$
\begin{aligned}
|N(y)(t)| &\leq M\|\phi\| + M'\|\varphi\| + M\int_0^t |f(s, y_s)|ds \\
&\leq C + Mr\int_0^t k(s)ds.
\end{aligned}
$$

Then

$$|N(y)(t)| \to l \leq C + Mk^* r, \quad \text{as } t \to +\infty.$$

Hence,

$$|N(y)(t) - N(y)(+\infty)| \to 0, \quad \text{as } t \to +\infty.$$

As a consequence of Steps 1-5, with Lemma 2.8, we can conclude that $N : B_r \to B_r$ is continuous and compact. From Schauder's theorem, we deduce that N has a fixed point y^* which is a mild solution of the problem (1)-(2).

4. The state-dependent delay case

In this section by $BC := BC(-\infty, +\infty)$ we denote the Banach space of all bounded and continuous functions from $(-\infty, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC} = \sup_{t \in (-\infty, +\infty)} |y(t)|.$$

Finally, by $BC' := BC'([0,+\infty))$ we denote the Banach space of all bounded and continuous functions from $[0,+\infty)$ into E equipped with the standard norm

$$\|y\|_{BC'} = \sup_{t\in[0,+\infty)} |y(t)|.$$

Now we give our main existence result for problem (3)-(4). Before starting and proving this result, we give the definition of a mild solution.

Definition 4.1. *We say that a continuous function $y : (-\infty,+\infty) \to E$ is a mild solution of problem (3)-(4) if $y(t) = \phi(t)$, $t \in (-\infty, 0]$, $y(\cdot)$ is continuously differentiable and $y'(0) = \varphi$ and*

$$y(t) = C(t)\phi(0) + S(t)\varphi + \int_0^t C(t-s)f(s, y_{\rho(t,y_t)})ds,\ t \in J.$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\phi) : (s,\phi) \in J \times \mathcal{B},\ \rho(s,\phi) \le 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\phi_t\| \le \mathcal{L}^\phi(t)\|\phi\| \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 4.2. *The condition (H_ϕ), is frequently verified by functions continuous and bounded. For more details, see for instance [30].*

Lemma 4.3. *([29], Lemma 2.4) If $y : (-\infty,+\infty) \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \le (m + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} + l\sup\{|y(\theta)|; \theta \in [0, max\{0,s\}]\},\ s \in \mathcal{R}(\rho^-) \cup J,$$

where $\mathcal{L}^\phi = \sup_{t\in\mathcal{R}(\rho^-)} \mathcal{L}^\phi(t)$.

Let us introduce the following hypotheses:

(D_1) $C(t)$ is compact for $t > 0$.

(D_2) The function $f : J \times \mathcal{B} \to E$ is Carathéodory.

(D_3) There exists a continuous function $k : J \to [0,+\infty)$ such that:

$$|f(t,u)| \le k(t)\|u\|,\ t \in J,\ u \in \mathcal{B}$$

and

$$k^* := \sup_{t\in J} \int_0^t k(s)ds < \infty.$$

(D_4) For each bounded $B \subset BC'$ and $t \in J$ the set :

$$\left\{\int_0^t C(t-s)f(s, y_{\rho(t,y_t)})ds : y \in \mathcal{B}\right\}$$

is relatively compact in E.

Theorem 4.4. *Assume that $(D_1)-(D_5),(H_\phi)$ hold. If $K^*Ml<1$, then the problem (3)-(4) has at least one mild solution on BC.*

Proof. We transform the problem (3)-(4) into a fixed point problem. Consider the operator $N:BC\to BC$ define by:

$$N(y)(t)=\begin{cases}\phi(t), & \text{if } t\in(-\infty,0],\\[2mm] C(t)\,\phi(0)+S(t)\varphi+\displaystyle\int_0^t C(t-s)\,f(s,y_{\rho(t,y_t)})\,ds, & \text{if } t\in J.\end{cases}$$

Let $x(\cdot):(-\infty,+\infty)\to E$ be the function defined by:

$$x(t)=\begin{cases}\phi(t); & \text{if } t\in(-\infty,0];\\[2mm] C(t)\,\phi(0); & \text{if } t\in J,\end{cases}$$

then $x_0=\phi$. For each $z\in BC$ with $z(0)=0,y'(0)=\varphi=z'(0)=\varphi_1$, we denote by $\overline{z}$ the function

$$\overline{z}(t)=\begin{cases}0; & \text{if } t\in(-\infty,0];\\[2mm] z(t); & \text{if } t\in J.\end{cases}$$

If y satisfies $y(t)=N(y)(t)$, we can decompose it as $y(t)=z(t)+x(t),\ t\in J$, which implies $y_t=z_t+x_t$ for every $t\in J$ and the function $z(\cdot)$ satisfies

$$z(t)=S(t)\varphi_1+\int_0^t C(t-s)\,f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})ds,t\in J.$$

Set

$$BC_0'=\{z\in BC':z(0)=0\}$$

and let

$$\|z\|_{BC_0'}=\sup\{|z(t)|:t\in J\},\ z\in BC_0'.$$

BC_0' is a Banach space with the norm $\|\cdot\|_{BC_0'}$. We define the operator $\mathcal{A}:BC_0'\to BC_0'$ by:

$$\mathcal{A}(z)(t)=S(t)\varphi_1+\int_0^t C(t-s)\,f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})ds,\ t\in J.$$

We shall show that the operator $\mathcal{A}$ satisfies all conditions of Schauder's fixed point theorem. The operator A maps BC_0' into BC_0', indeed the map $\mathcal{A}(z)$ is continuous on $[0,+\infty)$ for any $z\in BC_0'$, and for each $t\in J$ we have

$$\begin{aligned}|\mathcal{A}(z)(t)| &\le M'\|\varphi_1\|+M\int_0^t|f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})|ds\\ &\le M'\|\varphi_1\|+M\int_0^t k(s)\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}ds\\ &\le M'\|\varphi_1\|+M\int_0^t k(s)(l|z(s)|+(m+\mathcal{L}^\phi+lMH)\|\phi\|_{\mathcal{B}})ds.\end{aligned}$$

Let

$$C=(m+\mathcal{L}^\phi+lMH)\|\phi\|_{\mathcal{B}}.$$

Then, we have:

$$\begin{aligned}|\mathcal{A}(z)(t)| &\le M'\|\varphi_1\|+MC\int_0^t k(s)ds+Ml\int_0^t k(s)|z(s)|ds\\ &\le M'\|\varphi_1\|+MCk^*+Ml\|z\|_{BC_0'}k^*.\end{aligned}$$

Hence, $\mathcal{A}(z) \in BC_0'$.

Moreover, let $r > 0$ be such that $r \geq \frac{M'\|\varphi_1\|+MCk^*}{1-Mlk^*}$, and B_r be the closed ball in BC_0' centered at the origin and of radius r. Let $y \in B_r$ and $t \in [0, +\infty)$. Then,

$$|\mathcal{A}(z)(t)| \leq \quad M'\|\varphi_1\| + MCk^* + Mlk^* r.$$

Thus,

$$\|\mathcal{A}(z)\|_{BC_0'} \leq r,$$

which means that the operator N transforms the ball B_r into itself.

Now we prove that $\mathcal{A} : B_r \to B_r$ satisfies the assumptions of Schauder's fixed theorem. The proof will be given in several steps.

Step 1: $\mathcal{A}$ is continuous in B_r.
Let $\{z_n\}$ be a sequence such that $z_n \to z$ in B_r. At the first, we study the convergence of the sequences $(z^n_{\rho(s,z^n_s)})_{n\in\mathbb{N}}, s \in J$.
If $s \in J$ is such that $\rho(s, z_s) > 0$, then we have,

$$\begin{aligned}\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} &\leq \|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z^n_s)}\|_{\mathcal{B}} + \|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}}\\ &\leq l\|z_n - z\|_{B_r} + \|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}},\end{aligned}$$

which proves that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s, z_s) > 0$. Similarly, is $\rho(s, z_s) < 0$, we get

$$\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} = \|\phi^n_{\rho(s,z^n_s)} - \phi_{\rho(s,z_s)}\|_{\mathcal{B}} = 0$$

which also shows that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s, z_s) < 0$. Combining the pervious arguments, we can prove that $z^n_{\rho(s,z_s)} \to \phi$ for every $s \in J$ such that $\rho(s, z_s) = 0$. Finally,

$$\begin{aligned}&|\mathcal{A}(z_n)(t) - \mathcal{A}(z)(t)|\\ \leq\ & M\int_0^t |f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds.\end{aligned}$$

Then by (D_2) we have

$$f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) \to f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}), \text{ as } n \to \infty,$$

and by the Lebesgue dominated convergence theorem we get,

$$\|\mathcal{A}(z_n) - \mathcal{A}(z)\|_{BC_0'} \to 0, \text{ as } n \to \infty.$$

Thus $\mathcal{A}$ is continuous.

Step 2 : $\mathcal{A}(B_r) \subset B_r$ this is clear.

Step 3: $\mathcal{A}(B_r)$ is equicontinuous on every compact interval $[0, b]$ of $[0, +\infty)$ for $b > 0$. Let $\tau_1, \tau_2 \in [0, b]$

with $\tau_2 > \tau_1$, we have

$$\begin{aligned}
& |\mathcal{A}(z)(\tau_2) - \mathcal{A}(z)(\tau_1)| \\
\leq\ & \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}\|\varphi_1\| \\
+\ & \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}|f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)})|ds \\
+\ & \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)}|f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)})|ds \\
\leq\ & \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}\|\varphi_1\| \\
+\ & \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}|f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)})|ds \\
+\ & \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)}|f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)})|ds \\
\leq\ & \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}\|\varphi_1\| \\
+\ & C\int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}k(s)ds \\
+\ & lr\int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}k(s)ds \\
+\ & C\int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)}k(s)ds \\
+\ & lr\int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)}k(s)ds.
\end{aligned}$$

When $\tau_2 \to \tau_2$, the right-hand side of the above inequality tends to zero, since $C(t)$ are a strongly continuous operator and the compactness of $C(t)$ for $t > 0$, implies he continuity in the uniform operator topology (see [40, 41]). This proves the equicontinuity of the set $\mathcal{A}(B_r)$.

Step 4:$N(B_r)$ is relatively compact on every compact interval of $[0, \infty)$. This is a consequence of (D_5).

Step 5: $N(B_r)$ is equiconvergent.
Let $y \in B_r$, we have:

$$\begin{aligned}
|\mathcal{A}(z)(t)| &\leq M'\|\varphi_1\| + M\int_0^t |f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)})|ds \\
&\leq M'\|\varphi_1\| + MCk^* + Mrl\int_0^t k(s)ds.
\end{aligned}$$

Then

$$|\mathcal{A}(z)(t)| \to l_1 \leq M'\|\varphi_1\| + Mk^*(C + lr), \quad \text{as } t \to +\infty.$$

Hence,

$$|\mathcal{A}(z)(t) - \mathcal{A}(z)(+\infty)| \to 0, \quad \text{as } t \to +\infty.$$

As a consequence of Steps 1-5, with Lemma 2.8, we can conclude that $\mathcal{A} : B_r \to B_r$ is continuous and compact. we deduce that $\mathcal{A}$ has a fixed point z^*. Then $y^* = z^* + x$ is a fixed point of the operators N, which is a mild solution of the problem (3)-(4).

5. Examples

Example 1. Consider the functional partial differential equation of second order

$$\frac{\partial^2}{\partial t^2}z(t, x) = \frac{\partial^2}{\partial x^2}z(t, x) + \frac{1}{2e^t}\frac{|z(t-1, x)|}{1 + |z(t-1, x)|},\ x \in [0, \pi],\ t \in J = [0, +\infty), \tag{6}$$

$$z(t,0) = z(t,\pi) = 0,\ t \in J, \tag{7}$$

$$z(t,x) = \phi(t),\quad \frac{\partial z(0,x)}{\partial t} = w(x),\ t \in [-1,0],\ x \in [0,\pi]. \tag{8}$$

Take $E = L^2[0,\pi]$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E; \omega, \omega' \text{are absolutely continuous},\ \omega'' \in E,\ \omega(0) = \omega(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$on E, respectively. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$ with corresponding normalized eigenvectors

$$z_n(\tau) := \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin n\tau,$$

and the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E.

(b) If $y \in E$, then $Ay = -\sum_{n=1}^{\infty} n^2 < y, z_n > z_n$.

(c) For $y \in E, C(t)y = \sum_{n=1}^{\infty} \cos(nt) < y, z_n > z_n$, and the associated sine family is

$$S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < y, z_n > z_n$$

which implies that the operator $S(t)$ is compact for all $t > 0$ and that

$$\|C(t)\| = \|S(t)\| \leq 1,\ \text{for all } t \geq 0.$$

(d) If Φ denotes the group of translations on E defined by

$$\Phi(t)y(\xi) = \tilde{y}(\xi + t),$$

where $\tilde{y}$ is the extension of y with period 2π, then

$$C(t) = \frac{1}{2}(\Phi(t) + \Phi(-t)); A = B^2,$$

where B is the infinitesimal generator of the group Φ on

$$X = \{y \in H^1(0,\pi) : y(0) = x(\pi) = 0\}.$$

For more details, see [20].

Then the problem (1)-(2) in an abstract formulation of the problem (6)-(8). It is clear that $(H_1) - (H_4)$ are satisfied with $k^* = \frac{1}{2}$. Theorem 3.3 implies that the problem (6)-(8) has at least one mild solution.

Example 2. Take $E = L^2[0,\pi]; \mathcal{B} = C_0 \times L^2(g, E)$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E; \omega, \omega' \text{are absolutely continuous},\ \omega'' \in E,\ \omega(0) = \omega(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$on E, respectively. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$ with corresponding normalized eigenvectors

$$z_n(\tau) := \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin n\tau,$$

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(b) If $y \in E$, then $Ay = -\sum_{n=1}^{\infty} n^2 < y, z_n > z_n$.

(c) For $y \in E, C(t)y = \sum_{n=1}^{\infty} \cos(nt) < y, z_n > z_n$, and the associated sine family is

$$S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < y, z_n > z_n$$

which implies that the operator $S(t)$ is compact, for all $t \in J$ and that

$$\|C(t)\| = \|S(t)\| \leq 1, \text{ for all } t \in \mathbb{R}.$$

(d) If Φ denotes the group of translations on E defined by

$$\Phi(t)y(\xi) = \tilde{y}(\xi + t),$$

where $\tilde{y}$ is the extension of y with period 2π. Then

$$C(t = \frac{1}{2}(\Phi(t) + \Phi(-t)); A = B^2,$$

where B is the infinitesimal generator of the group Φ on

$$X = \{y \in H^1(0, \pi) : y(0) = x(\pi) = 0\}.$$

For more details, see [20].

Consider the functional partial differential equation of second order

$$\frac{\partial^2}{\partial t^2} z(t, x) = \frac{\partial^2}{\partial x^2} z(t, x) + \int_{-\infty}^{0} a(s - t) z(s - \rho_1(t)\rho_2(|z(t)|), x) ds, \ x \in [0, \pi], \ t \in J, \tag{9}$$

$$z(t, 0) = z(t, \pi) = 0, \ t \in J, \tag{10}$$

$$z(t, x) = \phi(t), \quad \frac{\partial z(0, x)}{\partial t} = \omega(x), \ t \in [-r, 0], \ x \in [0, \pi], \tag{11}$$

where $J := [0, +\infty)$, $\rho_i : [0, \infty) \to [0, \infty)$, $a : \mathbb{R} \to \mathbb{R}$ be continuous, and

$$L_f = \left(\int_{-\infty}^{0} \frac{a^2(s)}{g(s)} ds \right)^{\frac{1}{2}} < \infty.$$

We define the functions $f : J \times \mathcal{B} \to E, \rho : J \times \mathcal{B} \to \mathbb{R}$ by

$$f(t, \psi)(x) = \int_{-\infty}^{0} a(s)\psi(s, x) ds,$$

$$\rho(s, \psi) = s - \rho_1(s)\rho_2(\|\psi(0)\|).$$

We have $\|f(t, \cdot)\|_{\mathcal{B}} \leq L_f$.

Then the problem (3)-(4) in an abstract formulation of the problem (9)-(11). Theorem 4.4 implies that the problem (9)-(11) has at least one mild solution.

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